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SOME WEAKER FORMS OF COUNTABLE COMPACTNESS

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1. **Introduction.** C. E. Aull [1] has introduced a new class of topological spaces called E_1-spaces. This class generalizes the class of Hausdorff spaces. A space is said to be an E_1-space if every point is the intersection of a countable number of closed neighbourhoods. It is easy to see that a continuous function from a countably compact space into an E_1-space is closed, since countable compactness is a weakly hereditary property preserved under continuous maps and a countably-compact subset of an E_1-space is closed [1]. In the present note we consider a class of spaces called functionally countably compact. A space is said to be functionally countably compact if whenever $\mathcal{U}$ is a countable open filterbase on X such that the intersection A of the elements of $\mathcal{U}$ is equal to the intersection of the closures of the elements of $\mathcal{U}$, then $\mathcal{U}$ is a base for the neighbourhoods of A. Functionally countably compact E_1-spaces are characterized by the property: Every continuous function defined on them into an E_1-space is closed. Another class of spaces called countably C-compact has been considered. A space $(X, \mathcal{T})$ is countably C-compact if every countable $\mathcal{T}$-open cover of every closed subset has a finite subfamily, the closures of whose members cover the set. The following relationship exists:

$$\text{countably-compact} \Rightarrow \text{countably } C\text{-compact}$$
$$\Rightarrow \text{functionally countably compact.}$$

Also functionally countably compact + E_1 $\Rightarrow$ minimal E_1. That these implications are not reversible is shown by the following examples.

EXAMPLE 1.1. A countably C-compact space need not be countably compact.

Let Z represent the set of positive integers, let Y denote the subset of the plane consisting of all points of the form $(1/n, 1/m)$ and the points of the form $(1/n, 0)$ for n and m in Z. Let $X = Y \cup \{\infty\}$. Topologize X as follows: Let each point of the form $(1/n, 1/m)$ be open. Partition Z into infinitely many infinite equivalence classes, $\{Z_i\}_{i=1}^{\infty}$. Let a neighbourhood system for the point $(1/i, 0)$ be composed of all sets of the form $G \cup F \cup \{1/i, 0\}$ with

$$G = \{(1/i, 1/m) | m \geqq k\}$$

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and

$$F = \{(1/n, 1/m) | m \in Z_i \text{ and } n \geqq k\}$$

for some $k \in Z$. Let a neighbourhood system for the point ∞ be composed of all sets of the form $X - T$ where

$$T = \left\{\left(\frac{1}{n}, 0\right) \middle| n \in Z\right\} \cup \bigcup_{i=1}^{k}\left(\left\{\left(\frac{1}{i}, \frac{1}{m}\right) \middle| m \in Z\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{m}\right) \middle| m \in Z_i, n \in Z\right\}\right)$$

for some $k \in Z$. This neighbourhood system defines a topology on X which is, by construction, Hausdorff.

Now X is not countably compact since the closed subset $\{(1/n, 0) | n \in Z\}$ is not countably compact. It has been shown in [4] that X is C-compact and hence it is countably C-compact.

Example 1.2. A functionally countably compact space need not be countably C-compact.

Let $I = [0, 1]$. For each integer $n \geqq 2$, let $\{a_n^j\}_{j=1}^{\infty}$ be a strictly decreasing sequence in $(1/n, 1/(n-1))$ converging to $1/n$. Let $X = I \sim \bigcup_{j \geqq 1; n \geqq 2}\{a_n^j\}$. Topologize X as follows: Let $X \sim (\bigcup\{1/n\}_{n=1}^{\infty} \cup \{0\})$ retain the usual topology. Let a neighbourhood system of the point 0 be composed of all sets of the form $\{x \in X \mid |x| < 1/m\} \sim \{1/n\}_{n=1}^{\infty}$, m an integer. Let a neighbourhood system of the point $1/n$ be composed of all sets of the form $G \cap X$ where G is an open set in I with $\{1/n, a_{n-1}^1, \cdots, a_2^{(n-1)/2}\} \subset G$ in the case that n is odd, and with $\{1/n, a_{n-1}^1, a_{n-3}^2, \cdots, a_3^{n/2-1}\} \subset G$ in the case that n is even. For $n = 2$, we simply have $\{\frac{1}{2}\}$.

