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PRINCIPAL AND ANTIPRINCIPAL SOLUTIONS OF SELFADJOINT DIFFERENTIAL SYSTEMS AND THEIR RECIPROCALS[1]

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ABSTRACT. The concepts of principal and antiprincipal solutions for disconjugate differential systems are applied to the reciprocal and obverse systems. Relationships are obtained between these types of solutions for such systems and their reciprocals.

1. **Introduction.** It has recently been observed [2, Theorem 3.1] that, under certain "variational" assumptions, differential systems of the form

$$y' = Gz, \qquad z' = -Fy \quad \text{on } (0, \infty) \tag{1.1}$$

are disconjugate in a neighborhood of ∞ if and only if the "reciprocal" system

$$y' = Fz, \qquad z' = -Gy \tag{1.2}$$

is disconjugate in a neighborhood of ∞.

From the work of Hartman [5], and Reid [10], it is known that under suitable hypotheses, system (1.1) is disconjugate in a neighborhood of ∞ if and only if there exists a principal solution of (1.1) at ∞. Consequently, one is led to the study of principal solutions of (1.2) and the relationships between these and related concepts.

Section 2 gives introductory lemmas and definitions of principal, coprincipal, antiprincipal, and anticoprincipal solutions of differential systems. In particular the definition of principal solution differs from that given previously [5], [10], but is shown to be equivalent in certain cases to the definitions of Hartman and Reid. The main results of the paper are given in §4. In particular, relationships are obtained between various types of solutions. §5 applies these results to scalar second order linear equations.

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Matrix notation is used throughout; in particular, matrices of one column are called vectors, all identity matrices are denoted by E, and 0 is used indiscriminately for the zero matrix of any dimensions. If H and K are $n \times r$ matrices, or functions, and the symbol $(H; K)$ is used to denote the $2n \times r$ partitioned matrix $(H^* \ K^*)^*$, where $*$ denotes conjugate transpose. If H and K are $n \times n$ hermitian matrices, we write $H \geqq K$ $[H > K]$ to indicate that $\pi^*(H - K)\pi$ is nonnegative [positive] for every unit column vector π. The symbol $\lambda(H)$ is used to denote the least proper value of an $n \times n$ hermitian matrix, or function, H. An $n \times n$ hermitian matrix valued function H on an interval X is said to be nondecreasing, [increasing], if and only if $x_1 < x_2$ implies $H(x_1) \leqq H(x_2)$ $[H(x_1) < H(x_2)]$; also H is said to be nonnegative on X if $H(x) \geqq 0$ for all x in X. Matrix valued functions will be said to have a property of differentiability, continuity, or integrability if and only if each entry of the matrix has that property.

2. **Definitions and preliminary material.** Suppose that X is a subinterval of the real line and, for each point x in X, we have $n \times n$ complex matrices $A(x)$, $B(x)$, and $C(x)$. For future reference, we state the following hypothesis:

(**H**) The $n \times n$ matrices $B(x)$ and $C(x)$ are hermitian for each point x in X, and the functions A, B, and C are Lebesgue integrable on arbitrary compact subintervals of X.

Consider the differential system

$$\mathcal{J}y' = \mathfrak{A}y, \tag{2.1}$$

on X, where $\mathcal{J}$ and $\mathfrak{A}$ are $2n \times 2n$ complex matrix valued functions defined on X by

$$\mathcal{J} = \begin{bmatrix} 0 & -E \\ E & 0 \end{bmatrix}, \quad \text{and} \quad \mathfrak{A} = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix}.$$

For r a natural number, a $2n \times r$ matrix valued function y on X is said to be a *solution* of (2.1) on X if and only if y is absolutely continuous (a.c.) on arbitrary compact subintervals of X and equation (2.1) is satisfied almost everywhere (a.e.) on X. Equivalently, if y is denoted by $(U; V)$, where $U(x)$ and $V(x)$ are $n \times r$ complex matrices for every x in X, then y is a solution of (2.1) on X if and only if $(U; V)$ is a.c. on arbitrary compact subintervals of X and the equations

$$U' = AU + BV, \qquad V' = CU - A^*V \tag{2.2}$$

are satisfied a.e. on X.

LEMMA 2.1. *Suppose that hypothesis* (**H**) *holds and* y_1 *and* y_2 *are respectively* $2n \times r_1$ *and* $2n \times r_2$ *solutions of* (2.1) *on* X. *Then the* $r_1 \times r_2$ *complex matrix valued function* $y_1^* \mathcal{J}^* y_2$ *is constant on* X.

If hypothesis (**H**) holds, then a $2n \times n$ solution y of (2.1) is called *self-conjoined* if and only if the constant function $y^* \mathcal{J}^* y$ is 0 on X. If X_0 is a nondegenerate subinterval of X, then system (2.1) is said to be *normal on* X_0 if and only if the only $2n \times 1$ solution of (2.1) of the form $(u; v)$ with u being the $n \times 1$ zero on X_0 is the zero solution. System (2.1) is said to be *identically normal on* X if and only if system (2.1) is normal on every nondegenerate subinterval of X.

