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LARGE ABELIAN SUBGROUPS OF SOME INFINITE GROUPS

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ABSTRACT. A generalization of the following conjecture of W. R. Scott is proved. If $(H_\alpha)_{\alpha \leqq \delta}$ is a well-ordered descending chain of subgroups of a group with the property that $H_\beta = \bigcap_{\alpha<\beta} H_\alpha$ for limit ordinals, then $[H_0 : H_\delta] \leqq \prod_{\alpha<\delta}[H_\alpha : H_{\alpha+1}]$. Using this, we show that the members of certain classes of infinite groups are guaranteed to have large abelian subgroups.

1. **Introduction.** Following Kurosh [5, p. 171], a totally ordered system $\mathcal{U}$ of subgroups of a group is said to be complete if for an arbitrary subsystem of $\mathcal{U}$, the unions and the intersections of the subgroups forming the subsystem belong to $\mathcal{U}$. W. R. Scott [8, p. 21] has conjectured that, if $(H_\alpha)_{\alpha \leqq \delta}$ is a well-ordered descending complete system of subgroups of a group H_0, then

$$(1) \qquad [H_0 : H_\delta] \leqq \prod_{\alpha<\delta} [H_\alpha : H_{\alpha+1}].$$

In a private communication, Scott has shown that this is indeed true for $\delta = \omega$, the first infinite ordinal, and has stated that under these same conditions both he and, independently, A. Kruse have proved that

$$[H_0 : H_\delta] \leqq \left[\prod_{\alpha<\delta} [H_\alpha : H_{\alpha+1}] \right]^{|\delta|}.$$

In §3 we shall establish a generalized form of Scott's conjecture from which the latter can be deduced. In addition, we shall find a lower bound for $[H_0 : H_\delta]$ which will be useful in §4.

If m is a cardinal number, we define $\exp m = \exp^1 m = 2^m$. Inductively, if n is any positive integer, we define $\exp^{n+1} m = \exp \exp^n m$. In §4 we utilize equation (1) to investigate the existence of large abelian subgroups of certain infinite groups. For example, C. R. Kulatilaka has shown [4, p. 241] that every infinite SI^*-group G

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has an infinite abelian subgroup A such that $A \lhd^2 G$ (A is normal in its normal closure, A^G). We shall show that *every infinite SI^*-group G has an abelian subgroup A such that* $\exp^2|A| \geqq |G|$, in fact, Theorem 5 states that G *need only have an ascending invariant series with FC factors.* For similar results on large discrete subspaces of topological spaces, the reader should see [2] and [3].

2. **Notation.** Let S and T be sets. $S < T$ will always mean strict inclusion. The cardinality of S will be denoted by $|S|$. If G is a group and H is a subgroup, we write $H \lhd^n G$ if there is an ascending normal series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

from H to G. $C(H) = C\ (H \leqq G)$ denotes the centralizer of H in G, while $N(H) = N\ (H \leqq G)$ denotes the normalizer of H in G. A *transversal* for H in G is a full set of distinct coset representatives for H in G; if a transversal contains the identity, it is said to be *normalized.*

Let $\operatorname{sym}|S| = \operatorname{sym}^1|S|$ be the order of the symmetric group on $|S|$ letters; if n is a positive integer, define $\operatorname{sym}^{n+1}|S| = \operatorname{sym}\operatorname{sym}^n|S|$. We shall frequently make use of the fact that

$$|\operatorname{Aut}(G)| \leqq \operatorname{sym}|G|,$$

where $\operatorname{Aut}(G)$ is the group of automorphisms of G.

We assume the terminology of §57 and §63 of [5] to denote various classes of generalized solvable and nilpotent groups.