It is easy to see that X is Hausdorff. To see that X is not countably C-compact, consider the closed set $\{1/2n \mid n > 1\}$. The countable open cover $\{O_{2n} \mid n > 1\}$, where $O_{2n} = \{x \in X \mid |x - 1/2n| < 1/3n\} \cup \bigcup_{i=1}^{n-1} \{x \in X \mid |x - a_{2n-2i+1}^i| < 1/3n\}$, of $\{1/2n \mid n > 1\}$ has no finite subfamily the closures of whose members cover the space. In [6] it has been shown that X is functionally compact and hence X is functionally countably compact. Thus, this is a functionally countably compact space which is not countably C-compact.

Example 1.3. An E_1 space can be minimal E_1 without being functionally countably compact.

Let $X = \{a, b, a_{ij}, b_{ij}, C_i \mid i, j = 1, 2, 3, \cdots\}$. Let each point a_{ij} and b_{ij} be isolated. Let $\{U^K(C_i) \mid K = 1, 2, \cdots\}$ be the fundamental system of neighbourhoods of C_i where $U^K(C_i) = \{C_i, a_{ij}, b_{ij} \mid j \geqq K\}$ and let $\{V^K(a) \mid K = 1, 2, \cdots\}$ and $\{V^K(b) \mid K = 1, 2, \cdots\}$ be that of a and

b, respectively, where $V^K(a) = \{a, a_{ij} \mid i \geqq K, j = 1, 2, \cdots\}$ and $V^K(b) = \{b, b_{ij} \mid i \geqq K, j = 1, 2, \cdots\}$. This space is a minimal E_1-space [7] but is not functionally countably compact.

We have obtained a few new characterizations of minimal E_1-spaces given in §2. §3 deals with countably C-compact spaces, while functionally countably compact spaces are considered in §4.

We shall denote the set of natural numbers as well as countable index sets by N.

2. Characterizations of minimal E_1-spaces.

Definition 2.1 [2]. A space is said to be lightly-compact if every locally finite family of open sets is finite or equivalently if every countable open cover of the space has a finite subfamily, the closures of whose members cover the space.

Theorem 2.1 [7]. *An E_1-space X is minimal-E_1 iff any of the following equivalent conditions is satisfied*:

(a) *X is semiregular and lightly-compact.*

(b) *Every countable open filterbase which has a unique adherent point is convergent.*

Theorem 2.2. *An E_1-space is minimal-E_1 if and only if for every point $x \in X$ and every countable open filterbase $\mathcal{U}$ on X such that $\{x\} = \bigcap \{U \mid U \in \mathcal{U}\}$ and $\{x\} = \bigcap \{\bar{U} \mid U \in \mathcal{U}\}$, $\mathcal{U}$ is a base for the neighbourhoods of x.*

Proof. Let $(X, \mathcal{T})$ be a minimal-E_1-space. Let $\mathcal{U}$ be a countable open filterbase on X such that $\{x\} = \bigcap \{U \mid U \in \mathcal{U}\} = \bigcap \{\bar{U} \mid U \in \mathcal{U}\}$. Let R be any open set containing x. Now $\mathcal{U}$ is a countable open filterbase with a unique adherent point x and hence by Theorem 2.1 converges to x. Therefore, there exists a $U \in \mathcal{U}$ such that $U \subset R$ and hence $\mathcal{U}$ is a base for the neighbourhoods of x.

