ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 2, Spring 1972

INVERSES, LOGARITHMS, AND IDEMPOTENTS IN $M(G)$[1]

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Let $M(R)$ denote the measure algebra on the line considered as a Banach algebra under convolution. In [**16**] we proved that if $\mu \in M(R)$ and μ is invertible, then μ has a factorization $\mu = \eta^k * \delta_x * \exp(\omega)$, where $k \in Z$, $x \in R$, $\omega \in M(R)$, and η can be chosen to be any measure in $L^1(R) + \mathbf{C}\delta_0$ whose Fourier transform is non-vanishing and has winding number one about zero. This result implies that the group $M(R)^{-1}/\exp(M(R))$ is isomorphic to $Z \oplus R$. Since the numbers k and x in the above factorization can be explicitly determined from μ, this result completely characterizes the invertible measures in $M(R)$ which have logarithms in $M(R)$.

The above result is a special case of a general factorization theorem proved in [**16**] for any commutative convolution measure algebra — in particular, for all algebras $M(G)$ for G a locally compact abelian (l.c.a.) group or $M(S)$ for S a locally compact abelian topological semigroup. This theorem is proved using the Arens-Royden theorem [**1**], [**8**], and a result in [**16**] which characterizes the cohomology groups of the maximal ideal space of any measure algebra. Another consequence of this result is a new proof of Cohen's idempotent theorem [**3**].

In [**17**] using some of the same techniques we proved that if a measure $\mu \in M(G)$ is invertible in $M(G)$ then its inverse must lie in a certain "small" subalgebra of $M(G)$ containing μ. This greatly simplifies the problem of determining the spectrum of an element of $M(G)$.

Unfortunately, the above results rely heavily on the specialized machinery developed in [**11**], [**12**], [**13**], and [**14**] for the study of convolution measure algebras. Also, the proof of the factorization theorem in [**16**] uses a considerable amount of sheaf theory and algebraic topology. Thus, the student of harmonic analysis who wishes to understand these results is faced with a discouraging amount of machinery to wade through.

Received by the editors October 13, 1970.
AMS 1970 *subject classifications*. Primary 43A10, 43A05.
[1]Research partially supported by the United States Air Force, Air Force Office of Scientific Research, under AF-AFOSR 1313-67.
[2]The author currently holds an Alfred P. Sloan Foundation fellowship.

The objective of this paper is to bring together some of the above results and prove them with a minimum of machinery. To this end, we shall restrict attention to $M(G)$ and not work with general convolution measure algebras. We shall use Šreĭder's generalized characters to describe the maximal ideal space of $M(G)$ rather than using the description given in [11]. Also, we have been able to eliminate the sheaf theory and algebraic topology from the proof of the factorization theorem of [16].

The results we present here all flow — with varying degrees of technical difficulty — from one general result (Theorem 1) concerning the maximal ideal space of $M(G)$. After a preliminary discussion in §1, we present Theorem 1 in §2. One portion of its proof (Lemma 1) is delayed until §6 in order to maintain the continuity of the discussion. Except for §6 — which relies heavily on some combinatorial machinery from [13] — the paper should appear self-contained to those who are familiar with Rudin's book [9].

In §3 we use Theorem 1 to give a new proof of Cohen's idempotent theorem. In §4 we present the factorization theorem for measures in $M(G)^{-1}$ and give several of its consequences. In §5 we prove that if $\mu \in M(G)$ then μ is invertible in $M(G)$ if and only if it is invertible in each subalgebra of a certain kind which contains μ. We use this to give a characterization of the spectrum of a continuous measure in $M(R)$.

As a corollary to Lemma 1, in §6 we prove the main theorem from [13] — which characterizes those L-subalgebras of $M(G)$ having $\hat{G}$ as maximal ideal space.

1. **Preliminaries.** Throughout the paper G will denote a locally compact abelian group and $M(G)$ its algebra of measures. Elements of $M(G)$ will be denoted by Greek letters. If $\mu, \nu \in M(G)$ then their convolution product will be denoted $\mu * \nu$. We shall use additive notation for the group operation in G.

If $\mu, \nu \in M(G)$ then "$\nu << \mu$" will mean "ν is absolutely continuous with respect to μ" and "$\mu \perp \nu$" will mean "μ and ν are mutually singular". If $\mathfrak{M}$ is a closed subspace (subalgebra, ideal) of $M(G)$ then $\mathfrak{M}$ will be called an L-subspace (L-subalgebra, L-ideal) provided $\mu \in \mathfrak{M}$ and $\nu << \mu$ imply $\nu \in \mathfrak{M}$. If $\mathfrak{M}$ is an L-subspace and $\mu \in M(G)$, then we say $\mu \perp \mathfrak{M}$ provided $\mu \perp \nu$ for each $\nu \in \mathfrak{M}$. We set $\mathfrak{M}^{\perp} = \{\mu \in M(G) : \mu \perp \mathfrak{M}\}$.

If $\mu \in M(G)$ then $|\mu|$ will denote its total variation measure and $\|\mu\|$ its total variation norm.

The following proposition is well known and follows directly from the Lebesgue Decomposition Theorem:

PROPOSITION 1.1. *If $\mathfrak{M}$ is an L-subspace of $M(G)$, then so is $\mathfrak{M}^{\perp}$ and $M(G) = \mathfrak{M} \oplus \mathfrak{M}^{\perp}$.*

The space $M(G)$ is a commutative Banach algebra with identity δ_0, where δ_0 is the unit mass at the origin. Hence, the maximal ideal space of $M(G)$ is a compact Hausdorff space. We shall use Šreĭder's description of this space.

DEFINITION 1.1 (ŠREĬDER [**10**]). A generalized character on G is a collection $f = \{f_\mu : \mu \in M(G)\}$ with $f_\mu \in L^\infty(\mu)$ for each $\mu \in M(G)$ and such that

(1) $\mu << \nu$ implies $f_\mu = f_\nu$ a.e./μ, and

(2) $f_\mu(x + y) = f_\mu(x)f_\mu(y)$ a.e./$\mu \times \mu$ for each $\mu \in M(G)$.

If f and g are generalized characters then we set $f = g$ provided $f_\mu = g_\mu$ a.e./μ for each $\mu \in M(G)$. The space of all generalized characters on G will be denoted $\Delta(G)$ or simply Δ.

In [**10**] Šreĭder shows that each complex homomorphism of $M(G)$ has the form $\mu \rightarrow \int f_\mu d\mu$ for some $f \in \Delta$. If $f \in \Delta$ and $\mu \in M(G)$ we shall denote the number $\int f_\mu d\mu$ by either $\mu\hat{}(f)$ or $\int f d\mu$.

The key to the study of Δ is to notice that it has a great deal of structure not generally enjoyed by maximal ideal spaces. It has a semigroup structure, an order structure, and two important topologies. In addition, it acts as a semigroup of endomorphisms of $M(G)$.

DEFINITION 1.2. If $f, g \in \Delta$, we define elements fg, $\bar{f}$, and $|f|$ in Δ by $(fg)_\mu = f_\mu g_\mu$, $(\bar{f})_\mu = \bar{f}_\mu$, and $|f|_\mu = |f_\mu|$, where these operations are defined pointwise in $L^\infty(\mu)$ for each $\mu \in M(G)$.

It is easy to see that these operations do indeed yield new elements of Δ.

DEFINITION 1.3. We set $\Delta^+ = \{f \in \Delta : f_\mu \geqq 0$ a.e./μ for each $\mu \in M(G)\}$. If $f, g \in \Delta^+$ then $f \leqq g$ will mean $f_\mu \leqq g_\mu$ for each $\mu \in M(G)$ and $f < g$ will mean $f \leqq g$ but $f \neq g$.

If 1 denotes the identically one function, then $1 \in \Delta^+$ and $f \leqq 1$ for all $f \in \Delta^+$.

The following facts concerning these notions have elementary proofs:

PROPOSITION 1.2. *If $f, g \in \Delta$ then $|f| \leqq |g|$ if and only if $f = gh$ for some $h \in \Delta$. In particular, for each $f \in \Delta$ there is a polar decomposition $f = |f|h$ with $h \in \Delta$ and $|h|^2 = |h|$.*

If $f \in \Delta^+$ then $f^z \in \Delta$ for each $z \in \mathbf{C}$ with $\operatorname{Re} z \geqq 0$, where $(f^z)_\mu = f_\mu{}^z$. Unless $f^2 = f$, the map $z \rightarrow f^z$ induces nontrivial analytic structure in the maximal ideal space of $M(G)$ (cf. [**11**, §3]).

DEFINITION 1.4. If $f \in \Delta$ and $\mu \in M(G)$ then we define $f\mu \in M(G)$ to be the measure $\nu << \mu$ such that $d\nu = f_\mu d\mu$.

PROPOSITION 1.3. *If* $f \in \Delta$ *then the map* $\mu \to f\mu : M(G) \to M(G)$ *is a bounded linear map (of norm one) and an algebra homomorphism. If* $g \in \Delta$ *also, then* $(f\mu)\hat{}(g) = \mu\hat{}(fg)$.

An important feature of our study of Δ is the relationship between two natural topologies on Δ. We describe these below.

DEFINITION 1.5. The weak topology on Δ is the weakest topology such that each of the maps $f \to \mu\hat{}(f)$ $(\mu \in M(G))$ is continuous. The strong topology on Δ is the weakest topology such that each of the maps $f \to f\mu$ $(\mu \in M(G))$ is continuous from Δ to $M(G)$ with the norm topology.

Note that the weak topology is the Gelfand topology for Δ if Δ is considered the maximal ideal space of $M(G)$. Obviously the strong topology dominates the weak topology since $\mu\hat{}(f) = (f\mu)\hat{}(1) = f\mu(G)$.

Unless G is discrete, the map $(f, g) \to fg : \Delta \times \Delta \to \Delta$ is not jointly continuous in the weak topology and the map $f \to |f| : \Delta \to \Delta$ is not weakly continuous. However, we have:

PROPOSITION 1.4. (a) *The map* $(f, g) \to fg : \Delta \times \Delta \to \Delta$ *is jointly continuous in the strong topology and separately continuous in the weak topology*;

(b) *the map* $f \to \bar{f}$: $\Delta \to \Delta$ *is continuous in both topologies*;

(c) *the map* $f \to |f| : \Delta \to \Delta^+$ *is strongly continuous*;

(d) *the set* Δ^+ *is closed in both topologies, as is any subset of the form* $\{g \in \Delta^+ : g \leqq f\}$ *or* $\{g \in \Delta^+ : g \geqq f\}$ *for* $f \in \Delta^+$.

In this paper we shall consider the group algebra $L^1(G)$ to be the subspace of $M(G)$ consisting of all absolutely continuous measures. With this agreement, $L^1(G)$ is an L-ideal of $M(G)$ (cf. [9]). In fact, it is the unique minimal L-ideal of $M(G)$.

We denote the dual group of G by $\hat{G}$. Each $\gamma \in \hat{G}$ determines an element of Δ — which we also denote by γ — by $\gamma_\mu = \gamma$ for every $\mu \in M(G)$. Hence, we consider $\hat{G}$ to be a subset of Δ.