LEMMA 2.2. *Suppose that hypothesis* (**H**) *holds and* y_0 *and* y *are* $2n \times n$ *solutions of* (2.1) *on* X *such that* $y_0^* \mathcal{J}^* y$ *is nonsingular. Then, for each point* x *in* X, *the columns of* $y(x)$ *are linearly independent. If, in addition,* X_0 *is a nondegenerate subinterval of* X *such that system* (2.1) *is normal on* X_0, *and* y *is denoted by* $(U; V)$, *then the columns of* U *are linearly independent functions on* X_0.

LEMMA 2.3. *Suppose that hypothesis* (**H**) *holds, and* y_1 *and* y_2 *are* $2n \times n$ *solutions of* (2.1) *on* X. *Then we have*

$$(2.3) \qquad \mathcal{J}(y_1\ y_2)^* \mathcal{J}^* (y_1\ y_2) = \begin{bmatrix} y_2^* \mathcal{J} y_1 & y_2^* \mathcal{J} y_2 \\ y_1^* \mathcal{J}^* y_1 & y_1^* \mathcal{J}^* y_2 \end{bmatrix},$$

on X. *If* y_2 *is self-conjoined and the columns of* y_2 *are linearly independent on* X, *then there exists a* $2n \times n$ *self-conjoined solution* y_0 *of* (2.1) *such that* $y_0^* \mathcal{J}^* y_2$ *is* E. *For any such* y_0, *the matrix* $(y_0(x)\ y_2(x))$ *is nonsingular at each point* x *of* X, *and*

$$(y_0\ y_2)^{-1} = \mathcal{J}(y_0\ y_2)^* \mathcal{J}^*,$$

on X.

LEMMA 2.4. *Suppose that hypothesis* (**H**) *holds, and* y_1 *and* y_2 *are* $2n \times n$ *self-conjoined solutions of* (2.1) *on* X. *Then the following conditions are equivalent.*

(i) *The constant function* $y_1^* \mathcal{J}^* y_2$ *is nonsingular.*

(ii) *If* C_1 *and* C_2 *are* $n \times n$ *matrices of complex numbers such that* $y_1 C_1 + y_2 C_2$ *is* 0 *on* X, *then* C_1 *and* C_2 *are both zero.*

(iii) *The* $2n \times 2n$ *solution* $(y_1\ y_2)$ *of* (2.1) *on* X *is fundamental.*

LEMMA 2.5. *Suppose that hypothesis* (**H**) *holds, and* y_0 *and* y *are* $2n \times n$ *self-conjoined solutions of* (2.1) *on* X *such that* $y_0^* \mathcal{J}^* y$ *is* E. *Then a* $2n \times n$ *function* y_1 *is a solution of* (2.1) *if and only if* y_1 *is expressible in the form*

$$y_1 = y_0 C_0 + y C_1, \tag{2.4}$$

on X, where C_0 and C_1 are $n \times n$ complex matrices. In particular, if y_1 is a $2n \times n$ solution of (2.1) *related to y_0 and y by* (2.4), *then C_0 and C_1 are uniquely determined as $y^* \mathcal{J} y_1$ and $y_0^* \mathcal{J}^* y_1$, respectively, and $y_1^* \mathcal{J}^* y_1$ is $C_0^* C_1 - C_1^* C_0$.*

The results in Lemma 2.3 were known to Radon [**9**, pp. 292-293]. The solution y_0 is obtained by letting y_1 be a $2n \times n$ solution such that $(y_1\ y_2)$ is fundamental and by determining C_1 and C_2 as $n \times n$ constant matrices such that y_0 defined as $y_1 C_1 + y_2 C_2$ has the desired properties. This may be done by observing that the right-hand member of (2.3) is nonsingular, hence $y_1^* \mathcal{J}^* y_2$ is nonsingular, and one may choose C_1 as $(y_2^* \mathcal{J} y_1)^{-1}$ and C_2 as $C_1^* y_1^* \mathcal{J} y_1 C_1/2$.

If y_1 and y_2 are $2n \times n$ solutions of (2.1), the constant $y_1^* \mathcal{J}^* y_2$ of Lemma 2.1 was referred to by Barrett [3, p. 80] as the wronskian in the case of real solutions of differential systems with real coefficients. Lemma 2.4 illustrates that when restricted to self-conjoined solutions, the function $y_1^* \mathcal{J}^* y_2$ has properties which are analogous to those of the wronskian of solutions of the scalar equation $y'' + f(t)y = 0$. If one considers the set of $2n \times n$ solutions of (2.1), together with the operations of pointwise addition and right scalar multiplication from the ring C_{nn} of $n \times n$ matrices of complex numbers, then we have the algebraic structure of a right unitary module over C_{nn}. See Warner [**16**, p. 253], for terminology. In this setting, condition (ii) is linear independence of y_1 and y_2. For terminology used in condition (iii) of Lemma 2.4, see [**6**, p. 47]. Moreover, conditions (ii) and (iii) are equivalent without the assumption that y_1 and y_2 are self-conjoined, and it can be shown that this module has well-defined dimension of 2.