Conversely, let $\mathcal{U}$ be a countable open filterbase with a unique adherent point, say x. We are required to prove that $\mathcal{U}$ converges to the point x. Since X is an E_1-space there exist countable families $\{F_i \mid i \in N\}$ and $\{G_i \mid i \in N\}$ of respectively closed and open neighbourhoods of x such that $\{x\} = \bigcap \{F_i \mid i \in N\} = \bigcap \{G_i \mid i \in N\}$ and $x \in G_i \subset F_i$ for each $i \in N$. Let $V_n = \bigcap \{G_i \mid i = 1, 2, \cdots, n\}$. Then $\{V_n \mid n \in N\}$ is a countable open filterbase on X such that $x \in V_n$ for all $n \in N$. Let $\mathcal{V} = \{U \cup V_n \mid U \in \mathcal{U}, n \in N\}$. Then $\mathcal{V}$ is a countable open filterbase on X such that $\{x\} = \bigcap \{V \mid V \in \mathcal{V}\} = \bigcap \{\bar{V} \mid V \in \mathcal{V}\}$, for if $x \neq y$, then there exists a $U \in \mathcal{U}$ such that $y \notin \bar{U}$, because x is the unique adherent point of $\mathcal{U}$. Also since F_i's are closed, $\{x\} = \bigcap \{\bar{G}_i \mid i \in N\}$ and hence there exists a V_m

such that $y \notin \bar{V}_m$. Then $y \notin \bar{U} \cup \bar{V}_m$ and hence $y \notin \bigcap \{\bar{V} \mid V \in \mathcal{V}\}$. Therefore, there exists a $V \in \mathcal{V}$ such that $V \subset R$ for each open set R containing x. Since each $V \in \mathcal{V}$ contains a $U \in \mathcal{U}$, therefore there exists a $U \in \mathcal{U}$ for each open set R containing x such that $U \subset R$. Thus $\mathcal{U}$ converges to x. Hence the result.

Corollary 2.1. *An E_1-space is minimal-E_1 if and only if for every point $x \in X$ and every countable regular-open filterbase $\mathcal{U}$ on X such that $\{x\} = \bigcap \{U \mid U \in \mathcal{U}\}$ and $\{x\} = \bigcap \{\bar{U} \mid U \in \mathcal{U}\}$, $\mathcal{U}$ is a base for the neighbourhoods of x.*

Theorem 2.3. *An E_1-space is minimal-E_1 if and only if given $p \in X$, a countable open cover $\mathcal{C}$ of $X \sim \{p\}$ and an open neighbourhood U of p, there exists a finite subfamily $C_i \in \mathcal{C}$, $1 \leqq i \leqq n$ such that $X = U \cup [\bigcup \{\bar{C}_i \mid i = 1, 2, \cdots, n\}]$.*

Proof. Let $(X, \mathcal{T})$ be a minimal-E_1-space. Let $p \in X$ and $\mathcal{C} = \{C_i \mid i \in N\}$ be a countable open cover of $X - \{p\}$ and U an open neighbourhood of p. Suppose that the closures of no finite subfamily of $\mathcal{C}$ cover $X \sim U$. Then $V_n \cap (X - U) \neq \emptyset$ for all $n \in N$, where $V_n = \bigcap \{X - \bar{C}_i \mid i = 1, 2, \cdots, n\}$. Since X is minimal-E_1, therefore it is semiregular in view of Theorem 2.1. Therefore, there exists a regular open set T such that $p \in T \subset U$. Now $V_n \cap (X - \bar{T}) \neq \emptyset$, because, if $\overline{V_n \cap (X - \bar{T})} = \emptyset$, then since V_n's are open, $V_n \cap (X - \bar{T}) = \emptyset$, that is $V_n \cap (X - (\bar{T})^0) = \emptyset$, that is, $V_n \cap (X - T) = \emptyset$, as T is regular open and this implies $V_n \cap (X - U) = \emptyset$ as $T \subset U$, which is a contradiction. Now $\{V_n \cap (X - \bar{T}) \mid n \in N\}$ is a countable open filterbase which has no adherent point, because $p \notin \overline{X - \bar{T}}$ and if q is any other point different from p, then there exists a $C \in \mathcal{C}$ such that $q \in C$ and hence does not belong to $\overline{X - \bar{C}}$. Thus, there exists a V_n such that $q \notin \overline{V_n}$. Since by Theorem 2.1, X is lightly-compact, this leads to a contradiction. Hence the result.