If $f \in \Delta$ and $f_\mu \neq 0$ for some $\mu \in L^1(G)$, then there is an element $\nu << \mu$ such that $\nu\hat{}(f) \neq 0$. It follows that $\omega\hat{}(f) = (\nu\hat{}(f))^{-1} \cdot (\nu * \omega)\hat{}(f)$ for every $\omega \in M(G)$. Since $L^1(G)$ is an ideal, $\nu * \omega \in L^1(G)$. Hence, f is determined by its values on $L^1(G)$. Since every complex homomorphism of $L^1(G)$ is determined by a group character, we conclude that $f \in \hat{G}$.

If $\gamma \in \hat{G}$ and $\mu \in L^1(G)$ with $\mu\hat{}(\gamma) \neq 0$, then $\{f \in \Delta : \mu\hat{}(f) \neq 0\}$ is a weakly open subset of Δ which is contained in $\hat{G}$ (since $f \notin \hat{G}$ implies $f_\mu = 0$). Hence, $\hat{G}$ is embedded in Δ as a weakly (hence strongly) open subset. Note also that if $f \in \Delta$

then $f \in \hat{G}$ if and only if $|f| = 1$. Summarizing, we have:

PROPOSITION 1.5. *The group $\hat{G}$ is an open subgroup (in both the weak and strong topologies) of the semigroup Δ. Likewise, 1 is an isolated point (in either topology) of Δ^+. The subset* $\Delta \backslash \hat{G} = \{f \in \Delta : f_\mu = 0$ *for* $\mu \in L^1(G)\} = \{f \in \Delta : f\mu = 0$ *for* $\mu \in L^1(G)\}$ $= \{f \in \Delta : \mu\hat{}(f) = 0$ *for* $\mu \in L^1(G)\} = \{f \in \Delta : |f| < 1\}$.

Since $L(G)^1$ is a closed ideal of $M(G)$, we can construct the factor algebra $M(G)/L^1(G)$. We denote this algebra by $M_s(G)$ and let $\pi : M(G) \to M_s(G)$ be the quotient map. Since $L^1(G)$ is an L-ideal of $M(G)$, we have $M(G) = L^1(G) \oplus L^1(G)^\perp$ by Proposition 1.1. It follows that, as a Banach space, $M_s(G)$ may be identified with $L^1(G)^\perp$ — the space of purely singular measures on G. Under this identification, the multiplication on $M_s(G)$ is convolution followed by projection back into $M_s(G)$.

Since $\Delta \backslash \hat{G} = \{f \in \Delta : |f| < 1\} = \{f \in \Delta : \mu\hat{}(f) = 0$ for $\mu \in L^1(G)\}$ we have:

PROPOSITION 1.6. *The maximal ideal space of $M_s(G)$ is $\Delta \backslash \hat{G}$.*

If G is nondiscrete there are measures $\mu \in L^1(G)^\perp$ such that $\mu^2 \in L^1(G)$ (cf. [7]). Hence, $M_s(G)$ is not a semisimple algebra. Its radical consists of the image under π of those measures in $M(G)$ which are in every maximal ideal containing $L^1(G)$. We denote this space of measures by $\mathrm{Rad}(L^1(G))$.

PROPOSITION 1.7. $\mathrm{Rad}(L^1(G))$ *is an L-ideal of $M(G)$ and can be characterized as* $\{\mu \in M(G) : \mu\hat{}(f) = 0$ *for* $f \in \Delta \backslash \hat{G}\} = \{\mu \in M(G) : f\mu = 0$ *for* $f \in \Delta \backslash \hat{G}\}$.

PROOF. Since $\Delta \backslash \hat{G}$ consists of those elements of Δ such that the corresponding maximal ideal contains $L^1(G)$, it is trivial that $\mathrm{Rad}(L^1(G)) = \{\mu \in M(G) : \mu\hat{}(f) = 0$ for $f \in \Delta \backslash \hat{G}\}$. It follows that $\mu \in \mathrm{Rad}(L^1(G))$ if and only if $(f\mu)\hat{}(\gamma) = \mu\hat{}(f\gamma) = 0$ for $f \in \Delta$ and $\gamma \in \hat{G}$. Since the Fourier transform separates points in $M(G)$ (cf. [9]), we have $\mu \in \mathrm{Rad}(L^1(G))$ if and only if $f\mu = 0$ for each $f \in \Delta \backslash \hat{G}$. It follows immediately that $\mathrm{Rad}(L^1(G))$ is an L-ideal.

2. **The critical points of Δ^+.** The two topologies on Δ introduced in the last section do not generally agree. In the weak topology Δ is compact but multiplication is not jointly continuous. In the strong topology multiplication is jointly continuous but Δ is not compact. However, there are certain subsets of Δ on which these topologies do agree, and this turns out to be very useful.

PROPOSITION 2.1. (a) *The weak and strong topologies agree when restricted to any totally ordered subset of* Δ^+; (b) *if* $\{f_\alpha\}$ *is a net in* Δ, $f \in \Delta$, *and* $|f_\alpha| \leqq |f|$ *for every* α, *then* $f_\alpha \to f$ *weakly if and only if* $f_\alpha \to f$ *strongly.*

PROOF. If F is a totally ordered subset of Δ^+, then $f, g \in F$ implies either $f \leqq g$ or $g \leqq f$. Hence, for such f and g and $\mu \in M(G)$ we have

$$\begin{aligned}\|f\mu - g\mu\| &= \int |f - g| \, d|\mu| = \left| \int f \, d|\mu| - \int g \, d|\mu| \right| \\ &= | \, |\mu|^\wedge(f) - |\mu|^\wedge(g)|.\end{aligned}$$

This implies that the weak topology dominates the strong topology on F and, hence, they agree.

If $|f_\alpha| \leqq |f|$ for every α, where f_α, $f \in \Delta$, then $\mu \in M(G)$ implies that

$$\begin{aligned}\|f\mu - f_\alpha\mu\|^2 &= \left[\int |f - f_\alpha| \, d|\mu|\right]^2 \leqq \|\mu\| \int |f - f_\alpha|^2 \, d|\mu| \\ &\leqq \|\mu\| \int (2|f|^2 - f\bar{f}_\alpha - \bar{f}f_\alpha) d|\mu| \\ &= 2\|\mu\| \operatorname{Im}[(\bar{f}|\mu|)^\wedge(f) - (\bar{f}|\mu|)^\wedge(f_\alpha)].\end{aligned}$$

Hence, if $\{f_\alpha\}$ converges to f weakly it also converges to f strongly.

The first part of the above proposition leads to the following:

PROPOSITION 2.2. *Each nonempty strongly closed subset of* Δ^+ *contains minimal and maximal elements.*

PROOF. If $\Lambda \subset \Delta^+$ is strongly closed and $F \subset \Lambda$ is totally ordered, then the weak compactness of Δ^+ implies that there is a point f in the weak closure of F in Δ^+ such that $f \leqq g$ for every $g \in F$. Since $\{f\} \cup F$ is totally ordered, Proposition 2.1 implies that f is in the strong closure of F—hence, $f \in \Lambda$. It follows from Zorn's lemma that Λ has a minimal element. The same argument yields that Λ has maximal elements.

If Λ is both strongly closed and strongly open in Δ^+, then a minimal element of Λ must have a very special form. This is the key to each of our main theorems. Deriving this form is the object of the present section.

DEFINITION 2.1. An element of Δ^+ is called a critical point if it cannot be weakly approximated by strictly smaller elements. In other words, $h \in \Delta^+$ is a critical point if h is isolated (weakly) in $h\Delta^+ = \{f \in \Delta^+ : f \leqq h\}$.

Note that if h is a critical point, then necessarily $h^2 = h$; for, otherwise, we would have $h^r < h$ for $r > 1$ and $\lim_{r \to 1} h^r = h$.

PROPOSITION 2.3. *If* $h \in \Delta^+$ *the following statements are equivalent*:

(1) *h is a critical point*;

(2) *h is strongly isolated in* $h\Delta^+$;

(3) *h is a minimal element of a strongly open and closed subset* $\Lambda \subset \Delta^+$;

(4) *the set* $H = \{f \in \Delta : |f| = h\}$ *is strongly open in* $h\Delta$;

(5) *H is weakly open in* $h\,\Delta$.

PROOF. Since weak open sets are also strongly open, (1) implies (2) trivially. If h is strongly isolated in $h\Delta^+$, then as before we conclude $h^2 = h$, and further that $\{f \in \Delta^+ : f \geqq h\} = \{f \in \Delta^+ : hf = h\}$ is both open and closed in the strong topology and contains h as a minimal element. Hence, (2) implies (3). In fact (2) and (3) are equivalent since (3) obviously implies (2).

That (2) implies (4) follows from the fact that, under the strongly continuous map $f \to |f| : \Delta \to \Delta^+$, H is the inverse image of $\{h\}$ and $h\Delta$ is the inverse image of $h\Delta^+$. That (4) implies (5) follows immediately from Proposition 2.1(b). Finally, that (5) implies (1) is trivial, since $h\Delta^+ = \Delta^+ \cap h\Delta$ and $\{h\} = \Delta^+ \cap H$.

Recall from Proposition 1.5 that 1 is an isolated point of Δ^+ and, hence, 1 is a critical point. If d is defined by $d_\mu = 1$ if μ is a discrete measure and $d_\mu = 0$ if μ is a continuous measure, then it is easily seen that $d \in \Delta^+$ and, in fact, d is the unique minimal element of Δ^+. Hence, d is also a critical point. Each of these examples is a special case of a general method for constructing critical points which we describe below.

By a continuous isomorphism $\alpha : G' \to G$ of an l.c.a. group G' onto G we shall mean a group isomorphism which is continuous but not necessarily a homeomorphism. If α is such a map, then $\tilde{\alpha} : M(G') \to M(G)$ will be the induced map ($\tilde{\alpha}\mu = \mu \circ \alpha^{-1}$) on measures. Note that $\tilde{\alpha}$ is an order preserving isomorphism-isometry of $M(G')$ onto an L-subalgebra, $\mathfrak{M}$, of $M(G)$. The space $\mathfrak{M}$ is exactly the set of $\mu \in M(G)$ for which $\sup\{|\mu|(K) : \alpha^{-1}(K)$ compact in $G'\} = \|\mu\|$, while $\mathfrak{M}^\perp$ is the set of $\mu \in M(G)$ for which $\alpha^{-1}(K)$ compact implies $|\mu|(K) = 0$. It follows that $\mathfrak{M}^\perp$ is an L-ideal.

The fact that $\mathfrak{M}$ is an L-subalgebra and $\mathfrak{M}^\perp$ is an L-ideal implies that $h \in \Delta^+$, where h is defined by $h_\mu = 1$ if $\mu \in \mathfrak{M}$ and $h_\mu = 0$ if $\mu \in \mathfrak{M}^\perp$. Clearly $h^2 = h$ and $\mathfrak{M}$ is the range of the map $\mu \to h\mu : M(G) \to M(G)$.

PROPOSITION 2.4. *With G', α, $\mathfrak{M}$, and h as above we have that* $f \to f \circ \alpha : h\Delta(G) \to \Delta(G')$ *is an order preserving homeomorphism and a semigroup isomorphism of* $h\Delta(G)$ *onto* $\Delta(G')$. *Furthermore,*

$\mu\hat{}(f \circ \alpha) = (\tilde{\alpha}\mu)\hat{}(f)$ for $\mu \in M(G')$ and $f \in h\Delta(G)$. The image of h under $f \to f \circ \alpha$ is 1.