Suppose that c is either (i) a point of X, (ii) the infimum of X (possibly $-\infty$) or (iii) the supremum of X (possibly $+\infty$). If c is a point of X, then by the phrase "neighborhood of c", we mean a subinterval X_0 of X such that c is in X_0, and X_0 is open in the relative topology of X. If c is the infimum [supremum] of X and c is not in X, then a "neighborhood of c" is understood to be an interval of the form (c, b) $[(b, c)]$ for some b in X. For the purposes of the following definitions, suppose that hypothesis (**H**) holds. A $2n \times n$ solution $(U; V)$ of (2.1) is said to be *principal at c* if and only if $(U; V)$ is self-conjoined and there exists a $2n \times n$ self-conjoined solution $(U_0; V_0)$ of (2.1) such that $U_0^* V - V_0^* U$ is nonsingular, U_0 is nonsingular in a neighborhood of c, and we have

$$\lim_{x\to c} U_0^{-1}(x)U(x) = 0. \tag{2.5}$$

A $2n \times n$ solution $(U; V)$ of (2.1) is said to be *antiprincipal at* c if and only if $(U; V)$ is self-conjoined and there exists a $2n \times n$ self-conjoined solution $(U_0; V_0)$ of (2.1) such that $U_0^*V - V_0^*U$ is nonsingular, U_0 is nonsingular in a neighborhood of c, and

$$\lim_{x\to c} U_0^{-1}(x)U(x) = E. \tag{2.6}$$

Consider the system "obverse" to (2.2), (see Reid [13, §2]),

$$U' = -A^*U + CV, \qquad V' = BU + AV, \tag{2.7}$$

on X. Under hypothesis (**H**), we refer to a $2n \times n$ solution $(U; V)$ of (2.1) as being a *coprincipal solution of* (2.1) *at* c if and only if the solution $(U_1; V_1)$ of (2.7) defined by $(U_1; V_1) = (V; U)$ is a principal solution of (2.7) at c. Similarly, a $2n \times n$ solution $(U; V)$ of (2.1) will be said to be an *anticoprincipal solution of* (2.1) *at* c if and only if $(U_1; V_1)$ as defined above is an antiprincipal solution of (2.7) at c. Equivalently, a $2n \times n$ solution $(U; V)$ of (2.1) may be defined as coprincipal [anticoprincipal] at c if and only if the function $(U_2; V_2)$ defined by $(U_2; V_2) = (V; -U)$ is principal [antiprincipal] at c for the system "reciprocal" [2, §2] to (2.1) defined by

$$U' = -A^*U - CV, \qquad V' = -BU + AV, \tag{2.8}$$

on X.

Notice that if $(U; V)$ is a $2n \times n$ solution of (2.1) which is principal at c, then there exists a $2n \times n$ self-conjoined solution $(U_0; V_0)$ of (2.1) such that $U_0^*V - V_0^*U$ is E, while U_0 is nonsingular in a neighborhood of c, and relation (2.5) holds. Also, in the definition of an antiprincipal solution, the solution $(U_0; V_0)$ can be chosen so that $U_0^*V - V_0^*U$ is E, if, instead of assuming that relation (2.6) holds, we have the existence of a nonsingular $n \times n$ matrix M such that $\lim_c U_0^{-1}U$ is M. Observe that if c is a point of X, then the $2n \times n$ solution $(U(\ , c);\ V(\ , c))$ $[(U_0(\ , c);\ V_0(\ , c))]$ which satisfies $(U(c, c);\ V(c, c)) = (0; E)$ $[(U_0(c, c);\ V_0(c, c)) = (E; 0)]$ is principal and anticoprincipal [antiprincipal and coprincipal] at c.

THEOREM 2.1. *Suppose that hypothesis* (**H**) *holds and* y, *denoted by* $(U; V)$, *is a* $2n \times n$ *solution of* (2.1) *which is principal at* c. *If* y_0, *denoted by* $(U_0; V_0)$, *is a* $2n \times n$ *self-conjoined solution of* (2.1) *such that* $y_0^*\mathcal{J}^*y$ *is nonsingular,* U_0 *is nonsingular in a neighborhood of* c, *and relation* (2.5) *holds, then* y_0 *is antiprincipal at* c.