Conversely, let $\mathcal{F}$ be a countable open filterbase with the unique adherent point p. Let U be an open neighbourhood of p. Now $\{X \sim \bar{F} \mid F \in \mathcal{F}\}$ is a countable open cover of $X \sim \{p\}$ and hence there exists a finite subfamily $\{F_i \in \mathcal{F} \mid 1 \leqq i \leqq n\}$ such that $X = U \cup [\bigcup \{\overline{X - \bar{F}_i} \mid i = 1, 2, \cdots, n\}]$. Since $\mathcal{F}$ is a filterbase, there exists an $F \in \mathcal{F}$ such that $F \subset \bigcap \{F_i \mid i = 1, 2, \cdots, n\}$. Now $X - U \subset \bigcup \{\overline{X \sim \bar{F}_i} \mid i = 1, 2, \cdots, n\}$. This implies $\bigcap \{\bar{F}_i^0 \mid i = 1, 2, \cdots, n\} \subset U$. Also $F_i \subset \bar{F}_i^0$ for $i = 1, 2, \cdots, n$. Then $F \subset U$ and hence $\mathcal{F}$ converges to p and X is minimal-E_1 by Theorem 2.1.

Corollary 2.2. *An E_1-space is minimal-E_1 if and only if given*

$p \in X$, a countable regular-open cover $\mathcal{C}$ of $X \sim \{p\}$ and an open neighbourhood U of p, there exists $C_i \in \mathcal{C}$, $1 \leqq i \leqq n$, such that $X = U \cup [\bigcup\{\bar{C}_i \mid i = 1, 2, \cdots, n\}]$.

3. Countably C-compact spaces.

DEFINITION 3.1. A space $(X, \mathcal{T})$ is said to be countably C-compact if given a closed set F of X and a countable $\mathcal{T}$-open cover $\mathcal{C}$ of F, there exists a finite subfamily $\{C_i \mid i = 1, 2, \cdots, n\}$ of $\mathcal{C}$ such that $F \subset \bigcup \{\bar{C}_i \mid i = 1, 2, \cdots, n\}$.

THEOREM 3.1. *Every continuous function from a countably C-compact space into an E_1-space is closed.*

PROOF. Let f be a continuous function from X into an E_1-space Y. Let C be a closed subset of X. Let $y \notin f(C)$. Since Y is an E_1-space, there exists a countable family $\{F_i \mid i \in N\}$ of closed neighbourhoods of y such that $\{y\} = \bigcap \{F_i \mid i \in N\}$. Since f is continuous, $\{f^{-1}(Y - F_i) \mid i \in N\}$ is a countable open cover of the closed subset C of the countably C-compact space X. Therefore, there exists a finite subfamily $\{f^{-1}(Y \sim F_{ij}) \mid j = 1, 2, \cdots, n\}$ such that $C \subset \bigcup \{\overline{f^{-1}(Y \sim F_{ij})} \mid j = 1, 2, \cdots, n\}$. Then $\bigcap \{F^0_{ij} \mid j = 1, 2, \cdots, n\} \subset Y \sim f(C)$. Since F_{ij}'s are neighbourhoods of y, $y \in \bigcap \{F^0_{ij} \mid j = 1, 2, \cdots, n\}$. Hence $f(C)$ is a closed subset of Y or f is a closed map.

COROLLARY 3.1. *Every continuous function from a countably compact space to an E_1-space is closed.*

PROOF. Every countably compact space is countably C-compact.