PROOF. Note that $f \to f \circ \alpha : h\Delta(G) \to \Delta(G')$ is simply the injection of $h\Delta(G)$ into $\Delta(G)$ followed by the adjoint map of $\tilde{\alpha}$ from the maximal ideal space of $M(G)$ to that of $M(G')$.

Each of the statements of the proposition is trivial except possibly that $f \to f \circ \alpha$ is onto. However, if $g \in \Delta(G')$ we simply define $f \in h\Delta(G)$ by $f_\mu = 0$ for $\mu \in \mathfrak{M}^\perp$ and $f_\mu = g_\nu \circ \alpha^{-1}$ for $\mu = \tilde{\alpha}\nu \in \mathfrak{M}$. The fact that $\mu = \tilde{\alpha}\nu$ implies that $g_\nu \circ \alpha^{-1} \in L^\infty(\mu)$. Clearly $f \in h\Delta$ and $f \circ \alpha = g$.

Note, it follows from the above that, since 1 is a critical point of $\Delta^+(G')$, h is a critical point of $\Delta^+(G)$. The object of this section is to show that every critical point of $\Delta^+(G)$ arises in this way from some map $\alpha : G' \to G$.

To this end, let $h \in \Delta^+$ be a critical point and set $\mathfrak{M} = \{\mu \in M(G) : h_\mu = 1 \text{ a.e.}/\mu\}$. Note that $\mathfrak{M}$ is an L-subalgebra of $M(G)$ and, since $h^2 = h$, $\mathfrak{M}^\perp = \{\mu \in M(G) : h_\mu = 0 \text{ a.e.}/\mu\}$ and $\mathfrak{M}^\perp$ is an L-ideal. Hence, to show that h has the form described in the preceding discussion, we need only find an l.c.a. group G' and a continuous isomorphism $\alpha : G' \to G$ such that $\tilde{\alpha}M(G') = \mathfrak{M}$. It is quite easy to find G' and α and show that they almost have the right property:

PROPOSITION 2.5. *With h and $\mathfrak{M}$ as above, there is an l.c.a. group G', a continuous surjective homomorphism $\alpha: G' \to G$, and an L-subalgebra $\mathfrak{M}' \subset M(G')$ such that*

(a) *$\tilde{\alpha} : M(G') \to M(G)$ maps $\mathfrak{M}'$ isometrically onto $\mathfrak{M}$;*

(b) *there is a continuous homomorphism $\varphi : M(G') \to \mathfrak{M}'$ such that φ is the identity on $\mathfrak{M}'$ and $h\tilde{\alpha}\mu = \tilde{\alpha}\varphi\mu$ for $\mu \in M(G')$;*

(c) *the map $\gamma \to F_\gamma$ ($F_\gamma(\mu) = \mu\hat{}(\gamma)$ for $\mu \in \mathfrak{M}'$) embeds $\hat{G}'$ homeomorphically as an open subset of the maximal ideal space of $\mathfrak{M}'$.*

PROOF. Let $H = \{f \in \Delta(G) : |f| = h\}$. Note that H is weakly open in the weakly compact set $h\Delta(G)$ by Proposition 2.3. It follows that H is locally compact. By Proposition 2.1(b) the weak and strong topologies agree on H. Now H is a group (with identity h and conjugation as inversion) and the operations are strongly continuous; hence, H is an l.c.a. group. Let $G' = \hat{H}$ be its dual group. Let $\eta : H \to \hat{G}'$ be the natural isomorphism from H to its second dual.

Let G_d be G with the discrete topology and consider $\hat{G}_d$ to be embedded in $\Delta(G)$ by identifying $\gamma \in \hat{G}_d$ with the generalized character f such that $f_\mu = 0$ if μ is continuous and $f_\mu = \gamma$ if μ is discrete. Recall that $d \in \Delta^+(G)$ is the element corresponding to the identity of $\hat{G}_d$.

There are continuous homomorphisms $\hat{\alpha} : \hat{G} \to \hat{G}'$ and $\hat{\beta} : \hat{G}' \to \hat{G}_d$ defined by $\hat{\alpha}(\gamma) = \eta(h\gamma)$ and $\hat{\beta}(\gamma') = d\eta^{-1}(\gamma')$. Note that $\hat{\beta} \circ \hat{\alpha}$: $\hat{G} \to \hat{G}_d$ is the dual of the identity map $G_d \to G$. If we take the duals of $\hat{\alpha}$ and $\hat{\beta}$ we obtain maps $\alpha : G' \to G$ and $\beta : G_d \to G'$ such that $\alpha \circ \beta =$ id: $G_d \to G$. Hence, α is onto.

Having constructed G' and $\alpha : G' \to G$, we now determine $\mathfrak{M}'$. Let D denote the algebra of functions on H of the form $\mu^\wedge|_H$ for $\mu \in \mathfrak{M}$. If $\mu \geqq 0$ then $\mu^\wedge|_H$ is positive definite and continuous on H. It follows from Bochner's Theorem (cf. [9, Chapter 1]) that $D \subset B(H)$. Since H contains a copy of $\hat{G}$ ($\{\gamma h : \gamma \in \hat{G}\}$) it follows that D is isomorphic and isometric in the $B(H)$ norm to the algebra $\mathfrak{M}$. Since $\eta : H \to \hat{G}'$ is an isomorphism and homeomorphism, the map $f \to f \circ \eta^{-1}$: $B(H) \to B(\hat{G}')$ maps D isomorphically and isometrically onto a subalgebra D' of $B(\hat{G}')$. Hence, there is a subalgebra $\mathfrak{M}'$ of $M(G')$ such that D' is the space of all Fourier transforms of elements of $\mathfrak{M}'$. Since $\mathfrak{M}$ is an L-subspace of $M(G)$, the space D is translation invariant on H. Hence, D' is translation invariant on $\hat{G}'$, and $\mathfrak{M}'$ is an L-subalgebra of $M(G')$. If $\mu \in \mathfrak{M}'$ and $\gamma \in \hat{G}$ then $(\tilde{\alpha}\mu)^\wedge(\gamma) = \mu^\wedge(\hat{\alpha}\gamma) = \mu^\wedge(\eta(h\gamma)) = \nu^\wedge(h\gamma)$, where ν is the element of $\mathfrak{M}$ which maps to μ under the composition of the isometries $\mathfrak{M} \to D \to D' \to \mathfrak{M}'$. Since $\nu \in \mathfrak{M}$, we conclude that $\nu^\wedge(h\gamma) = (h\nu)^\wedge(\gamma) = \nu^\wedge(\gamma)$ and, hence, $\tilde{\alpha}\mu = \nu$. Thus, $\tilde{\alpha}$ maps $\mathfrak{M}'$ isometrically onto $\mathfrak{M}$ and we have proved part (a).

Part (b) follows immediately; since we have $h\tilde{\alpha}\mu \in \mathfrak{M}$ for each $\mu \in M(G')$, the equation $h\tilde{\alpha}\mu = \tilde{\alpha}\varphi\mu$ uniquely defines a homomorphism φ which is the identity on $\mathfrak{M}'$.

The maximal ideal space of $\mathfrak{M}$ — hence of $\mathfrak{M}'$ — is the space $h\Delta(G)$, which contains H as an open subset since h is critical. With this identification of the maximal ideal space of $\mathfrak{M}'$, the embedding of $\hat{G}'$ in part (c) is just the homeomorphism $\eta^{-1} : \hat{G}' \to H$. This completes the proof.

We will complete our characterization of critical points by proving that $\mathfrak{M}' = M(G')$. This forces α to be one to one since $\tilde{\alpha} : \mathfrak{M}' \to \mathfrak{M}$ is an isometry.

The hard part of proving $\mathfrak{M}' = M(G')$ is showing that $L^1(G') \subset \mathfrak{M}'$. We defer this task until §6. In Lemma 1 of §6 we shall prove that an L-subalgebra satisfying (c) of Proposition 2.5 must contain $L^1(G')$. Assuming this, we have $\mu * \varphi(\nu) = \varphi(\mu * \nu) = \mu * \nu$ for every $\mu \in L^1(G')$ and every $\nu \in M(G')$. This clearly forces $\varphi(\nu) = \nu$ for $\nu \in M(G')$ and, hence, $M(G') = \mathfrak{M}'$. Thus, we have:

THEOREM 1. *If $h \in \Delta^+(G)$ then h is a critical point if and only if there is an l.c.a. group G' and a continuous isomorphism $\alpha : G' \to G$, such that if $\mathfrak{M} = \tilde{\alpha}M(G')$ then $h_\mu = 0$ for $\mu \in \mathfrak{M}^\perp$ and $h_\mu = 1$ for $\mu \in \mathfrak{M}$.*

If h_1 and h_2 are critical points of Δ^+ with $h_1 \leqq h_2$ and if $\alpha_1 : G_1 \to G$ and $\alpha_2 : G_2 \to G$ are corresponding group maps with $\mathfrak{M}_1 = \tilde{\alpha}_1 M(G_1)$ and $\mathfrak{M}_2 = \tilde{\alpha}_2 M(G_1)$, then $\mathfrak{M}_1 \subset \mathfrak{M}_2$ and $\tilde{\beta} = \tilde{\alpha}_2{}^{-1} \circ \tilde{\alpha}_1$ determines a homomorphism $\hat{\beta}$ of $M(G_1)$ into $M(G_2)$. Clearly, $\tilde{\beta}$ preserves positivity and norm. It follows that $\tilde{\beta}$ is induced by a continuous group isomorphism $\beta : G_1 \to G_2$ such that $\alpha_1 = \alpha_2 \circ \beta$. Hence, $h_1 \leqq h_2$ if and only if α_1 factors through α_2. If $h_1 = h_2$ then the map β is a homeomorphism and α_1 and α_2 are equivalent in the obvious sense.

It follows from the above that the set of critical points of Δ^+ is in one to one correspondence with the set of equivalence classes of continuous isomorphisms $\alpha : G' \to G$. For each critical point h we fix a corresponding group G_h and map $\alpha_h : G_h \to G$. We set $\alpha_h \leqq \alpha_k$ if α_h factors through α_k. Then $\alpha_h \leqq \alpha_k$ if and only if $h \leqq k$.

If h and k are critical points let $G' = [G_h \oplus G_k]/K$, where $K = \{(x, y) : \alpha_h(x) = -\alpha_k(y)\}$, and define $\alpha' : G' \to G$ by $\alpha' \circ \pi = \alpha_h \oplus \alpha_k$, where $\pi : G_h \oplus G_k \to G'$ is the quotient map. Then $\alpha' : G' \to G$ is a continuous isomorphism and each of α_h and α_k factors through α'. It follows that there is a critical point $h \vee k$ such that α' is equivalent to $\alpha_{h \vee k}$ and $h \vee k \geqq h$, $h \vee k \geqq k$. Furthermore, $h \vee k$ is the minimal critical point with this property.

Definition 2.2. If h is a critical point and $\alpha_h : G_h \to G$ the corresponding continuous isomorphism, let $M_h(G) = \tilde{\alpha}_h M(G_h)$ and $L_h(G) = \tilde{\alpha}_h L^1(G_h)$. Let $\mathcal{L}(G)$ denote the closed linear span in $M(G)$ of the spaces $L_h(G)$.