Let $F_1(G)$ be the set of all elements in G which have at most a finite number of conjugates in G. As in [1], we define the *upper FC-series* of G to be the ascending characteristic series

$$E = F_0(G) \lhd F_1(G) \lhd \cdots \lhd F_\alpha(G) \lhd \cdots$$

where $F_{\alpha+1}(G)/F_\alpha(G) = F_1(G/F_\alpha(G))$, and if β is a limit ordinal, then $F_\beta(G) = \bigcup_{\alpha<\beta} F_\alpha(G)$. If $F_1(G) = G$, G is said to be an *FC-group.* If $F_\alpha(G) = G$ for some α, then G is a *ZFC-group*; if α is an integer, G is *FC-nilpotent.*

If m is an infinite cardinal, let $M_1(G)$ be the set of all elements in G which have at most m conjugates in G. By analogy with the upper FC-series, we define the *upper mC-series* of G to be the ascending characteristic series

$$E = M_0(G) \lhd M_1(G) \lhd \cdots \lhd M_\alpha(G) \lhd \cdots$$

where $M_{\alpha+1}(G)/M_\alpha(G) = M_1(G/M_\alpha(G))$, and if β is a limit ordinal, then $M_\beta(G) = \bigcup_{\alpha<\beta} M_\alpha(G)$. If $M_1(G) = G$, G is an *mC-group*; if $M_\alpha(G) = G$ for some α, then G is a *ZmC-group.*

If $\mathfrak{X}$ is a class of groups, we let $\mathfrak{X}I^*$ be the class of all groups having an ascending invariant series with factors in $\mathfrak{X}$. If every subgroup and every homomorphic image of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group, then the same holds true for $\mathfrak{X}I^*$-groups [5, §56].

Let $\mathcal{U} = (A_\alpha)_{\alpha \in W}$ be a complete ordered system of subgroups of a group G. We shall suppose that W is ordered by a relation $<$ such that $\alpha < \beta$ implies that $A_\alpha < A_\beta$. If α has an immediate successor in W, we denote it by $\alpha + 1$. Let $\mathcal{J}$ be the set of all α in W for which $\alpha + 1$ exists. If W is well ordered, $\mathcal{U}$ is an ascending series; if W with the inverse ordering is well ordered, $\mathcal{U}$ is a descending series.

If α is an ordinal, we say α is of the *first kind* if it has an immediate predecessor; otherwise, α is of the *second kind*. A *limit ordinal* is any nonzero ordinal of the second kind.

3. The index theorems.

Theorem 1. *Let $\mathcal{U} = (A_\alpha)_{\alpha \in W}$ be a complete ordered system of subgroups of a group G containing the whole group $G = A_\mu$ and some $A_0 = \bigcap_{\alpha \in W} A_\alpha$. Then*

$$[G : A_0] \leqq \prod_{\alpha \in \mathcal{J}} [A_{\alpha+1} : A_\alpha].$$

Proof. Let R_α be a transversal for A_α in $A_{\alpha+1}$ for each $\alpha \in \mathcal{J}$. Let T_α be a transversal for A_α in G for each $\alpha \in W$. For each $g \in T_0$, let g_α be the unique element in T_α such that $A_\alpha g_\alpha = A_\alpha g$. If $\alpha \in \mathcal{J}$, then consider $A_\alpha g_\alpha (g_{\alpha+1})^{-1}$. Since $A_{\alpha+1} g_{\alpha+1} = A_{\alpha+1} g$ and $A_\alpha g_\alpha = A_\alpha g$, it follows that $g(g_{\alpha+1})^{-1} \in A_{\alpha+1}$ and that $g_\alpha g^{-1} \in A_\alpha \leqq A_{\alpha+1}$; and, consequently, that $g_\alpha (g_{\alpha+1})^{-1} \in A_{\alpha+1}$. Thus we can define a unique point F_g in the cartesian product, $\prod_{\alpha \in \mathcal{J}} R_\alpha$, by the two conditions

(i) $F_g(\alpha) \in R_\alpha$,

(ii) $A_\alpha F_g(\alpha) = A_\alpha g_\alpha (g_{\alpha+1})^{-1}$,

for all $\alpha \in \mathcal{J}$. If the function F taking g to F_g is one-to-one, then $|T_0| \leqq \prod_{\alpha \in \mathcal{J}} |R_\alpha|$, the conclusion of the theorem.