DEFINITION 3.2 [5]. A filterbase $\mathcal{G}$ is said to be (*regular*) *adherent convergent* if every (regular) open neighbourhood of the adherent set of $\mathcal{G}$ contains an element of $\mathcal{G}$.

THEOREM 3.2. *A space is lightly-compact iff every countable open filterbase is regular adherent convergent.*

PROOF. Let $(X, \mathcal{T})$ be a lightly-compact space, $\mathcal{U}$ a countable open filterbase. A the adherent set of $\mathcal{U}$ and R a regular open neighbourhood of A. Suppose that no element of $\mathcal{U}$ is contained in R, that is, $U \cap (X \sim R) \neq \emptyset$ for each $U \in \mathcal{U}$. Since R is regularly-open, $U \cap (X \sim R) \neq \emptyset \Rightarrow U \cap (X \sim \bar{R}) \neq \emptyset$. Now $\{U \cap (X \sim \bar{R}) \mid U \in \mathcal{U}\}$ is a countable open filterbase with empty adherence. Hence the contradiction. The converse follows from the fact that empty set is a regular open set.

THEOREM 3.3. *A space is countably C-compact iff every countable open filterbase is adherent convergent.*

PROOF. Let $(X, \mathcal{T})$ be a countably C-compact space and $\mathcal{G}$ be a countable open filterbase with the adherent set A. Let R be an open set containing A. Then $\{X \sim \bar{F} \mid F \in \mathcal{G}\}$ is a countable open cover of the closed set $X \sim R$ and hence there exists a finite subfamily $\{X \sim \bar{F}_i \mid i = 1, 2, \cdots, n\}$ such that

$$X \sim R \subset \bigcup \{\overline{X \sim \bar{F}_i} \mid i = 1, 2, \cdots, n\} \subset \bigcup \{X \sim F_i \mid i = 1, 2, \cdots, n\}$$

and hence $\bigcap \{F_i \mid i = 1, 2, \cdots, n\} \subset R$. Since $\mathcal{G}$ is a filterbase there exists an F such that $F \subset \bigcap \{F_i \mid i = 1, 2, \cdots, n\} \subset R$. Hence the result. Conversely, suppose that $(X, \mathcal{T})$ is not countably C-compact and that every countable open filterbase is adherent convergent. Therefore, there exists a closed set D and a countable open cover $\mathcal{C}$ of D such that $D \not\subset \bigcup \{\bar{C}_i \mid i = 1, 2, \cdots, n; C_i \in \mathcal{C}\}$ for any finite subfamily of $\mathcal{C}$. Let $V_n = \bigcap \{X \sim \bar{C}_i \mid i = 1, 2, \cdots, n\}$. Then $\{V_n \mid n \in N\}$ is a countable open filter base. Now $\bigcap \{\bar{V}_n \mid n \in N\} = \bigcap \{\overline{X \sim \bar{C}} \mid C \in \mathcal{C}\} \subset \bigcap \{X \sim C \mid C \in \mathcal{C}\} \subset X \sim D$. Hence there exists a V_n contained in $X \sim D$ which is not possible. Hence the result.

COROLLARY 3.2. *Every countably C-compact space is lightly-compact.*

DEFINITION 3.3 [5], [8]. A space is said to be seminormal if given a closed set C and an open subset G containing C there exists a regular open set R with $C \subset R \subset G$.

THEOREM 3.4. *A seminormal space is lightly-compact iff it is countably C-compact.*

PROOF. In view of Corollary 3.2, 'the only if' part alone need be proved.

Let $(X, \mathcal{T})$ be a seminormal lightly-compact space. Let $\mathcal{G}$ be a countable open filterbase with the adherent set A. Let G be an open set containing the closed set A. Since $(X, \mathcal{T})$ is seminormal, there exists a regular open set R such that $A \subset R \subset G$. Now since it is lightly-compact, there exists an F such that $F \subset R$ and hence $F \subset G$. Hence the result.