Proposition 2.6. *The space $\mathcal{L}(G)$ is an L-subalgebra of $M(G)$ which is a symmetric algebra under the involution $\mu \to \tilde{\mu}$ $(\tilde{\mu}(E) = \bar{\mu}(-E))$ on $M(G)$.*

Proof. Since each $L_h(G)$ is an L-subalgebra of $M(G)$, it follows that $\mathcal{L}(G)$ is an L-subspace and will be an L-subalgebra if we can prove that for each pair of critical points h, k there is a critical point g such that $L_h(G) * L_k(G) \subset L_g(G)$. However, it is trivial to see that $g = h \vee k$ has this property.

To see that $\mathcal{L}(G)$ is symmetric, note that each complex homomorphism F of $\mathcal{L}(G)$ is — when restricted to $L_h(G)$ for some critical point h — either zero or given by an element $\gamma_h \in \hat{G}_h$; i.e., $F(\mu) = \int \gamma_h \circ \alpha_h{}^{-1} d\mu$ for $\mu \in L_h(G)$. It follows that $F(\tilde{\mu}) = \overline{F(\mu)}$ for $\mu \in L_h(G)$. Since $\mathcal{L}(G)$ is the closed linear span of the spaces $L_h(G)$, we conclude that $\mathcal{L}(G)$ is symmetric.

We close this section by pointing out that if $\alpha : G' \to G$ is a continuous isomorphism, then we can think of G' as being the group G with a locally compact group topology which is at least as strong as the original topology of G. It follows that the critical points of $\Delta^+(G)$ are in a one to one, order preserving correspondence with the locally compact group topologies on G which dominate the original one.

3. **Cohen's idempotent theorem.** In this section we show how — assuming Theorem 1 — to give a short proof of Cohen's idempotent theorem [3], [9]. Of course, since the proof of Theorem 1 is so complicated, we are actually giving a very long proof of Cohen's idempotent theorem.

Let H^0 be the additive subgroup of $M(G)$ generated by the idempotents of $M(G)$. Equivalently, H^0 is the set of all $\mu \in M(G)$ for which $\mu\hat{}$ is integer valued on Δ. Let $H_1{}^0$ be the subgroup of H^0 generated by elements of the form $\gamma\mu$, where $\gamma \in \hat{G}$ and μ is Haar measure on some compact subgroup of G. Our object is to prove that $H^0 = H_1{}^0$.

Note that any two distinct elements $\mu, \nu \in H^0$ must differ by at least one in norm since $\mu\hat{} - \nu\hat{}$ is integer valued and not identically zero.

If $f \in \Delta$ and $\mu^2 = \mu \in M(G)$, then $(f\mu)^2 = f\mu^2 = f\mu$. Hence, if $\nu \in H^0$ then $f\nu \in H^0$ for all $f \in \Delta$.

PROPOSITION 3.1. *If $\mu \in H^0$ and $f\mu \in H_1{}^0$ for each $f \in \Delta^+$ with $f < 1$, then $\mu \in H_1{}^0$.*

PROOF. For $f \in \Delta^+$ with $f < 1$ we have $f\mu \in H_1{}^0$, $\mu - f\mu \in H^0$, and $\|\mu - f\mu\| + \|f\mu\| = \int(1-f)\,d|\mu| + \int f d|\mu| = \|\mu\|$. Hence, either $f\mu = 0$ or $\|\mu - f\mu\| \leqq \|\mu\| - 1$. By a simple iteration we can find $\nu \in H_1{}^0$ and $\omega \in H^0$ such that $\mu = \nu + \omega$ and $f\omega = 0$ for every $f \in \Delta^+$ with $f < 1$.

It follows that $\omega \in \mathrm{Rad}\, L^1(G)$ (cf. Proposition 1.7). Since the maximal ideal space of $\mathrm{Rad}\, L^1(G)$ is $\hat{G}$, it follows that the Fourier transform of ω is constant off some open-compact subset S of $\hat{G}$. It is then a simple matter to show that S is the coset ring of $\hat{G}$ and, hence, $\omega \in H_1{}^0$ (cf. [9, 2.4.3]).

The general case of the idempotent theorem is now just a simple reduction based on Theorem 1.

THEOREM 2 (COHEN [3]). *The additive group in $M(G)$ generated by elements of the form $\gamma\mu$ with $\gamma \in \hat{G}$ and μ the Haar measure of a compact subgroup of G is exactly the group generated by the idempotents of $M(G)$.*

PROOF. Let μ be an element of H^0 and set $\Lambda = \{f \in \Delta^+ : f\mu \notin H_1{}^0\}$. Since $f \to f\mu$ is strongly continuous and H^0 is a discrete group, we have Λ is open and closed in the strong topology. If h is a minimal element of Λ, then h is a critical point by Proposition 2.3.

By Theorem 1, there is a continuous isomorphism $\alpha_h : G_h \to G$ and

$h_\nu = 1$ for $\nu \in M_h(G) = \tilde{\alpha} M(G_h)$ and $h_\nu = 0$ for $\nu \in M_h(G)^\perp$. Now, $h\mu \in M_h(G)$ and $h\mu \notin H_1{}^0$; however, $f\mu \in H_1{}^0$ for $f \in \Delta^+$ with $f < h$.

Hence, if $\nu \in M(G_h)$ and $\tilde{\alpha}\nu = h\mu$, we conclude that ν satisfies the hypothesis of Proposition 4.1 as an element of $M(G_h)$ but does not satisfy the conclusion. It follows that Λ has no minimal elements and, hence, by Proposition 2.2, Λ is empty. We conclude $\mu \in H_1{}^0$.

The proofs of the theorems in the next two sections will follow the same pattern. Each time, we first prove that if the conclusion is true of $f\mu$ for each $f \in \Delta^+$ with $f < 1$, then it is true of μ as well. We then let Λ be the set of $f \in \Delta^+$ such that the statement is not true of $f\mu$. If Λ is nonempty we choose a minimal element and use Theorem 1 to return to the previous case.

4. **Logarithms of measures.** If A is a commutative Banach algebra with identity, we will denote by A^{-1} the group of invertible elements of A and by $\exp(A)$ the subgroup consisting of the range of the exponential function. The subgroup $\exp(A)$ is exactly the connected component of the identity in A^{-1} and is an open subgroup of A^{-1} (with the norm topology). The Arens-Royden Theorem [**1**], [**8**], [**5**] states that $A^{-1}/\exp(A)$ is isomorphic to the first Čech cohomology group—with integer coefficients—of the maximal ideal space of A.

In [**16**] we proved that for p bigger than zero the Čech cohomology group $H^p(\Delta, Z)$ is isomorphic to $\Sigma \oplus H^p(\Delta_h, Z)$, where Δ is the maximal ideal space of $M(G)$ and Δ_h is the maximal ideal space of $L_h(G) + \mathbf{C}\delta_0$ for each critical point $h \in \Delta^+$. This and the Arens-Royden Theorem yield a factorization theorem for invertible measures in $M(G)$ which characterizes those invertible measures which have logarithms in $M(G)$. Here we propose to use Theorem 1 to prove this factorization theorem without using the machinery of sheaf theory or algebraic topology.

We shall assume the following proposition—which is actually a restatement of the Arens-Royden Theorem:

Proposition 4.1 (Cf. [5, III 6.2 and 7.2]). *Let A be a commutative Banach algebra with maximal ideal space X. Then,*

(a) *if $a \in A^{-1}$ and $\hat{a} = \exp(f)$ for some $f \in C(X)$, then $a = \exp(b)$ for some $b \in A$;*

(b) *if $f \in C(X)^{-1}$, then $f = \hat{a} \cdot \exp(g)$ for some $a \in A^{-1}$, $g \in C(X)$.*

We proceed to develop our factorization theorem.

Proposition 4.2. *If $\omega \in M(G)$ and $\exp(\omega) \in \mathcal{L}(G)$ then $\omega \in \mathcal{L}(G)$.*

Proof. By Proposition 2.6, $\mathcal{L}(G)$ is a symmetric algebra. Hence, each complex homomorphism of $\mathscr{L}(G)$ extends to a complex homomorphism of $M(G)$ and is, therefore, given by an element of Δ. It follows that since $\mu = \exp(\omega)$ is invertible in $M(G)$ it is also invertible in $\mathscr{L}(G)$.

If we can prove that $\mu = \exp(\nu)$ for some $\nu \in \mathscr{L}(G)$, then $(2\pi i)^{-1}(\omega - \nu)$ will be an element of H^0 — hence, an element of $\mathscr{L}(G)$ by Theorem 2 — and it will follow that $\omega \in \mathscr{L}(G)$.

Suppose first that $f\mu \in \exp(\mathscr{L}(G))$ for every $f \in \Delta^+$ with $f < 1$. Since $f\mu = \exp(f\omega)$ it follows as in the preceding paragraph that $f\omega \in \mathscr{L}(G)$ for every $f \in \Delta^+$ with $f < 1$. We conclude that if $\omega = \omega_1 + \omega_2$ with $\omega_1 \in \operatorname{Rad} L^1(G)$ and $\omega_2 \in [\operatorname{Rad} L^1(G)]^\perp$, then $\omega_2 \in \mathscr{L}(G)$. Hence, $\exp(\omega_1) = \mu * \exp(-\omega_2)$ is an invertible element of $\mathscr{L}(G) \cap [\operatorname{Rad} L^1(G) + \mathbf{C}\delta_0] = L^1(G) + \mathbf{C}\delta_0$. Furthermore, on the maximal ideal space of $L^1(G) + \mathbf{C}\delta_0$, $\exp(\omega_1)\hat{}$ has a continuous logarithm (given by the Fourier transform of ω_1). Hence, by Proposition 4.1, $\exp(\omega_1) = \exp(\omega_3)$ for some $\omega_3 \in L^1(G) + \mathbf{C}\delta_0 \subset \mathscr{L}(G)$. As before, $\omega_1 - \omega_3 \in \mathscr{L}(G)$, and we conclude that $\omega = \omega_1 + \omega_2 \in \mathscr{L}(G)$.

We reduce the general case to the above case by using Theorem 1. If $\Lambda = \{f \in \Delta^+ : f\mu \notin \exp(\mathscr{L}(G))\}$ then Λ is strongly open and closed in Δ^+ since $\exp(\mathscr{L}(G))$ is open and closed in $\mathscr{L}(G)^{-1}$. If h is a minimal element of Λ, then $h\mu \notin \exp(\mathscr{L}(G))$ and $f\mu \in \exp(\mathscr{L}(G))$ for $f \in \Delta^+$ with $f < h$. If $\alpha_h : G_h \to G$ is the map corresponding to the critical point h as in Theorem 1 and if $\nu \in M(G_h)$ with $\tilde{\alpha}_h\nu = h\mu$, then $\nu \notin \exp(\mathscr{L}(G_h))$ and $f\nu \in \exp(\mathscr{L}(G_h))$ for $f \in \Delta^+(G_h)$ with $f < 1$. Also, $\nu = \exp(\rho)$ where $\rho \in M(G_h)$ with $\tilde{\alpha}\rho = h\omega$. However, this situation is impossible by the above paragraph. We conclude that $\Lambda = \emptyset$ and, hence, $\omega \in \mathscr{L}(G)$.