Suppose that C_0 and C_1 are defined as E and $y^*\mathcal{J}y_0$, respectively. Let y_1, denoted by $(U_1; V_1)$, be defined by equation (2.4) on X. Then y_1 is self-conjoined, $y_1^*\mathcal{J}^*y_0$ is nonsingular, and in a neighborhood of c, we have

$$U_0^{-1}U_1 = E + U_0^{-1}UC_1,$$

and hence, $\lim_c U_0^{-1}U_1$ is E. Thus U_1 is nonsingular in a neighborhood of c and $\lim_c U_1^{-1}U_0$ is E.

COROLLARY. *Suppose that hypothesis* (H) *holds and* y, *denoted by* $(U; V)$, *is a* $2n \times n$ *solution of* (2.1) *which is coprincipal at* c. *If* y_0, *denoted by* $(U_0; V_0)$ *is a* $2n \times n$ *self-conjoined solution of* (2.1) *such that* $y_0^*\mathcal{J}^*y$ *is nonsingular,* V_0 *is nonsingular in a neighborhood of* c, *and* $\lim_c V_0^{-1}V$ *is* 0, *then* y_0 *is anticoprincipal at* c.

THEOREM 2.2. *Suppose that hypothesis* (H) *holds, system* (2.1) *is identically normal on* X, *and* B *is nonnegative on* X. *If* y, *denoted by* $(U; V)$, *is a* $2n \times n$ *solution of* (2.1) *on* X *which is principal at* c, *and* y_1, *denoted by* $(U_1; V_1)$, *is a* $2n \times n$ *solution of* (2.1) *such that* $y_1^*\mathcal{J}^*y$ *is nonsingular, then* U_1 *and* U *are nonsingular in a deleted neighborhood of* c, *and* $\lim_c U_1^{-1}U$ *is* 0.

The word deleted is redundant unless c is in X. Suppose that y_0, denoted by $(U_0; V_0)$, is a $2n \times n$ self-conjoined solution such that U_0 is nonsingular in a neighborhood of c, $y_0^*\mathcal{J}^*y$ is E, and relation (2.5) holds. From Lemma 2.5, there exist $n \times n$ matrices C_0 and C_1 such that equation (2.4) holds on X. Also, C_0 is nonsingular. The nonsingularity of U_1 in a neighborhood of c follows from the result that $U_0^{-1}(x)U_1(x) \to C_0$, as $x \to c$. From Lemma 2.2, we are assured that the columns of U are linearly independent functions on X. The sturmian separation theorem for systems assures that U is nonsingular in a deleted neighborhood X_1 of c. For the case in which B is positive definite on X, the sturmian separation theorem may be found in [8, p. 64, Corollary 1], and in the generality needed here, in [14, Lemma VII 7.1]. For each constant n-vector π, we have

$$U^{-1}(x)U_1(x)\pi = U^{-1}(x)U_0(x)C_0\pi + C_1\pi$$

for each x in X_1. Thus, for each x in X_1, we have

$$\min_{\pi^*\pi=1} |U^{-1}(x)U_1(x)\pi| \geqq \min_{\pi^*\pi=1} |U^{-1}(x)U_0(x)C_0\pi| - \max_{\pi^*\pi=1} |C_1\pi|.$$

Suppose that for each x in X_1, the unit vector $\pi(x)$ is such that $\min |U^{-1}(x)U_0(x)C_0\pi| = |U^{-1}(x)U_0(x)C_0\pi(x)|$. Let $\eta(x)$ be $C_0\pi(x)$, and $\zeta(x) = \eta(x)/|\eta(x)|$, for x in X_1. Then

$$|U^{-1}(x)U_0(x)\eta(x)| = |\eta(x)|\,|U^{-1}(x)U_0(x)\zeta(x)|,$$

and we have $1 = |\pi(x)| \leqq |C_0{}^{-1}|\ |\eta(x)|$, and $|\eta(x)| \geqq |C_0{}^{-1}|^{-1}$, where $|C| = (\sum_{i=1}^{n}|C_{ij}|^2)^{1/2}$. Thus

$$\min_{\pi^*\pi=1} |U^{-1}(x)U_1(x)\pi| \geqq \Big(\min_{\pi^*\pi=1} |C_0{}^{-1}|^{-1} \cdot |U^{-1}(x)U_0(x)\pi|\Big) - \max_{\pi^*\pi=1} |C_1\pi|,$$

where $\min_{\pi^*\pi=1}|U^{-1}(x)U_0(x)\pi| \to \infty$, as $x \to c$. Thus

$$\max_{\pi^*\pi=1} |U_1{}^{-1}(x)U(x)\pi| \to 0,$$

as $x \to c$, (see [**10**, p. 153], and $U_1{}^{-1}(x)U(x) \to 0$, as $x \to c$.

COROLLARY. *Suppose that hypothesis* (**H**) *holds, system* (2.7) *is identically normal on X, either* $C(x)$ *is nonnegative for each* x *in* X, *or nonpositive for each* x *in* X. If $(U; V)$ *is a* $2n \times n$ *solution of* (2.1) *which is coprincipal at* c *and* $(U_1; V_1)$ *is a* $2n \times n$ *solution of* (2.1) *such that* $U_1{}^*V - V_1{}^*U$ *is nonsingular, then* V_1 *and* V *are nonsingular in a deleted neighborhood of* c *and we have* $\lim_{x\to c} V_1{}^{-1}(x)V(x) = 0$.