If $F_g = F_h$ for g and h in T_0, then by (ii)

$$(*) \qquad A_\alpha g_\alpha (g_{\alpha+1})^{-1} = A_\alpha h_\alpha (h_{\alpha+1})^{-1}$$

for all $\alpha \in \mathcal{J}$. Let $P(\alpha)$ be the statement that $g_\alpha = h_\alpha$ for all $\alpha \in W$. Since there is only one element in T_μ, $P(\mu)$ obviously holds. If $[S(1), S(2)]$ is a Dedekind section taken in W having the property that $P(\alpha)$ holds for all elements $\alpha \in S(2)$, then we can easily find some $\beta \in S(1)$ for which $P(\beta)$ holds. Suppose this were not the case, then $g_\alpha \neq h_\alpha$ for every $\alpha \in S(1)$. Since $A_\alpha g = A_\alpha g_\alpha \neq A_\alpha h_\alpha = A_\alpha h$, then $gh^{-1} \notin A_\alpha$ for every $\alpha \in S(1)$; and so $gh^{-1} \notin \bigcup_{\alpha \in S(1)} A_\alpha$. But by assumption, $gh^{-1} \in A_\alpha$ for all $\alpha \in S(2)$, and thus $A_\gamma = \bigcup_{\alpha \in S(1)} A_\alpha$ must be the last element of S(1). Similarly, $A_\alpha g = A_\alpha h$ for each $\alpha \in S(2)$ implies that $gh^{-1} \in \bigcap_{\alpha \in S(2)} A_\alpha$. Hence $\bigcap_{\alpha \in S(2)} A_\alpha$ cannot be in S(1) and, therefore, must be the first element of S(2), namely $A_{\gamma+1}$. But (*) and $g_{\gamma+1} = h_{\gamma+1}$ together yield $A_\gamma g_\gamma = A_\gamma h_\gamma$, contradicting the assumption that $g_\gamma \neq h_\gamma$.

Let $S'(2) = \{\alpha \in W \mid \alpha \geqq \beta\}$ and let $S'(1)$ be its complement in W. Then $[S'(1), S'(2)]$ is a section in W with $S'(2) > S(2)$. Since $g_\beta = h_\beta$, we have $gh^{-1} \in A_\beta \leqq A_\alpha$ for every $\alpha \geqq \beta$, that is, $g_\alpha = h_\alpha$ for all $\alpha \in S'(2)$. Thus, by induction on the ordered set W [9, p. 264], $P(\alpha)$ holds for all $\alpha \in W$.

To complete the proof, we note that $g_\alpha = h_\alpha$ for all $\alpha \in W$ implies that $gh^{-1} \in A_\alpha$ for all α, and hence that $gh^{-1} \in \bigcap_{\alpha \in W} A_\alpha = A_0$. Thus $g = h$ and F is one-to-one.

Statement (1) in the introduction is easily seen to be that special case of Theorem 1 in which W with the inverse ordering is a well-ordered set.

THEOREM 2. *Let $\mathcal{U} = (A_\alpha)_{\alpha \in W}$ be a complete ordered system of subgroups of a group G containing the whole group $G = A_\mu$ and some $A_0 = \bigcap_{\alpha \in W} A_\alpha$. Then*

$$\sum_{\alpha \in \mathcal{J}} [A_{\alpha+1} : A_\alpha] \leqq [G : A_0].$$

PROOF. If $\mathcal{J}$ is finite, one can easily establish by induction that

$$\sum_{\alpha \in \mathcal{J}} [A_{\alpha+1} : A_\alpha] \leqq \prod_{\alpha \in \mathcal{J}} [A_{\alpha+1} : A_\alpha],$$

the desired result in this case.

If $\mathcal{J}$ is infinite, we shall establish the somewhat stronger statement

$$\sum_{\alpha \in \mathcal{J}} [A_{\alpha+1} : A_0] \leqq [G : A_0]. \tag{2}$$

Let R_α be a normalized transversal for A_α in $A_{\alpha+1}$ for each $\alpha \in \mathcal{J}$. Let L_α be a normalized transversal for A_0 in A_α for each $\alpha \in \mathcal{J}$. The main steps in the proof consist of showing that:

(i) $L_\alpha R_\alpha$ is a transversal for A_0 in $A_{\alpha+1}$ for each $\alpha \in \mathcal{J}$;

(ii) $\{1\} \cup_{\alpha \in \mathcal{J}} (L_\alpha R_\alpha \backslash L_\alpha)$ is a transversal for A_0 in G, and this union is disjoint;

(iii) $|L_\alpha R_\alpha| \leqq 2|L_\alpha R_\alpha \backslash L_\alpha|$.

Since $\mathcal{J}$ is infinite, (iii) implies (2). Details are omitted.