THEOREM 3.5. *A space X is countably C-compact if and only if given a closed subset F of X and a countable open cover $\mathcal{C}$ of $X \sim F$ and an open neighbourhood U of F there exists $C_i \in \mathcal{C}$, $i = 1, 2, \cdots, n$, such that $X = U \cup [\bigcup \{\bar{C}_i \mid i = 1, 2, \cdots, n\}]$.*

PROOF. Let X be a countably C-compact space, F a closed subset of X, U an open neighbourhood of F and $\mathcal{C}$ a countable open cover of $X \sim F$. Since $F \subset U$, therefore $\mathcal{C}$ is a countable open cover of $X \sim U$

also and consequently there exists a finite number of elements of $\mathcal{C}$, say C_i, $i = 1, 2, \cdots, n$, such that $X \sim U = \bigcup \{\bar{C}_i \mid i = 1, 2, \cdots, n\}$. Then $X = U \cup [\bigcup \{\bar{C}_i \mid i = 1, 2, \cdots, n\}]$.

Conversely, let $\mathcal{G}$ be a countable open filter base with the adherence set A and let R be an open set containing A. Then $\{X \sim \bar{F} \mid F \in \mathcal{G}\}$ is a countable open cover of the set $X \sim A$ and consequently there exists a finite subfamily $\{X \sim \bar{F}_i \mid i = 1, 2, \cdots, n\}$ such that $X = R \cup [\bigcup \{X \sim \bar{F}_i \mid i = 1, 2, \cdots, n\}]$. Since $\mathcal{G}$ is a filterbase, there exists an $F \in \mathcal{G}$ such that $F \subset \bigcap \{F_i \mid i = 1, 2, \cdots, n\}$. Now $F \subset R$ and hence $\mathcal{G}$ is adherent convergent and thus from Theorem 3.3, X is countably C-compact.

Corollary 3.3. *Every countably C-compact E_1-space is minimal-E_1.*

Proof. Observe that in an E_1-space every singleton is closed and apply Theorem 2.3.

Corollary 3.4. *A space X is countably C-compact if and only if given a closed subset F of X and a countable regular open cover $\mathcal{C}$ of $X \sim F$ and an open neighbourhood U of F there exists $C_i \in \mathcal{C}$, $i = 1, 2, \cdots, n$, such that $X = U \cup [\bigcup \{\bar{C}_i \mid i = 1, 2, \cdots, n\}]$.*

Proof. Obvious.

4. Functionally countably compact spaces.

Definition 4.1. A space X is said to be functionally countably compact if whenever $\mathcal{U}$ is a countable open filterbase on X such that the intersection A of the elements of $\mathcal{U}$ is equal to the intersection of the closures of the elements of $\mathcal{U}$, then $\mathcal{U}$ is a base for the neighbourhoods of A.

Theorem 4.1. *Every functionally countably compact E_1-space is minimal-E_1 and hence semiregular.*

Proof. That every functionally countably compact space is minimal-E_1, follows from Theorem 2.2 and Definition 4.1, semiregularity follows from Theorem 2.1. An independent proof for semiregularity can however be given as follows: Let $(X, \mathcal{T})$ be a functionally countably compact E_1-space. Let $x \in X$ and let G be an open set containing x. Since X is an E_1-space there exist countable families $\{F_i \mid i \in N\}$ and $\{G_i \mid i \in N\}$ respectively, of closed and open neighbourhoods of x such that $x \in G_i \subset F_i$ and $\{x\} = \bigcap \{G_i \mid i \in N\} = \bigcap \{F_i \mid i \in N\}$. Let $V_n = \bigcap \{\bar{G}_i^0 \mid i = 1, 2, \cdots, n\}$. Then $\{V_n \mid n \in N\}$ is a countable open filterbase such that $\{x\} = \bigcap \{V_n \mid n \in N\} = \bigcap \{\bar{V}_n \mid n \in N\}$ and hence there exists a V_n such that $x \in V_n \subset G$.