3. **Equivalent formulations for the identically normal case.** Distinct points, a and b, of X are said to be *conjugate* relative to (2.1) if there exists a $2n \times 1$ solution $(u; v)$ of (2.1) such that $u(a) = 0 = u(b)$ holds, and u is not the zero function on (a, b). For X_0 a nondegenerate subinterval of X, we say that (2.1) is disconjugate on X_0 if and only if X_0 contains no pairs of conjugate points.

LEMMA 3.1. *Suppose that hypothesis* (**H**) *holds,* X_0 *is a subinterval of* X *and* $(U; V)$ *is a* $2n \times n$ *self-conjoined solution of* (2.1) *for which* U *is nonsingular on* X_0. *If* a *is in* X_0, *and* $(U_0(x); V_0(x))$ *is defined at each point* x *of* X_0 *by the relations,*

$$U_0(x) = -U(x)\int_a^x U^{-1}(t)B(t)U^{*-1}(t)\,dt,$$

$$V_0(x) = -U^{*-1}(x) - V(x)\int_a^x U^{-1}(t)B(t)U^{*-1}(t)\,dt, \tag{3.1}$$

then $(U_0; V_0)$ *is a* $2n \times n$ *self-conjoined solution of* (2.1) *such that* $U_0{}^*V - V_0{}^*U$ *is* E.

THEOREM 3.1. *Suppose that hypothesis* (**H**) *holds,* B *is nonnegative on* X, *and system* (2.1) *is identically normal on* X. *Suppose also that* c *is* $\sup X$, *and* c *is not in* X. *Let* $(U; V)$ *be a* $2n \times n$ *self-conjoined solution of* (2.1) *for which* U *is nonsingular on* $[a, c)$. *Then the function* $S(\ , a; U)$, *whose value at each point* x *of* $[a, c)$ *is* $\int_a^x U^{-1}(t)B(t)U^{*-1}(t)\,dt$, *is nonsingular on* (a, c) *and*

(i) $(U; V)$ *is principal at* c *if and only if* $\lim_c S^{-1}(\ , a; U)$ *is* 0,
(ii) $(U; V)$ *is antiprincipal at* c *if and only if there exists a nonsingular matrix* M *of complex numbers such that* $\lim_c S(\ , a; U)$ *is* M.

Suppose that b is a point of (a, c), and π is an n-vector such that $S(b, a; U)\pi$ is 0. Let $(u; v)$ be defined as $(U_0\pi; V_0\pi)$ on $[a, b]$, where $(U_0; V_0)$ is defined as in Lemma 3.1. Then $u(a) = 0 = u(b)$ holds, and u is 0 on $[a, b]$, since a and b cannot be conjugate [**11**, Theorem 5.2, p. 678]. From normality, v must be 0 on $[a, b]$, and we have $0 = V_0(b)\pi = -U^{*-1}(b)\pi$, and π is 0. Conclusion (i) follows from Lemma 3.1 and Theorem 2.2 upon noting that $S^{-1}(\ , a; U)$ is $-U_0{}^{-1}U$ on (a, c). If $\lim_c S(\ , a; U)$ is a nonsingular matrix M, then $(U_0; V_0)$, defined as above, is such that $\lim_c U_0{}^{-1}U$ is $-M^{-1}$ and $(U; V)$ is antiprincipal at c. Conversely, suppose that $(U; V)$ is antiprincipal at c, and $(U_1; V_1)$ is a $2n \times n$ self-conjoined solution of (2.1) such that $U_1{}^*V - V_1{}^*U$ is nonsingular, U_1 is nonsingular in a neighborhood of c and $\lim_c U_1{}^{-1}U$ is E. Suppose that C_0 and C_1 are $n \times n$ constant matrices such that U_1 is $U_0C_0 + UC_1$ on $[a, c)$. Such constants exist as a consequence of Lemma 2.5, and furthermore, C_0 is nonsingular. Thus $U^{-1}U_1$ is $-S(\ , a; U)C_0 + C_1$ and $\lim_c S(\ , a; U)$ exists. However, $S(\ , a; U)$ is hermitian positive definite, and increasing on (a, c); hence $\lim_c S(\ , a; U)$ is positive definite.