PROPOSITION 4.3. *If $\mu \in M(G)^{-1}$ and for each $f \in \Delta^+$ with $f < 1$ the measure $f\mu$ has a factorization of the form $f\mu = \nu * \exp(\omega)$ with $\nu \in \mathscr{L}(G)^{-1}$ and $\omega \in M(G)$, then μ also has a factorization of this form.*

PROOF. For each idempotent $k = k^2 \in \Delta^+$ with $k < 1$ we choose $\nu_k \in \mathscr{L}(G)^{-1}$ and $\omega_k \in M(G)$ such that $k\mu = \nu_k * \exp(\omega_k)$ and $\omega_k \perp \mathscr{L}(G)$. The condition that $\omega_k \perp \mathscr{L}(G)$ forces this factorization to be unique. In fact, if $\nu_k * \exp(\omega_k) = \nu_k' * \exp(\omega_k')$, then $\exp(\omega_k - \omega_k') = \nu_k' * \nu_k^{-1}$ which is in $\mathscr{L}(G)$. Hence, $\omega_k - \omega_k' \in \mathscr{L}(G)$ by Proposition 4.2, and $\omega_k - \omega_k' = 0$ since $\omega_k - \omega_k' \perp \mathscr{L}(G)$. Note that the uniqueness of ν_k and ω_k forces $k\nu_k = \nu_k$ and $k\omega_k = \omega_k$ since $k\mu = (k\nu_k) * \exp(k\omega_k)$ also.

We define two functions v and w on $\Delta \backslash \hat{G} = \{f \in \Delta : |f| < 1\}$ as follows: If $f \in \Delta\backslash\hat{G}$ we choose any $k = k^2 \in \Delta^+$ with $|f| \leqq k < 1$ (for example, $|f|^0 = \lim_{r\to 0} |f|^r$ will do) and define $v(f) = \nu_k\hat{}(f)$ and $w(f) = \omega_k\hat{}(f)$. The definitions of $v(f)$ and

$w(f)$ are independent of the choice of k. In fact if $k \geqq |f|$ and $j \geqq |f|$ then $jk \geqq k|f| = |f|$ and $jk\mu = j\nu_k * \exp(j\omega_k) = k\nu_j * \exp(k\omega_j)$. The uniqueness implies that $j\nu_k = k\nu_j$ and $j\omega_k = k\omega_j$. Hence,

$$\nu_j\hat{}(f) = \nu_j\hat{}(kf) = (k\nu_j)\hat{}(f) = (j\nu_k)\hat{}(f) = \nu_k\hat{}(jf) = \nu_k\hat{}(f)$$

and, similarly, $\omega_j\hat{}(f) = \omega_k\hat{}(f)$.

We now prove that v and w are continuous in the weak topology. If $k = k^2 \in \Delta^+\backslash\{1\}$ then v and w are continuous on the set $\{f \in \Delta : |f| \leqq k\}$, since on this set they are given by the fixed functions $\nu_k\hat{}$ and $\omega_k\hat{}$. Unfortunately it may take infinitely many of these sets to cover $\Delta\backslash\hat{G}$. The solution to this difficulty is simple, but it escaped us for several months.

Let $\{f_\alpha\} \subset \Delta\backslash\hat{G}$ be a net converging weakly to $f \in \Delta\backslash\hat{G}$. For each α we choose $k_\alpha = k_\alpha^2 \in \Delta^+\backslash\{1\}$ with $|f_\alpha| \leqq k_\alpha$. We let $g \in \Delta^+\backslash\{1\}$ be a weak cluster point of the net $\{k_\alpha\}$ and choose $k = k^2 \in \Delta^+\backslash\{1\}$ with $g \leqq k$. Note that $kf_\alpha \rightarrow kf$ weakly and $|kf_\alpha| \leqq k$ for every α. Hence, $v(kf_\alpha) \rightarrow v(kf)$ and $w(kf_\alpha) \rightarrow w(kf)$. We shall prove that the nets $\{v(f_\alpha) - v(kf_\alpha)\}$ and $\{w(f_\alpha) - w(kf_\alpha)\}$ cluster to zero.

Choose $\epsilon > 0$. Since $\{k_\alpha\}$ clusters to $g \leqq k$, the set S of all α for which $\|k_\alpha\mu - kk_\alpha\mu\| = \int (k_\alpha - kk_\alpha)\, d|\mu| < \epsilon$ is a cofinal set. Furthermore, $\|(k_\alpha\mu)^{-1}\| = \|k_\alpha\mu^{-1}\| \leqq \|\mu^{-1}\|$ and $\|kk_\alpha\mu\| \leqq \|\mu\|$. It follows that for $\epsilon' > 0$ we can choose ϵ small enough that for $\alpha \in S$, $k_\alpha\mu = (kk_\alpha\mu) * \exp(\rho_\alpha)$ with $\|\rho_\alpha\| < \epsilon'$. Hence, for $\alpha \in S$ we have $\exp(\omega_{kk_\alpha} - \omega_{k_\alpha} + \rho_\alpha) = \nu_{k_\alpha} * \nu_{kk_\alpha}^{-1} \in \mathcal{L}(G)$. It follows that $\omega_{kk_\alpha} - \omega_{k_\alpha} = \rho_\alpha'$, where ρ_α' is the part of ρ_α lying in $\mathcal{L}(G)^\perp$. Hence $\|\omega_{kk_\alpha} - \omega_{k_\alpha}\| \leqq \|\rho_\alpha'\| < \epsilon'$ for $\alpha \in S$. We conclude that for $\alpha \in S$,

$$\begin{aligned} |w(f_\alpha) - w(kf_\alpha)| &= |\hat{\omega}_{k\alpha}(f_\alpha) - \hat{\omega}_{kk_\alpha}(kf_\alpha)| \\ &= |\hat{\omega}_{k_\alpha}(f_\alpha) - \hat{\omega}_{kk_\alpha}(f_\alpha)| < \epsilon'. \end{aligned}$$

A similar conclusion holds for $|v(f_\alpha) - v(kf_\alpha)|$. Hence, these nets cluster to zero and v and w are continuous.

Recall from Proposition 1.6 that $\Delta\backslash\hat{G}$ is the maximal ideal space of $M_s(G) = M(G)/L^1(G)$. The maximal ideal space of $\mathcal{L}(G)/L^1(G)$ is the space of equivalence classes of elements of $\Delta\backslash\hat{G}$ under the relation $f \sim g$ if $\nu\hat{}(f) = \nu\hat{}(g)$ for $\nu \in \mathcal{L}(G)$. Note that if $f \sim g$ and $|f| \leqq k \in \Delta^+\backslash\{1\}$, $|g| \leqq j \in \Delta^+\backslash\{1\}$, then $v(f) = \nu_k\hat{}(f) = \nu_k\hat{}(f) + \nu_j\hat{}(f) - \nu_{jk}\hat{}(f) = \nu_k\hat{}(g) + \nu_j\hat{}(g) - \nu_{jk}\hat{}(g) = \nu_j\hat{}(g) = v(g)$. Hence, v determines a continuous function on the maximal ideal space of $\mathcal{L}(G)/L^1(G)$. Note that v does not vanish since each $\nu_k\hat{}$ is invertible.

By Proposition 4.1, there is a measure $\nu_1 \in \mathcal{L}(G)$ and a continuous function φ on $\Delta\backslash\hat{G}$ such that $v(f) = \nu_1\hat{}(f)\exp(\varphi(f))$ for $f \in \Delta\backslash\hat{G}$. Then $\exp(\varphi(f) + w(f)) = (\mu * \nu_1^{-1})\hat{}(f)$ for $f \in \Delta\backslash\hat{G}$. By Proposition 4.1 applied to $M_s(G)$, there is $\omega \in M(G)$ with $\mu * \nu_1^{-1} * \exp(-\omega) = \delta_0 + \rho$ and $\rho \in L^1(G)$. If $\nu = (\delta_0 + \rho) * \nu_1$ then $\nu \in \mathcal{L}(G)$ and $\mu = \nu * \exp(\omega)$. This completes the proof.

Proposition 4.4. *If $\mu \in M(G)^{-1}$ then $\mu = \nu * \exp(\omega)$ for some $\nu \in \mathcal{L}(G)$ and $\omega \in M(G)$.*

Proof. We use Theorem 1 to reduce to the case covered by Proposition 4.3. We note that $\Lambda = \{f \in \Delta^+ : f\mu$ has no such factorization$\}$ is strongly open and closed, and proceed as before.

Proposition 4.5. *If $\mu \in \mathcal{L}(G)^{-1}$ then $\mu = \mu_1 * \cdots * \mu_n * \exp(\omega)$ with $\omega \in \mathcal{L}(G)$ and each μ_i an element of $(L_{h_i}(G) + \mathbf{C}\delta_0)^{-1}$ for some critical point $h_i \in \Delta^+$.*

Proof. The union of those L-subalgebras of $\mathcal{L}(G)$ which are finite sums of algebras $L_h(G)$ is dense in $\mathcal{L}(G)$. It follows that $\mu = \mu' * \exp(\rho)$ for some μ' in such an L-subalgebra. Hence, without loss of generality we may assume that $\mu \in L_d(G) + L_{h_1}(G) + \cdots + L_{h_n}(G)$ and $L_{h_i}(G) * L_{h_j}(G) \subset L_{h_k}(G)$ (i.e., $h_i \vee h_j = h_k$) for each i, j and some k. Also, we may assume that the discrete part of μ is δ_0, since, otherwise, $\mu = \lambda * [\delta_0 + \lambda^{-1} * \eta]$, where $\mu = \lambda + \eta$ is the decomposition of μ into discrete part λ and continuous part η.

Thus, let $\mu = \delta_0 + \nu_1 + \cdots + \nu_n$ with $\nu_i \in L_{h_i}(G)$. We can assume h_1 has been chosen minimal among $h_1, \cdots, h_n$. It follows that $h_1\mu = \delta_0 + \nu_1$ and so $\delta_0 + \nu_1$ is invertible in $\mathbf{C}\delta_0 + L_{h_1}^{h_1}(G)$. Hence, $\mu = (\delta_0 + \nu_1)[\delta_0 + \nu_2' + \cdots + \nu_n']$ with $\nu_j' = (\delta_0 + \nu_1)^{-1}\nu_j$ and $h_1\nu_j' = 0$. It follows that $\nu_j' \perp L_{h_1}(G)$ for each j and so $\nu_2' + \cdots + \nu_n'$ is in an L-subalgebra of $\mathcal{L}(G)$ containing only $n-1$ algebras $L_h(G)$. This forms the basis of an induction which yields the required factorization.

If we combine Proposition 4.4 and 4.5 we obtain our main theorem:

Theorem 3. *If $\mu \in M(G)^{-1}$ then μ has a factorization $\mu = \nu_1 * \cdots * \nu_n * \exp(\omega)$ with $\omega \in M(G)$ and each $\nu_i \in (\mathbf{C}\delta_0 + L_{h_i}(G))^{-1}$ for some critical point h_i.*

Recall that $B(\hat{G})$ is the algebra of Fourier transforms of elements of $M(G)$. We have the following corollary of Theorem 3:

Corollary 4.6. *If $\varphi \in B(\hat{G})$, $\varphi^{-1} \in B(\hat{G})$, and $\varphi > 0$, then $\log \varphi \in B(\hat{G})$.*

Proof. Let φ be the Fourier transform of $\mu \in M(G)^{-1}$. Since

$\varphi > 0$ we have $\tilde{\mu} = \mu$. If $\mu = \nu * \exp(\omega)$ with $\nu \in \mathcal{L}(G)$ and $\omega \in M(G)$, then $\mu^2 = \nu * \tilde{\nu} * \exp(\omega + \tilde{\omega})$. Since $\mathcal{L}(G)$ is symmetric it follows that $\nu * \tilde{\nu} = \exp(\rho)$ for some $\rho \in \mathcal{L}(G)$ with $\tilde{\rho} = \rho$. Hence, $\mu = \exp[\frac{1}{2}(\omega + \tilde{\omega} + \rho)]$. Clearly $\log \varphi$ is the Fourier transform of $\frac{1}{2}[\omega + \tilde{\omega} + \rho]$.