Now V_n is a regularly-open set and hence $(X, \mathcal{T})$ is semiregular.

THEOREM 4.2. *An E_1-space X is functionally countably compact if and only if every continuous mapping of X into any E_1-space is closed.*

PROOF. Suppose that X is a functionally countably compact E_1-space and let f be a continuous mapping from X into an E_1-space Y. Let C be a closed set in X. Let $y \notin f(C)$. Let $\{V_i \mid i \in N\}$ and $\{U_i \mid i \in N\}$ be countable collections of respectively closed and open neighbourhoods of y such that $y \in U_i \subset V_i$ for $i \in N$ and $\{y\} = \bigcap \{U_i \mid i \in N\} = \bigcap \{V_i \mid i \in N\}$. Let $G_n = \bigcap \{U_i \mid i = 1, 2, \cdots, n\}$. Let $\mathcal{G} = \{f^{-1}(G_n) \mid n \in N\}$.

Since f is continuous, $\mathcal{G}$ is a countable open filterbase. Also $f^{-1}(y) = \bigcap \{f^{-1}(G_n) \mid n \in N\} = \bigcap \{\overline{f^{-1}(G_n)} \mid n \in N\}$. Therefore $\mathcal{G}$ is a base for the neighbourhoods of $f^{-1}(y)$, that is, for each open set $R \subset X$ containing $f^{-1}(y)$, there exists an open set G_n such that $f^{-1}(y) \subset f^{-1}(G_n) \subset R$, that is, $y \in G_n \subset f(R)$. In particular, $y \in G_n \subset f(X - C) \subset y - f(C)$, since C is closed. Therefore there exists an open set G_n containing y which does not intersect $f(C)$. Hence $f(C)$ is closed.

Conversely, suppose that every continuous mapping of the E_1-space X into an E_1-space is closed. Let $\mathcal{U}$ be a countable open filterbase on X such that the intersection A of the elements of $\mathcal{U}$ equals the intersection of the closures of the elements of $\mathcal{U}$. Suppose further that there exists an open set R of X containing A such that for every $U \in \mathcal{U}$, $(X - R) \cap U \neq \emptyset$. Let Y be the decomposition of X whose only nondegenerate element is A and let f be the natural transformation of X onto Y defined by $x \in f(x)$. We topologize Y by defining a base $\mathcal{B}$ for a topology as follows: $B \in \mathcal{B}$ if and only if (i) $f^{-1}(B)$ is an open subset of $X - A$ or (ii) $f^{-1}(B) \in \mathcal{U}$.

Y with this topology is an E_1-space for $A = \bigcap \{\bar{U} \mid U \in \mathcal{U}\}$, where U is a closed neighbourhood of A in Y. If $y \in Y$ and $y \neq A$ then $f^{-1}(y)$ is a single point. Since $(X, \mathcal{T})$ is an E_1-space, there exists a countable family $\{F_i \mid i \in N\}$ of closed neighbourhoods of $f^{-1}(y)$ in X, such that $f^{-1}(y) = \bigcap \{F_i \mid i \in N\}$. Let I be a subset of N such that $F_i \cap A = \emptyset$ for $i \in I$. Now since $y \neq A$, $f^{-1}(y) \notin A$ and hence there exists a $U \in \mathcal{U}$ such that $f^{-1}(y) \notin \bar{U}$. Now $\{y\} = \bigcap_{i \in I} \{f(F_i) \cap \overline{X \sim \bar{U}}\} \cap \{\bigcap_{i \in N-I} f(F_i)\}$, where $\{f(F_i) \cap \overline{X \sim \bar{U}}\}$ and $f(F_i)$ are closed neighbourhoods of y in Y. Now f is a mapping of X onto Y which is continuous. By our hypothesis, f should be closed, but $f(X \sim R)$ is not closed since $f(A)$ is a limit point of $f(X \sim R)$ and $f(A) \notin f(X \sim R)$. This is contradiction. This completes the proof.

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