COROLLARY. *Suppose that hypothesis* (**H**) *holds,* C *is nonnegative on* X, *or* C *is nonpositive on* X, *system* (2.7) *is identically normal,* c *is* $\sup X$, *and* c *is not in* X. *Let* $(U; V)$ *be a* $2n \times n$ *self-conjoined solution of* (2.1) *for which* V *is nonsingular on* $[a, c)$. *Then the function* $S_0(\ , a; V)$, *whose value at each point* x *of* $[a, c)$ *is* $\int_a^x V^{-1}(t)C(t)V^{*-1}(t)\,dt$, *is nonsingular on* (a, c), *and*
(i) $(U; V)$ *is coprincipal at* c *if and only if* $\lim_c S_0{}^{-1}(\ , a; V)$ *is* 0,
(ii) $(U; V)$ *is anticoprincipal at* c *if and only if there exists a nonsingular matrix* M *of complex numbers such that* $\lim_c S_0(\ , a; V)$ *is* M.

The condition $\lim_c S^{-1}(\ , a; U) = 0$ of (i) of Theorem 3.1 is equivalent under those hypotheses to saying that all proper values of $S(\ , a; U)$ are unbounded on (a, c). The latter condition was used as the definition of a principal solution by Hartman [**5**, p. 28]. The existence of $\lim_c S(\ , a; U)$ is equivalent to saying that all proper values of $S(\ , a; U)$ are bounded. Hartman called such a solution nonprincipal, and subsequently Tomastik [**15**, p. 62], changed that terminology to antiprincipal. In Reid's definition of principal solutions for more general settings [**12**, p. 408], an assumption is always

made that $(U; V)$ has U nonsingular in a neighborhood of c. Observe that if A, B, and C are all 0 on X, then $(0; E)$ is a $2n \times n$ solution of (2.1) which is principal at every point of X, and at $\inf X$, and $\sup X$, according to the definition used here. Thus the definitions differ in the abnormal cases. Reid did not make the assumption that a principal solution was self-conjoined, but under the hypotheses of the following theorem, the definitions agree [**10**, p. 159].

THEOREM 3.2 [**11**, p. 677]. *Suppose that hypothesis* (**H**) *holds, B is nonnegative on X, and system* (2.1) *is identically normal on X. Suppose also that c is* $\sup X$, *and c is not in X. Then, system* (2.1) *is disconjugate in a neighborhood of c if and only if there exists a $2n \times n$ solution of* (2.1) *which is principal at c.*

THEOREM 3.3. *Under the hypotheses of Theorem* 3.2, *system* (2.1) *is disconjugate in a neighborhood of c if and only if there exists a $2n \times n$ solution of* (2.1) *which is antiprincipal at c.*

If system (2.1) is disconjugate in a neighborhood of c, then there exists a principal solution and due to the definition and Theorem 2.1, there exists an antiprincipal solution. Conversely, if $(U; V)$ is antiprincipal at c, then $(U; V)$ is self-conjoined with U nonsingular in a neighborhood of c, and system (2.1) is disconjugate in a neighborhood of c [**11**, Theorem 5.2, p. 678].

4. Relationships between principal and coprincipal solutions.

LEMMA 4.1. *If hypothesis* (**H**) *holds, c is* $\sup X$, *and $(U; V)$ is a $2n \times n$ self-conjoined solution of* (2.1) *such that U and V are nonsingular on $[a, c)$, then we have the identity*

$$\int_a^x U^{-1}BU^{*-1}\,dt + U^{-1}(x)V^{*-1}(x) = \int_a^x V^{-1}(-C)V^{*-1}\,dt + U^{-1}(a)V^{*-1}(a), \tag{4.1}$$

for $a \leqq x < c$.

Indeed, the lemma follows upon differentiation of both members of (4.1).

THEOREM 4.1. *Suppose that hypothesis* (**H**) *holds, B is nonnegative on X, C is nonpositive on X, and c is* $\sup X$, *where* $\sup X$ *is not in X. Suppose also that systems* (2.1) *and* (2.8) *are identically normal on X, and for D a fundamental solution of $D' = AD$, we have the condition*

$$\pi^*\left(\int_a^x D^{-1}BD^{*-1}\,dt\right)\pi \to \infty, \quad \text{as } x \to \infty,$$

for every constant unit vector π. *If* $(U; V)$ *is a* $2n \times n$ *solution of* (2.1) *which is principal at* c, *then* $(U; V)$ *is coprincipal at* c.

For D as in the hypotheses of Theorem 4.1, define Y and Z by $Y = D^{-1}U$ and $Z = D^*V$ on X. Let G and F be defined by $G = D^{-1}BD^{*-1}$ and $F = -D^*CD$, respectively. Then $(Y; Z)$ is a $2n \times n$ self-conjoined solution of the system

$$Y' = GZ, \qquad Z' = -FY, \quad \text{on } X.$$

Suppose that U is nonsingular on $[a_1, c)$ and W is defined as ZY^{-1} on $[a_1, c)$. Then W is an $n \times n$ hermitian solution of $W' = -F - WGW$, on $[a_1, c)$, and from the normality assumption on system (2.8) together with the nonnegativeness of F, we conclude that $a_1 \leqq x_1 < x_2 < c$ implies that $\int_{x_1}^{x_2} F(t)\,dt$ is positive definite, (see, for example [**1**, Lemma 2.1, p. 274]). Thus, W is decreasing on $[a_1, c)$, and each proper value of W is decreasing on $[a_1, c)$. Therefore, each proper value of W can have at most one zero on $[a_1, c)$, and due to finite dimensionality, W is nonsingular in a neighborhood of c. Thus Z, and consequently V, is nonsingular in a neighborhood of c.