REMARK. If $\mathcal{U}$ is a complete ordered system as in the theorem, let

$$\Sigma(\mathcal{U}) = \sum_{\alpha \in \mathcal{J}} [A_{\alpha+1} : A_\alpha], \qquad \Pi(\mathcal{U}) = \prod_{\alpha \in \mathcal{J}} [A_{\alpha+1} : A_\alpha].$$

If the complete ordered system $\mathcal{U}'$ is a refinement of $\mathcal{U}$, then it is easily shown that

$$\Sigma(\mathcal{U}') \leqq \Sigma(\mathcal{U}) \leqq [G : A_0] \leqq \Pi(\mathcal{U}) \leqq \Pi(\mathcal{U}'),$$

that is, the finer the system becomes, the worse the approximation to $[G : A_0]$. However, using a general fact from cardinal arithmetic [9, p. 418], we obtain

$$\Pi(\mathcal{U}) \leqq [\Sigma(\mathcal{U})]^{|\mathcal{J}|}$$

By examining first the case where $\mathcal{J}$ is finite, and then the case where $\mathcal{J}$ is infinite, one gets

$$\Sigma(\mathcal{U}) \leqq [G : A_0] \leqq \Pi(\mathcal{U}) \leqq \exp \Sigma(\mathcal{U}) \leqq \exp [G : A_0].$$

Hence both $\Sigma(\mathcal{U})$ and $\Pi(\mathcal{U})$ are limited in range. In fact, if G is infinite and we assume the generalized continuum hypothesis, there are only two possible values for $[G : A_0]$: $\Sigma(\mathcal{U})$ or $\exp \Sigma(\mathcal{U})$; also, there are only two possible values for $\Pi(\mathcal{U})$: $[G : A_0]$ or $\exp [G : A_0]$. The only combination of these possibilities which cannot occur is $[G : A_0] = \exp \Sigma(\mathcal{U})$ and $\Pi(\mathcal{U}) = \exp [G : A_0]$.

4. Large abelian subgroups.

LEMMA [4, p. 240]. *If A is a maximal normal abelian subgroup of a group G and if G/A is a ZA-group, then* $A = C(A)$.

COROLLARY 1. *If A is a maximal normal abelian subgroup of an infinite group G and if G/A is a ZA-group, then* $\exp |A| \geqq |G|$.

PROOF [4, p. 240]. By the lemma, $A = C(A)$. Hence G/A is isomorphic to a subgroup of the group of automorphisms of A; and so if A were finite, G would also be finite. Thus A must be infinite and $|\mathrm{Aut}\,(A)| \leqq \exp |A|$. It follows that

$$|G| = |A|\,|G/A| \leqq |A|\,|\mathrm{Aut}\,(A)| \leqq \exp |A|.$$

THEOREM 3. *If A_1 is a normal abelian subgroup of an infinite group G and if G/A_1 is a ZFC-group, then G has an abelian subgroup A containing A_1 such that* $\exp |A| \geqq |G|$.

PROOF. Let $A_0 = E$, $H_0 = G$ and $H_1 = C(A_1 \leqq G)$. Inductively, suppose that we have defined the ascending chain $(A_\alpha)_{\alpha<\beta}$ and the descending chain $(H_\alpha)_{\alpha<\beta}$ such that $A_\alpha \leqq Z(H_\alpha)$ for all $\alpha < \beta$. If β is a limit ordinal, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and let $H_\beta = \bigcap_{\alpha<\beta} H_\alpha$. If $\beta = \alpha + 1$ for some ordinal α, let the superscript $^-$ denote homomorphic images in $H_\alpha/Z(H_\alpha)$. Since $\bar{H}_\alpha$ is a ZFC-group, if it is not trivial, there exists an element $x_{\alpha+1}$ in $H_\alpha \backslash Z(H_\alpha)$ such that $\bar{x}_{\alpha+1}$ is in $F_1(\bar{H}_\alpha)$. Let $A_{\alpha+1} = \langle x_{\alpha+1}\rangle Z(H_\alpha)$, and let $H_{\alpha+1} = C\ (A_{\alpha+1} \leqq H_\alpha)$. Hence for all α, $A_\alpha < A_{\alpha+1}$ unless $H_\alpha = Z(H_\alpha)$, that is, unless H_α is abelian. Let γ be the first ordinal such that H_γ is abelian.