COROLLARY 4.7. *If* $\varphi \in B(\hat{G})^{-1}$ *then* $|\varphi| \in B(\hat{G})^{-1}$.

PROOF. Since $|\varphi|^2 = \varphi\bar{\varphi} \in B(\hat{G})^{-1}$ we have $\log|\varphi| = \frac{1}{2}\log|\varphi|^2 \in B(\hat{G})$ and, hence, $|\varphi| \in B(\hat{G})^{-1}$.

We now specialize to the case where G is the real line R. There are only two continuous isomorphisms of l.c.a. groups onto R (to within equivalence): id: $R \to R$ and $\alpha_d : R_d \to R$, where R_d is R with the discrete topology. Hence, there are only two algebras $L_h(R)$ ($L^1(R)$ and $L_d(R) = M_d(R)$).

It follows from Proposition 4.1 that if $\mu \in [L^1(R) + \mathbf{C}\delta_0]^{-1}$, then μ has a logarithm in $M(G)$ if and only if its Fourier transform has winding number zero about the origin as a function on $R \cup \{\infty\}$. Also, by a result of Bohr [2], if $\mu \in M_d(G)^{-1}$ then $\mu = \delta_x * \exp(\omega)$ for a unique $x \in R$ and some $\omega \in M_d(G)$. Hence, we have the following corollary to Theorem 3:

COROLLARY 4.8. If $\mu \in M(R)^{-1}$ *then there are unique numbers* $k \in Z$ *and* $x \in R$ *such that* $\mu = \eta^k * \delta_x * \exp(\omega)$ *for some* $\omega \in M(R)$, *where* η *can be chosen to be any element of* $(L^1(R) + \mathbf{C}\delta_0)^{-1}$ *whose Fourier transform has winding number one.*

One choice for η in the above corollary is the measure whose Fourier transform is $(1 - it)(1 + it)^{-1}$. If we rephrase Corollary 4.8 in terms of Fourier transforms we have:

COROLLARY 4.9. *If* $\varphi \in B(R)^{-1}$ *then there are unique numbers* $k \in Z$ *and* $x \in R$ *such that the function*

$$\psi(t) = \log\varphi(t) - k[\log(1 - it) - \log(1 + it)] - ixt$$

is an element of $B(R)$.

Finally, we should mention that Corollary 4.8 leads to a characterization of the spectrum of the Wiener-Hopf operator W_μ acting on $L^1(R^+)$, where $\mu \in M(R)$ and $W_\mu f(x) = \int_0^\infty f(t)\, d\mu(x - t)$ $(x \geqq 0)$ for $f \in L^1(R^+)$ (cf. [4]).

5. **The spectrum of a measure.** Since the maximal ideal space of $M(G)$ is the set Δ of generalized characters, and since generalized characters are clearly impossible to understand, the problem of deciding when a given measure is in no maximal ideal—hence, is

invertible — appears to be very difficult. Unless a measure has some very special form — e.g., is absolutely continuous, discrete, or concentrated on a thin set [9, Chapter 5] — there seem to be no reasonable methods for computing its spectrum.

The techniques we have been employing turn out to be of some use on this problem. They yield a considerable simplification of the problem of computing the spectrum of a measure.

DEFINITION 5.1. Let $\mathfrak{M}$ be an L-subalgebra of $M(G)$ which contains the identity δ_0. We shall say $\mathfrak{M}$ is balanced if $\mathfrak{M} \cap \mathcal{L}(G)$ is closed under the involution $\mu \to \tilde{\mu}$. Equivalently, $\mathfrak{M}$ is balanced if $\mathfrak{M} \cap L_h(G)$ is closed under involution for each critical point $h \in \Delta^+$.

We shall prove that if $\mathfrak{M}$ is balanced and $\mu \in \mathfrak{M}$, then μ is invertible in $M(G)$ if and only if it is invertible in $\mathfrak{M}$.

PROPOSITION 5.1. *Let $\mathfrak{M}$ be a balanced L-subalgebra of $M(G)$. Suppose $\nu \in \mathcal{L}(G)^{-1}$ and $\omega \in M(G)$ have the properties that $\nu *$ $\exp(\omega) \in \mathfrak{M}$, $\omega \perp \mathcal{L}(G)$, and $f\nu$, $f\omega \in \mathfrak{M}$ for all $f \in \Delta^+$ with $f < 1$. Then $\nu \in \mathfrak{M}$ and $\omega \in \mathfrak{M}$.*

PROOF. We first prove that we may as well assume that either $L^1(G) \subset \mathfrak{M}$ or $L^1(G) \cap \mathfrak{M} = (0)$.

If $L^1(G) \cap \mathfrak{M} \neq (0)$ then it is an L-ideal of $\mathfrak{M}$ and a symmetric L-subalgebra of $L^1(G)$. It follows that the support of $L^1(G) \cap \mathfrak{M}$ is an open subgroup G_0 of G and $L^1(G) \cap \mathfrak{M} = L^1(G_0)$. Since $L^1(G_0)$ is an ideal of $\mathfrak{M}$ we conclude that $\mathfrak{M} \subset M(G_0)$. It is easily seen that each generalized character in $\Delta(G_0)$ is the restriction of a generalized character in $\Delta(G)$. (See the proof of Proposition 6.2.) Since $\mu =$ $\nu * \exp(\omega) \in \mathfrak{M}$ is invertible in $M(G)$, it is invertible in $M(G_0)$. By Theorem 3, $\mu = \nu' * \exp(\omega')$ with $\nu' \in \mathcal{L}(G_0)$ and $\omega' \in M(G_0)$ and $\omega' \perp \mathcal{L}(G_0)$. However, as before, this forces $\omega = \omega'$ and $\nu = \nu'$. Each element of $\Delta^+(G_0)\backslash\{1\}$ is the restriction of an element of $\Delta^+(G)\backslash\{1\}$, so we have $f\nu \in \mathfrak{M}$ and $f\omega \in \mathfrak{M}$ for each $f \in \Delta^+(G_0)\backslash\{1\}$. Hence, ν, ω, and $\mathfrak{M}$ satisfy the hypothesis of the proposition as elements of $M(G_0)$. Thus, if $L^1(G) \cap \mathfrak{M} \neq (0)$ we may as well assume that $L^1(G) \subset \mathfrak{M}$.

Let $\nu = \nu_1 + \nu_2$ and $\omega = \omega_1 + \omega_2$ where ν_1, $\omega_1 \in \operatorname{Rad} L^1(G)$ and $\nu_2, \omega_2 \in (\operatorname{Rad} L^1(G))^\perp$. Note that since $f\nu = f\nu_2 \in \mathfrak{M}$ and $f\omega = f\omega_2 \in \mathfrak{M}$ for $f \in \Delta^+\backslash\{1\}$, we have $\nu_2, \omega_2 \in \mathfrak{M}$ (cf. Proposition 1.7). Also, we have $\nu_1 \in \mathcal{L}(G)$ and so $\nu_1 \in \mathcal{L}(G) \cap \operatorname{Rad} L^1(G) =$ $L^1(G)$.

If $L^1(G) \subset \mathfrak{M}$ then $\nu \in \mathfrak{M}$ and $\rho = \exp(\omega_2) = \mu * \nu^{-1} * \exp(-\omega_1)$ $\in \mathfrak{M} \cap [\delta_0 + \operatorname{Rad} L^1(G)]$. It follows that the Gelfand transform of $\pi\rho \in \mathfrak{M}/L^1(G)$ has a continuous logarithm (determined by $\omega_2\hat{}$). By

Proposition 4.1, $\rho = \exp(\omega_3)(\delta_0 + \lambda)$ for some $\omega_3 \in \mathfrak{M}$ and $\lambda \in L^1(G)$. Hence, $\exp(\omega_2 - \omega_3) \in \mathcal{L}(G)$ and by Proposition 4.2 we conclude that ω_2 is the part of ω_3 in $\mathcal{L}(G)^\perp$. Hence $\omega_2 \in \mathfrak{M}$ and $\omega \in \mathfrak{M}$.

If $L^1(G) \cap \mathfrak{M} = (0)$, then each power of $|\mu|$ is purely singular and so $\pi|\mu|$ has spectral radius $\|\mu\|$ in $M(G)/L^1(G) = M_s(G)$. Hence, there is $g \in \Delta\backslash\hat{G}$ such that $| |\mu|\hat{\ }(g)| = \|\mu\|$. If $f = |g|$, then $f \in \Delta^+\backslash\{1\}$ and $f\mu = 1$ a.e./μ. It follows that $\mu = f\mu = (f\nu) * \exp(f\omega)$. However, the uniqueness of this factorization (with $\omega \perp \mathcal{L}(G)$) implies that $f\nu = \nu$ and $f\omega = \omega$. Hence, $\nu \in \mathfrak{M}$ and $\omega \in \mathfrak{M}$.

Theorem 4. *Let $\mathfrak{M}$ be a balanced L-subalgebra of $M(G)$. If $\mu \in \mathfrak{M}$ and $\mu \in M(G)^{-1}$, then $\mu^{-1} \in \mathfrak{M}$. Furthermore, the measures ν and ω in the factorization $\mu = \nu * \exp(\omega)$ of Proposition 4.4 must lie in $\mathfrak{M}$ if ω is chosen so that $\omega \perp \mathcal{L}(G)$.*

Proof. Let $\Lambda_1 = \{f \in \Delta^+ : f\mu^{-1} \notin \mathfrak{M}\} = \{f \in \Delta^+ : f\mu \notin \mathfrak{M}^{-1}\}$. The first description of Λ shows that it is strongly open while the second shows that it is strongly closed. Let $\Lambda_2 = \{f \in \Delta^+\backslash\Lambda_1 : f\mu$ does not have a factorization $\rho * \exp(\lambda)$ with $\rho \in \mathcal{L}(G) \cap \mathfrak{M}$ and $\lambda \in \mathfrak{M}\}$. Clearly Λ_2 is strongly open and closed in $\Delta^+\backslash\Lambda_1$ and, hence, in Δ^+. Furthermore, by uniqueness, we have $\Lambda_2 = \{f \in \Delta^+\backslash\Lambda_1 : f\omega \notin \mathfrak{M}$ or $f\nu \notin \mathfrak{M}\}$. Hence, to prove Theorem 4 we must show that $\Lambda_1 \cup \Lambda_2 = \emptyset$.

If $\Lambda_1 \cup \Lambda_2 \neq \emptyset$ we let h be a minimal element of it—hence, a critical point—apply Theorem 1 and Proposition 5.1, and obtain a contradiction.