Now suppose that U and Y are nonsingular on $[a, c)$. Then we have

$$U^{-1}V^{*-1} = Y^{-1}Z^{*-1} = Z^{-1}WZ^{*-1}, \quad \text{on } [a, c).$$

The assumption that $(U; V)$ is principal at c implies that for each constant unit vector π, we have $\pi^*(\int_a^x U^{-1}BU^{*-1}\,dt)\pi \to \infty$, as $x \to \infty$. The theorem will follow from the lemma once it is shown that W is positive definite on $[a, c)$. For W_1 defined as W^{-1} on $[a, c)$, it follows that W_1 is an $n \times n$ hermitian solution of $W_1' = G + W_1FW_1$ on $[a, c)$. Hence, W_1 is increasing and $\pi^*W_1(x)\pi \to \infty$, as $x \to c$, for each constant unit vector π. Therefore, $\lambda(W_1(x)) \to \infty$, as $x \to c$, and W_1 is positive definite in a neighborhood of c. Since the proper values of W_1 are continuous and never zero on $[a, c)$, we conclude that W_1, and hence W, is positive definite on $[a, c)$.

Reversing the roles of systems (2.1) and (2.8) produces the following result.

COROLLARY. *Suppose that hypothesis* (**H**) *holds, B is nonnegative and C is nonpositive on X, and c is* $\sup X$, *where* $\sup X$ *is not in X. Suppose also that systems* (2.1) *and* (2.8) *are identically normal on X, and for D a fundamental solution of* $D' = AD$, *we have the condition* $\pi^*(\int_a^x D^*CD\,dt)\pi \to -\infty$, *as* $x \to c$, *for every constant unit*

vector π. *If* $(U; V)$ *is a* $2n \times n$ *solution of* (2.1) *which is coprincipal at* c, *then* $(U; V)$ *is principal at* c.

LEMMA 4.2. *Suppose that the hypotheses of Theorem* 4.1 *hold, and* $(U; V)$ *is a* $2n \times n$ *solution of* (2.1) *which is antiprincipal at* c. *Then* U *and* V *are nonsingular in a neighborhood of* c *and* $\lim_c U^{-1}V^{*-1}$ *is* 0.

Let $(U_2; V_2)$ be defined by the equation

$$U_2(x) = U(x) \int_x^c U^{-1}BU^{*-1}\, dt,$$

$$V_2(x) = -U^{*-1}(x) + V(x) \int_x^c U^{-1}BU^{*-1}\, dt,$$

for x in a neighborhood of c. Then $(U_2; V_2)$ is a self-conjoined solution of (2.1) with $\lim_c U^{-1}U_2 = 0$, and $\lim_c (V^{-1}V_2 + V^{-1}U^{*-1}) = 0$. Since $U_2{}^*V - V_2{}^*U$ is nonsingular, it follows that $(U_2; V_2)$ is principal at c, and, hence coprincipal at c from Theorem 4.1. Therefore, $\lim_c V^{-1}V_2$ is 0 and the conclusion of the lemma follows.

THEOREM 4.2. *Suppose that the hypotheses of Theorem* 4.1 *hold. Then a* $2n \times n$ *solution of* (2.1) *is antiprincipal at* c *if and only if it is anticoprincipal at* c.

Suppose that $(U; V)$ is a $2n \times n$ solution of (2.1) which is anti-coprincipal at c. Application of part (ii) of the corollary to Theorem 3.1 assures that the right-hand member of equation (4.1) has a limit. Since the term $U^{-1}(x)V^{*-1}$ is hermitian and positive definite, the hermitian term $\int_a^x U^{-1}BU^{*-1}\, dt$ is bounded above, and consequently has a limit, since it is nondecreasing. Application of part (ii) of Theorem 3.1 assures that $(U; V)$ is antiprincipal at c. The converse follows in a similar manner upon applying Lemmas 4.1 and 4.2.

Reversing the roles of systems (2.1) and (2.8) produces the following corollary.

