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THE CONVERGENCE OF CERTAIN PADÉ APPROXIMANTS*

J. NUTTALL

1. **Introduction.** Although Padé approximants have been used with great success in all manner of numerical computations for many years, it is only for very restricted classes of functions that rigorous results about their convergence have been obtained. In the case of functions given by series of Stieltjes, which we shall not discuss here, convergence of the P.A.'s $[N, N \pm J]$ as $N \to \infty$ occurs in the cut plane. There are a number of generalizations of this basic result, all depending on the non-negative character of a certain function. The convergence proof relies on the fact that the poles of the P.A.'s fall on the cut of the function to be approximated and therefore are outside the region where convergence is to be proved.

For more general functions the location of the poles of P.A.'s is not well understood, and finding information on this point constitutes an important unsolved problem in the theory of the P.A. For instance, it is not obvious that a sequence of P.A.s converges even inside the circle of convergence of a power series. In fact, it is possible to construct an entire function for which the sequence $[N, N]$ diverges and is actually unbounded everywhere except the origin. This may be done because it has been shown that, given a sequence of integers n_ν, $\nu = 1, 2, \cdots$, with $n_\nu > 2n_{\nu-1}$, there exists an entire function for which $[n_\nu, n_\nu]$ has a pole at any given point $b_\nu \neq 0$ of the complex plane.

Numerical examples have been given of functions for which $[N, N]$ appears to be converging well inside a region of analyticity of the function, but then for a high value of N, the P.A. contains a pole together with a nearby zero, both spurious. These considerations have led to the conjecture of Baker, Gammel and Wills [2] that states that, for a function f analytic in $|z| \leqq 1$ except for a finite number of poles in $0 < |z| < 1$ and except for $z = 1$ where the function is continuous when only points satisfying $|z| \leqq 1$ are considered, then a subsequence of the sequence $[N, N]$ exists which converges uniformly in the domain $|z| \leqq 1$ with small disks centered at the poles of f removed. This conjecture has been neither proved nor disproved.

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Until recently the only results about the convergence of diagonal or near diagonal sequences (probably the more powerful type of P.A.) have involved assumptions about the number of poles or some other similar property of the P.A.'s. These theorems, which have been summarized by Baker, [1] lack utility because of the difficulty of determining whether a particular function of interest satisfies the conditions needed for the theorems to hold.

In view of this situation, it seemed natural to search for a convergence theorem that did not use in its proof any information about the location of the poles of the P.A.'s. This meant that the type of convergence to be demonstrated had to be changed. It had been previously shown in a scattering theory problem that a sequence of approximations derived from the application of a stationary variational principle converged in measure [6]. Since there is a close formal connection between P.A.'s and variational approximations [7], the same type of convergence was suggested for P.A.'s.

Convergence in measure allows for the possibility of any number of P.A. poles in the region of interest provided that there are also nearby zeros, so that over a large fraction of the region (which increases toward 100% as $N \to \infty$) the P.A. differs only slightly from the function being approximated. Convergence in measure has been proved for diagonal sequences of P.A.'s for meromorphic functions, [8] and there have been several extensions and generalizations of this result, which will be mentioned below in Sec. 3. An outline of the method of proof is given in Sec. 2.

It is perhaps not too surprising that, in return for convergence in a larger domain than is the case for a power series, the type of convergence of P.A.'s is weaker. In mathematics, as in the real world, one rarely gets something for nothing.

2. **Proof of Convergence in Measure.** The original proof depended on two lemmas, the first being,

LEMMA 1. *Let a function $f(z)$ (with no pole at the origin) have the form*

$$f = g + Q/R \tag{1}$$

where

$$g(z) = \sum_{j=0}^{2N+J} g_j z^j \tag{2}$$

and Q, R are polynomials of degree $m - 1$, m respectively. Write the

$[N, N+J]$ Padé approximant to f as B/D where B, D are polynomials of degree no higher than $N+J$, N respectively. Then, if $m < N+1$,

$$B/D - f = -\frac{[Dg]_{2N+J+1}}{D} + \frac{[R[Dg]^{2N+J}]_{2N+J+1}}{DR}. \tag{3}$$

When applied to a polynomial $Q(z) = \sum_{j=0}^{M} q_j z^j$, $[Q]_a^b$ means $[Q]_a^b = \sum_{j=a}^{b} q_j z^j$. If a or b is omitted, it is taken to be 0 or M respectively.

Although it was not made clear in [8], this Lemma uses the Frobenius definition of the P.A., in which

$$B - Df = 0(z^{2N+J+1}). \tag{4}$$

B/D always exists and is unique.

The second lemma states,

LEMMA 2. *Whatever the values of z_j, N, the inequality relating to the polynomial $D(z) = \prod_{j=1}^{N}(z - z_j)$, $|D(z)| \leqq x^N$, holds in a region of the complex z-plane whose area is never greater than πx^2.*

This lemma is a special case of a more general result of Cartan [3].

To prove convergence in measure for meromorphic functions within the region $|z| \leqq 1$ for the sequence $[N, N+J]$ as $N \to \infty$, we first choose x as small as we like, and then choose m large enough so that Q/R can represent exactly the contribution to the meromorphic function F from all poles within the circle $|z| \leqq \rho = 8/x$. Then $F - Q/R$ is given by a power series whose coefficients we shall identify with g_j of Lemma 1, and which possess the property $|g_j| \leqq C\rho^{-j}$. Since $F - f$ can be made very small, the theorem follows if the right-hand side of (3) is small. This is true for the first term since $[Dg]_{2N+J+1}$ has no more than $2^N N$ terms each containing a factor g_j with $j \geqq N+1$, and a similar argument holds for the second term. The point is that g_j, $j \geqq N+1$, are small enough to compensate for the fact that $1/D$ may be large, provided we avoid a region of the z-plane area no larger than πx^2. Thus we have the theorem.

THEOREM. If *$F(z)$ is meromorphic in the whole complex plane and does not have a pole at the origin, then the $[N, N \pm J]$ P.A. (using the Frobenius definition) of F converges in measure to F as $N \to \infty$ in any bounded subset of the complex plane.*

3. **Extensions and Generalizations.** Pommerenke [9] has extended the theorem of the preceding section by demonstrating convergence in capacity, which implies convergence in measure, for a larger class of functions, namely those analytic in the whole complex

plane except for a set of capacity zero. Capacity (or transfinite diameter) is defined in Hille's book [5]. Pommerenke considers sequences $[N, M]$ for which $N, M \to \infty$, $1/\lambda \leqq N/M \leqq \lambda$, $\lambda > 1$.

For certain classes of entire functions, the coefficients g_j decrease sufficiently rapidly that stronger results may be obtained, and Wallin [10] has obtained some results in this direction.

Zinn-Justin [12] has done some related work which will be described elsewhere in this volume.

Recently, Gammel and Nuttall [4] showed that Padé approximants to a certain class of quasi-analytic functions converge in measure, even outside the natural boundary. This subject is discussed in more detail by Gammel in his contribution to this volume.

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Department of Physics, University of Western Ontario, London, Ontario, Canada