Corollary 5.2. *If $\mu \in M(G)$ then the spectrum of μ as an element of $M(G)$ is the same as the spectrum of μ as an element of any balanced L-subalgebra containing μ.*

Corollary 5.3. *If $\mu \in M(G)$ and $|\mu|^n \perp \mathcal{L}(G)$ for each n, then $\delta_0 + \mu$ is invertible in $M(G)$ if and only if $\delta_0 + \mu = \exp(\omega)$ for some ω with $\omega << \sum_{n=0}^{\infty}(2\|\mu\|)^{-n}|\mu|^n$.*

Proof. If $\mathfrak{M} = \{\nu \in M(G) : \nu << \sum_{n=0}^{\infty}(2\|\mu\|)^{-n}|\mu|^n\}$ then $\mathfrak{M}$ is a balanced L-subalgebra containing μ. Furthermore, $\mathfrak{M} \cap \mathcal{L}(G) = \mathbf{C}\delta_0$, so in the factorization $\mu = \nu * \exp(\omega)$ of Theorem 4 we must have $\nu = \lambda\delta_0$ for some scalar λ.

We now restrict attention to the line. If $\mu \in M(R)$ and μ is continuous, we set $\rho = \sum_{n=0}^{\infty}(2\|\mu\|)^{-n}|\mu|^n$ and $\mathfrak{M} = \{\nu : \nu << \rho\} + L^1(R)$. Note that $\mathfrak{M}$ is a balanced L-subalgebra containing μ and $\mathfrak{M} \cap M_d(R) = \mathbf{C}\delta_0$, $\mathfrak{M} \cap L^1(R) = L^1(R)$.

Now each complex homomorphism of $\mathfrak{M}$ is determined by a bounded Borel function f on R such that $f(0) = 1$, $f(t) = e^{-itx}$ a.e./dt for some

$x \in R$ or $f = 0$ a.e./dt; and for each n, $f(t_1 + \cdots + t_n) = f(t_1) \cdots f(t_n)$ a.e./$\mu \times \cdots \times \mu$ on R^n.

Hence, applying Theorem 4 we have:

Corollary 5.4. *If $\mu \in M(R)$ and μ is continuous then the spectrum of μ in $M(G)$ is the set of numbers $\int f\, d\mu$ where f ranges over those bounded Borel functions for which*

(a) *$f(t) = e^{-ixt}$ a.e./dt for some $x \in R$ or $f = 0$ a.e./dt; and*

(b) *for each n, $f(t_1 + \cdots + t_n) = f(t_1) \cdots f(t_n)$ a.e./$\mu \times \cdots \times \mu$ on R^n.*

Corollary 5.5. *If μ is a continuous measure on R, $\mu \geqq 0$, and $\mu^n \perp \mu^m$ for $n \neq m$, then the spectrum of μ is the disc of radius $\|\mu\|$.*

Proof. Choose disjoint sets $E_1, E_2, \cdots$ of Lebesgue measure zero such that μ^n is concentrated on E_n. If $|z| \leqq \|\mu\|$ set $f_z(t) = (z/\|\mu\|)^n$ on E_n and $f_z(t) = 0$ if $t \notin \bigcup_{n=1}^{\infty} E_n$. Then f_z satisfies the conditions of Corollary 5.4 and $\int f_z(t)\, d\mu(t) = z$.

6. **The main lemma.** In this section we complete the proof of Theorem 1 by proving the following lemma:

Lemma 1. *Let $\mathfrak{M}$ be an L-subalgebra of $M(G)$ containing the identity. If the map $\gamma \to F_\gamma$ ($F_\gamma(\mu) = \mu\hat{}(\gamma)$ for $\mu \in \mathfrak{M}, \gamma \in \hat{G}$) embeds $\hat{G}$ homeomorphically as an open subset of the maximal ideal space of $\mathfrak{M}$, then $L^1(G) \subset \mathfrak{M}$.*

The proof relies heavily on the combinatorial machinery of [**14**]. We have been unable to simplify this part of the argument in any essential way. Our discussion here will consist of two preliminary propositions, an outline of the machinery we require from [**14**], and an argument to show how this machinery proves the lemma.

Proposition 6.1. *If $\mathfrak{M}$ satisfies the conditions of the lemma, then $\mathfrak{M}$ is weak-$*$ dense in $M(G)$.*

Proof. Let K be the smallest closed subset of G on which each measure in $\mathfrak{M}$ is supported. Since $\mathfrak{M}$ is an L-subalgebra, it suffices to prove that $K = G$. Clearly K is a closed subsemigroup of G. Also, K is not contained in any proper closed subgroup of G; if it were then the set of Fourier transforms of elements of $\mathfrak{M}$ would not separate points in $\hat{G}$.

Hence, if $K \neq G$ then K is a proper closed subsemigroup of G such that the closed group generated by K is G. By Lemma 2 of [**12**], there is a continuous homomorphism $\alpha: G \to R$ such that $(0) \neq \alpha(K) \subset R^+$. However, the functional $F_t(\mu) = \int \exp(-t\alpha(x))\, d\mu(x)$ $(\mu \in \mathfrak{M})$ is a complex homomorphism of $\mathfrak{M}$, for each $t \in R^+$, which clearly does not correspond to a point of $\hat{G}$ if $t > 0$. Also, $F_t(\mu) \to \mu\hat{}(1)$ as $t \to 0$ for each $\mu \in \mathfrak{M}$. This contradicts the assumption that $\hat{G}$ is open in the maximal ideal space of $\mathfrak{M}$. We conclude that $K = G$ and $\mathfrak{M}$ is weak-$*$ dense in $M(G)$.

Definition 6.1. Let Ω be the set of all $f = \{f_\mu : \mu \in \mathfrak{M}\}$ such that $f_\mu \in L^\infty(\mu)$ for each μ, $\nu << \mu$ implies $f_\nu = f_\mu$ a.e./μ, and $f_\mu(x+y) = f_\mu(x)f_\mu(y)$ a.e./$\mu \times \mu$ for each μ. We give Ω the weakest topology under which each map $f \to \mu\hat{}(f) = \int f_\mu \, d\mu$ $(\mu \in \mathfrak{M})$ is continuous.

Clearly Ω can be identified with the maximal ideal space of $\mathfrak{M}$ just as Δ was identified with the maximal ideal space of $M(G)$.

Note that the hypothesis on $\mathfrak{M}$ in Lemma 1 implies that $\hat{G}$ is open in Ω. It is a subgroup of the group $H = \{f \in \Omega : |f| = 1\}$. It follows that H is also open in Ω and is an l.c.a. group. If $\hat{G} \neq H$ we can proceed as in the proof of Proposition 2.5 and embed $\mathfrak{M}$ in $M(G')$ for some group G' for which $\hat{G}'$ is H and the hypothesis of Lemma 1 is still satisfied. Hence, we may as well—and we will—assume that $\hat{G} = H = \{f \in \Omega : |f| = 1\}$.

Note that, as in Proposition 2.3, the statement that $\hat{G}$ is open in Ω is equivalent to the statement that $\{1\}$ is an isolated point of $\Omega^+ = \{f \in \Omega : f \geqq 0\}$.

Proposition 6.2. *Lemma 1 is true for all l.c.a. groups G if it is true for groups of the form $R^n \times K$ with K compact.*

Proof. If G is an arbitrary l.c.a. group then it has an open subgroup G_0 of the form $R^n \times K$ [9, Chapter 2]. For each $x \in G$ we let $\mathfrak{M}_x$ denote the space of all measures in $\mathfrak{M}$ which are concentrated on the coset $x + G_0$. Note that, since $\mathfrak{M}$ is weak-$*$ dense, $\mathfrak{M}_x \neq (0)$ for each x. We shall show that $\mathfrak{M}_0$ satisfies the hypothesis of Lemma 1 as an L-subalgebra of $M(G_0)$.

Let Ω_0 be the space of Definition 6.1 for the algebra $\mathfrak{M}_0$. The restriction map $\Omega \to \Omega_0$ maps Ω onto a compact subset of Ω_0. To show that this map is onto it is sufficient to show that if $\mu_1, \cdots, \mu_n \in \mathfrak{M}_0$ and

$$(*) \qquad \mu_1 * \nu_1 + \cdots + \mu_n * \nu_n = \delta_0$$

has a solution for $\nu_1, \cdots, \nu_n \in \mathfrak{M}$ then it also has a solution in $\mathfrak{M}_0$. However, if $\nu_1, \cdots, \nu_n \in \mathfrak{M}$ satisfy $(*)$ and $\nu_1', \cdots, \nu_n'$ are the restrictions of these measures to G_0, then clearly $\nu_1', \cdots, \nu_n' \in \mathfrak{M}_0$ also give a solution to $(*)$. Hence, $\Omega \to \Omega_0$ is onto.

In particular, each $f \in \Omega_0$ with $|f| < 1$ is necessarily the restriction of some $g \in \Omega$ with $|g| < 1$. Hence, $\{f \in \Omega_0 : |f| < 1\}$ is in the image under restriction of the compact set $\{g \in \Omega : |g| < 1\}$. It follows that $\{f \in \Omega_0 : |f| = 1\}$ is open and is the image under restriction of $\hat{G}$. Hence, $\hat{G}_0 = \{f \in \Omega_0 : |f| = 1\}$ is open in Ω_0.

Since we are assuming that Lemma 1 holds for G_0, we must have $L^1(G_0) \subset \mathfrak{M}_0$. Since $\mathfrak{M}$ is weak-$*$ dense and $L^1(G) \cap \mathfrak{M}$ is an ideal of $\mathfrak{M}$ containing $L^1(G_0)$, a standard approximate identity argument yields $L^1(G) \subset \mathfrak{M}$.

Henceforth, we assume $G = R^n \times K$ for some compact group K. We write elements of G in the form (x, k) with $x \in R^n$ and $k \in K$.

Note that if $n = 0$ then $\hat{G} = \hat{K}$ is a discrete subset of the maximal ideal space of $\mathfrak{M}$, and the Shilov idempotent theorem [**5**, III.6.5] implies that Haar measure on G is an element of $\mathfrak{M}$. Hence, Lemma 1 is trivial in this case, and we may as well assume $n > 0$.

We now describe a class of algebras introduced in §2 of [**14**]. By $\mathfrak{M}_{\text{loc}}$ we shall mean the space of all (possibly unbounded) measures μ on the ring of bounded Borel sets of G, such that $\mu|_E \in \mathfrak{M}$ for each compact set E.

Definition 6.2. Let A be a compact, convex subset of R^n. Then,

(a) $\varphi_A(x, k) = \sup \{\exp(-x \cdot y) : y \in A\}$;

(b) $\|\mu\|_A = \int \varphi_A \, d|\mu|$ for $\mu \in \mathfrak{M}_{\text{loc}}$; and

(c) $\mathfrak{M}(A) = \{\mu \in \mathfrak{M}_{\text{loc}} : \|\mu\|_A < \infty\}$.

By Lemma 2.2 of [**14**] each $\mathfrak{M}(A)$ is a Banach algebra under convolution. Note that $A \subset B$ implies $\mathfrak{M}(B) \subset \mathfrak{M}(A)$ and $\mathfrak{M}(\{0\}) = \mathfrak{M}$. In fact $\mathfrak{M}(\{y\})$ is isomorphic to $\mathfrak{M}(\{0\}) = \mathfrak{M}$ for each $y \in R^n$, where $d\mu(x) \to \exp(-x \cdot y)\, d\mu(x)$ describes the isomorphism. Using this fact, Lemma 2.5 of [**14**], and Theorem 4.1 of [**14**] we conclude that the maximal ideal space of $\mathfrak{M}(A)$ is given by the set $\Omega(A)$ described below.