COROLLARY 1. *Suppose that the hypotheses of the corollary to Theorem* 4.1 *hold. Then the conclusion of Theorem* 4.2 *holds.*

COROLLARY 2. *Suppose that the hypotheses of Theorem* 4.1, *or of the corollary to Theorem* 4.1, *hold and* $(U; V)$ *is a* $2n \times n$ *solution of* (2.1) *which is coprincipal* [*principal*] *at* c. *Then there exists a point* a *in* X, *together with a constant unit vector* π, *such that* U *and* V *are nonsingular on* $[a, c)$, *and*

$$\pi^* \left(\int_a^x U^{-1}BU^{*-1}\, dt \right) \pi \to \infty, \quad \text{as } x \to c,$$

$$\left[\pi^*\left(\int_a^x V^{-1}CV^{*-1}\,dt\right)\pi \to -\infty, \text{ as } x \to c\right].$$

THEOREM 4.3. *Suppose that the hypotheses of Theorem* 4.1, *or the hypotheses of the corollary to Theorem* 4.1 *hold, and system* (2.1) *is disconjugate in a neighborhood of c. Then there exist* $2n \times n$ *self-conjoined solutions* y *and* y_0 *of* (2.1) *for which* y *is principal and coprincipal at* c *and* y_0 *is antiprincipal and anticoprincipal at* c *such that every* $2n \times r$ *solution* y_1 *of* (2.1) *is uniquely expressible in the form* (2.4), *where* C_0 *and* C_1 *are* $n \times r$ *complex matrices.*

Indeed, under the hypotheses of Theorem 4.1, there exists a $2n \times n$ solution y which is principal at c due to Theorem 3.1. Application of Theorem 4.1 implies that y is also coprincipal at c. It follows from the definition of y being principal at c that there exists a $2n \times n$ solution y_0 of (2.1) which satisfies the conditions imposed in Theorem 2.1. Consequently y_0 is antiprincipal at c, and it follows from Theorem 4.2 that y_0 is anticoprincipal at c. Application of Lemma 2.4 completes the proof in this case. The case in which the hypotheses of the corollary to Theorem 4.1 hold follows in a similar manner. Observe that in both cases y_0 may be chosen so that $y_0^* \mathcal{J}^* y$ is E.

5. **An application to second order selfadjoint scalar equations.** Let c be an extended real number such that $0 < c \leqq \infty$ holds, and suppose that r and f are continuous on $(0, c)$ with r positive and f of fixed sign on $(0, c)$. One may consider the differential equation

$$(y'/r)' + fy = 0, \tag{5.1}$$

on $(0, c)$, and the "reciprocal" equation [4, p. 558]

$$(y'/f)' + ry = 0, \tag{5.2}$$

on $(0, c)$. A differentiable function y on $(0, c)$ is a solution of (5.1) on $(0, c)$ if and only if y'/r is a solution of (5.2). A pair of solutions y_1 and y_2 of (5.1) on $(0, c)$ is linearly independent on $(0, c)$ if and only if y_1' and y_2' are linearly independent on $(0, c)$. A solution y of (5.1) is principal [coprincipal] at c if and only if y is real valued on $(0, c)$, and there exists a real solution y_0 such that y and y_0 are linearly independent, y_0 is nonzero in a neighborhood of c, and $\lim_c y/y_0$ [$\lim_c y'/y_0'$] exists and is 0.

THEOREM 5.1. *Suppose that* r *and* f *are continuous positive real valued functions on* $(0, c)$ *and* $\int_a^x r(t)\,dt \to \infty$, *or* $\int_a^x f(t)\,dt \to \infty$, *as* $x \to c$. *Then* ϕ *is a solution of* (5.1) *on* $(0, c)$ *which is principal at* c *if and only if* ϕ *is a solution of* (5.1) *on* $(0, c)$ *which is coprincipal at* c.

Notice that if $\int_a^c r(t)\,dt$ and $\int_a^c f(t)\,dt$ both exist, the conclusion of Theorem 5.1 does not follow. Indeed, let c be π and consider the solution sin of $y'' + y = 0$ on $(0, \pi)$ which is principal at π and anti-coprincipal at π. Examples may also be provided on an infinite interval by transforming this one.

Theorem 5.1 follows from Corollary 2 to Theorem 4.2 upon noting that the possibilities in conditions (i) and (ii) of Theorem 3.1 (or its corollary) exhaust the cases when n is 1. The question as to whether the converse of Theorem 4.1 is true in general remains open.

COROLLARY. *Suppose that r and f are as in Theorem* 5.1. *Consider two real linearly independent solutions y_1 and y_2 of* (5.1) *on* $(0, c)$ *which are nonzero in a neighborhood of c. Then* $\lim_c y_1/y_2$ *exists and is* 0 *if and only if* $\lim_c y_1'/y_2'$ *exists and is* 0. Also, $\lim_c y_1/y_2$ *exists and is nonzero if and only if* $\lim_c y_1'/y_2'$ *exists and is nonzero.*

The word coprincipal has been used in this section for what Leighton [7, p. 267] called a focal point solution. The above corollary extends Lemmas 2.5 and 2.6 of Leighton [7, pp. 261-262]. The behavior illustrated in Lemma 4.1 of Leighton [7, pp. 267-268] is expressing a basic property of a principal solution of (5.2), i.e., a coprincipal solution of (5.1).

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