Definition 6.3. Let $\Omega(A)$ be the space of all "functions" $e^{-y}f$, where $y \in A$, $f \in \Omega$, and $(e^{-y}f)_\mu(x, k) = \exp(-y \cdot x) f_\mu(x, k)$ for $\mu \in \mathfrak{M}$ with compact support. If $\mu \in \mathfrak{M}$ with compact support, we define $\mu\hat{\ }(e^{-y}f) = \int \exp(-x \cdot y) f_\mu(x, k)\, d\mu(x, k)$.

Note that we can extend the definition of $\mu\hat{\ }(e^{-y}f)$ to all $\mu \in \mathfrak{M}(A)$ by continuity and obtain a complex homomorphism $\mu \to \mu\hat{\ }(e^{-y}f)$ of $\mathfrak{M}(A)$ for each $e^{-y}f \in \Omega(A)$.

We let $\Gamma(A)$ be the subset of Ω consisting of those $e^{-y}f$ for which $|f| = 1$—i.e., for which $f \in \hat{G}$. Note that each element of $\Gamma(A)$ can be written in the form $(x, k) \to e^{-z \cdot x}\gamma(k)$ for some $z \in \mathbf{C}^n$ with $\operatorname{Re} z \in A$ and some $\gamma \in \hat{K}$. Hence, if $\mu \in \mathfrak{M}(A)$ we define $\mu\check{\ }(z, \gamma) = \mu\hat{\ }(g)$ with $g(x, k) = e^{-z \cdot x}\gamma(h)$, $g \in \Gamma(A)$. Note that for $\operatorname{Re} z$ in the interior of A and for fixed $\gamma \in \hat{K}$, $\mu\check{\ }(z, \gamma)$ is an analytic function of z. If μ has compact support, then $\mu \in \mathfrak{M}(A)$ for all A and $\mu\check{\ }(z, \gamma)$ is analytic on $\mathbf{C}^n$ for each γ.

PROPOSITION 6.3. *There is a measure μ in $\mathfrak{M}$ with compact support, and a compact convex set $A \subset R^n$ with zero as an interior point, such that $\mu\hat{}(1) = 0$ but $\mu\hat{}$ does not vanish on $\Omega(A)\backslash\Gamma(A)$.*

PROOF. By hypothesis, $\hat{G}$ is open in Ω. Hence, 1 is an interior point of $\hat{G}$. Choose $\epsilon > 0$ and $\mu_1, \cdots, \mu_n \in \mathfrak{M}$ such that $\{f \in \Omega : |\mu_i\hat{}(1) - \mu_i\hat{}(f)| < \epsilon$ for $i = 1, \cdots, n\}$ is contained in $\hat{G}$. Hence, if $\nu = |\mu_1| + \cdots + |\mu_n|$, we have

$$\int (1 - |f|)d\nu = \sum \int |1 - |f|\,|d|\mu_i| \geqq \sum |\mu_i\hat{}(1) - \mu_i\hat{}(|f|)| \geqq \epsilon$$

for $f \in \Omega$ with $|f| < 1$. Thus, $|\nu\hat{}(f)| \leqq \nu\hat{}(|f|) \leqq \nu\hat{}(1) - \epsilon$ for $f \in \Omega\backslash\hat{G}$. Since ν is a positive regular measure we can replace ν by a positive measure $\omega \in \mathfrak{M}$ which has compact support and satisfies $|\omega\hat{}(f)| \leqq \omega\hat{}(1) - \epsilon/2$ for $f \in \Omega\backslash\hat{G}$. We then set $\mu = \omega\hat{}(1)\delta_0 - \omega$. Note that $\mu\hat{}(1) = 0$ and $|\mu\hat{}(f)| \geqq \epsilon/2$ on $\Omega\backslash\hat{G}$.

Since μ has compact support, $\mu\hat{}(e^{-y}f)$ exists for all $y \in R^n$ and $f \in \Omega$. Clearly we can choose a compact convex neighborhood A of zero in R^n such that $|\mu\hat{}(e^{-y}f)| \geqq \epsilon/4$ for $f \in \Omega\backslash\hat{G}$ and $y \in A$. Hence, $\mu\hat{}$ does not vanish on $\Omega(A)\backslash\Gamma(A)$.

PROPOSITION 6.4. *There are an n-simplex $S \subset R^n$, with zero as an interior point, and measures $\mu_1, \cdots, \mu_n \in \mathfrak{M}(S)$ such that $\mu_1\hat{}$ does not vanish on $\Omega(S)\backslash\Gamma(S)$, $\mu_1\hat{}, \cdots, \mu_n\hat{}$ do not vanish simultaneously at any point $e^{-y}f$ with $y \in \partial S$, but $\mu_1\hat{}(1) = \cdots = \mu_n\hat{}(1) = 0$.*

PROOF. Let μ_1 be the measure μ of Proposition 6.3 and A the compact, convex set. We let $V = \{(z, x) \in C^n \times \hat{K} : \mu_1\check{}(z, x) = 0$ and $\operatorname{Re} z \in \operatorname{int} A\}$. Note $V \subset \{(z, x) \in C^n \times \hat{K} : \mu_1\check{}(z, x) = 0$ and $\operatorname{Re} z \in A\}$ and $\bar{V}$ is compact.

Since $\bar{V}$ is compact and $\hat{K}$ is discrete, there are only finitely many points $\gamma \in \hat{K}$ for which $V \cap (C^n \times \{\gamma\}) \neq \emptyset$. Let J be this finite set of points. Then V is a subvariety of the n-dimensional analytic space $X = \{(z, \gamma) \in C^n \times J : \operatorname{Re} z \in \operatorname{int} A\}$. If, for $i = 2, \cdots, n$, we let $\nu_i = \delta_0 - \delta_{(x_i, 0)}$, where $x_i = (0, 0, \cdots, 1, \cdots, 0)$ (1 is the ith position) is in R^n, and set $W = \{(z, \gamma) \in X : \nu_2\check{}(z, \gamma) = \cdots = \nu_n\check{}(z, \gamma) = 0\}$, then W is a one-dimensional submanifold of X. Clearly $\bar{V} \cap W$ is compact and is a subvariety of W. Hence, $V \cap W$ is finite (cf. [6, III.B.17]). In other words $\{(z, \gamma) \in X : \mu_1\check{}(z, \gamma) = \nu_2\check{}(z, \gamma) = \cdots = \nu_n\check{}(z, \gamma) = 0\}$ is a finite set.

Since $\mathfrak{M}$ is weak-$*$ dense in $M(G)$ there are measures $\mu_2, \cdots, \mu_n$ in $\mathfrak{M}$, with compact support, which are close enough to $\nu_2, \cdots, \nu_n$ so that $\{(z, \gamma) \in X : \mu_1\check{}(z, \gamma) = \mu_2\check{}(z, \gamma) = \cdots = \mu_n\check{}(z, \gamma) = 0\}$ remains a compact subvariety of X—hence, a finite set—and we still have $\mu_1\hat{}(1) = \cdots = \mu_n\hat{}(1) = 0$.

We now have $\{e^{-y}f \in \Omega(A) : y \in \text{int } A, \mu_1\hat{}(e^{-y}f) = \cdots = \mu_n\hat{}(e^{-y}f) = 0\}$ is a finite set which includes 1. Hence, we can choose a small simplex $S \subset \text{int } A$ with $0 \in \text{int } S$, such that $\mu_1\hat{}, \cdots, \mu_n\hat{}$ do not all vanish at any $e^{-y}f$ with $y \in \partial S$. This completes the proof.

Let $\mu_1, \cdots, \mu_n$ and S be as above, and let $S_1, \cdots, S_{n+1}$ be the $(n-1)$-faces of S. We have $\mu_1\hat{}, \cdots, \mu_n\hat{}$ do not vanish simultaneously on $\Omega(S_i)$ for each i. Also, for each i, $\mu_1, \cdots, \mu_n$ are elements of the Banach algebra $\mathfrak{M}(S_i)$, which has $\Omega(S_i)$ as a maximal ideal space. It follows that for each i there are measures $\nu_{1i}, \cdots, \nu_{ni} \in \mathfrak{M}(S_i)$ such that

$$\mu_1 * \nu_{1i} + \cdots + \mu_n * \nu_{ni} = \delta_0.$$

However, since $\mu_1\hat{}(1) = \cdots = \mu_n\hat{}(1) = 0$, and $1 \in \Omega(S)$, the equation cannot be solved in $\mathfrak{M}(S)$.

Under these circumstances, Theorem 4.2 and Lemma 5.3 of [14] imply that there is a nonzero measure $\rho \in \mathfrak{M}_{\text{loc}}$ which is absolutely continuous (also see Theorem 6.2 of [14]). The measure ρ can be described as

$$\rho = \det \begin{vmatrix} \delta_0 & \cdots & \delta_0 \\ \nu_{11} & \cdots & \nu_{1n+1} \\ \vdots & & \vdots \\ \nu_{n1} & \cdots & \nu_{nn+1} \end{vmatrix} = \sum(-1)^i \rho_i,$$

where ρ_i is the determinant of the matrix (ν_{jk}) with the ith column deleted. Note that the multiplications involved in computing each ρ_i are valid since for all $k \neq i$ and all j we have $\nu_{jk} \in \bigcap_{k \neq i}\mathfrak{M}(S_k) \subset \mathfrak{M}(\{p_i\})$, where $p_i = \bigcap_{k \neq i} S_k$ is the ith vertex of S.

To prove that ρ as defined above is both nonzero and absolutely continuous involves most of the combinatorial machinery of [14]. It turns out that if ρ were zero then the equation $\mu_1 * \nu_1 + \cdots + \mu_n * \nu_n = \delta_0$ could be solved in $\mathfrak{M}(S)$.

The proof of Lemma 1 is now essentially complete. The restriction of the measure ρ above, to any compact set in G, will be an element of $L^1(G) \cap \mathfrak{M}$. For some compact sets this restriction must be nonzero. Hence, $L^1(G) \cap \mathfrak{M}$ is a nonzero L-ideal of $\mathfrak{M}$. Since $\mathfrak{M}$ is weak-$*$ dense; it follows that $L^1(G) \cap \mathfrak{M} = L^1(G)$ and, hence, $L^1(G) \subset \mathfrak{M}$.

In addition to being the key step in the proof of Theorem 1, Lemma 1 also has the following consequence (cf. [13, Theorem 1]):

COROLLARY. *If $\mathfrak{M}$ is an L-subalgebra of $M(G)$ such that $\hat{G}$ is, in the natural way, the maximal ideal space of $\mathfrak{M}$ (both as a set and topologically), then $L^1(G) \subset \mathfrak{M} \subset$* Rad *$L^1(G)$.*

PROOF. If we set $\mathfrak{M}' = \mathbf{C}\delta_0 + \mathfrak{M}$ then $\mathfrak{M}'$ will satisfy the hypothesis of Lemma 1. Hence, we have $L^1(G) \subset \mathfrak{M}$. If it were not true that $\mathfrak{M} \subset \text{Rad } L^1(G)$ then some $f \in \Delta \backslash \hat{G}$ would determine a nonzero complex homomorphism of $\mathfrak{M}$ obviously not given by a character. Hence, we have $\mathfrak{M} \subset \text{Rad } L^1(G)$.

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