We note that since $\bar{x}_{\alpha+1}$ has only a finite number of conjugates in $\bar{H}_\alpha$, $n_{\alpha+1} = [H_\alpha : N(A_{\alpha+1} \leqq H_\alpha)] = [\bar{H}_\alpha : N(\bar{A}_{\alpha+1} \leqq \bar{H}_\alpha)]$ is finite. Thus

$$[H_\alpha : H_{\alpha+1}] = n_{\alpha+1}[N(A_{\alpha+1} \leqq H_\alpha) : C(A_{\alpha+1} \leqq H_\alpha)] \leqq n_{\alpha+1}\ |\mathrm{Aut}\ (A_{\alpha+1})|.$$

Hence by Theorem 1

$$|G| = |H_\gamma|\ [G : H_\gamma] \leqq |H_\gamma| \prod_{\alpha<\gamma} [H_\alpha : H_{\alpha+1}] \leqq |H_\gamma| \prod_{\alpha<\gamma} n_{\alpha+1}\ |\mathrm{Aut}\ (A_{\alpha+1})|;$$

and so if H_γ were finite, G would also be finite. Thus H_γ is infinite and it follows by equation (2) that

$$|G| \leqq |H_\gamma| \prod_{\alpha<\gamma} \aleph_0^{|A_{\alpha+1}|} = |H_\gamma|\,\aleph_0^{\Sigma_{\alpha<\gamma}|A_{\alpha+1}|} \leqq |H_\gamma|\,\aleph_0^{|H_\gamma|} = \exp |H_\gamma|.$$

THEOREM 4. *If A_1 is a normal FC-subgroup of an infinite FCI^*-group G, then G has a ZFC-subgroup H containing A_1 such that* $\exp |H| \geqq |G|$.

PROOF. Let $H_0 = G$ and let $A_0 = E$. If possible, let $A_{\alpha+1}$ be a normal subgroup of H_α such that $A_{\alpha+1}/A_\alpha$ is a nontrivial normal FC-subgroup of H_α/A_α, then let $H_{\alpha+1}$ be the normal subgroup of H_α such that $H_{\alpha+1}/A_\alpha = (A_{\alpha+1}/A_\alpha)C\ (A_{\alpha+1}/A_\alpha \leqq H_\alpha/A_\alpha)$. If β is a limit ordinal, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and let $H_\beta = \bigcap_{\alpha<\beta} H_\alpha$. Since H_α/A_α is an FCI^*-group, if it is not trivial, then it has a nontrivial normal FC-subgroup. Hence $A_{\alpha+1} > A_\alpha$ unless $H_\alpha = A_\alpha$. Thus there is a first ordinal γ such that $H_\gamma = A_\gamma$.

Since $A_{\alpha+1}/A_\alpha$ is an FC-group, using the definition of $H_{\alpha+1}$, we see that each element in $A_{\alpha+1}/A_\alpha$ has only a finite number of conjugates in A_γ/A_α. Thus A_γ is a ZFC-group. It follows by the method of the previous argument (Theorem 3) that

$$[G : A_\gamma] \leqq \prod_{\alpha<\gamma} [H_\alpha/A_\alpha : H_{\alpha+1}/A_\alpha] \leqq \prod_{\alpha<\gamma} |\mathrm{Aut}\ (A_{\alpha+1}/A_\alpha)|.$$

Again, A_γ must be infinite, so by Theorem 2

$$[G : A_\gamma] \leqq \prod_{\alpha<\gamma} \aleph_0^{|A_{\alpha+1}/A_\alpha|} \leqq \aleph_0^{\Sigma_{\alpha<\gamma}|A_{\alpha+1}/A_\alpha|} \leqq \aleph_0^{|A_\gamma|}.$$

Thus

$$|G| \leqq \aleph_0^{|A_\gamma|} = \exp |A_\gamma|.$$

Theorem 3 together with Theorem 4 gives

THEOREM 5. *If G is an infinite FCI^*-group and if A_1 is a normal abelian subgroup of G, then G has an abelian subgroup A containing A_1 such that* $\exp^2|A| \geqq |G|$.

We would like to show that an infinite group G, every subgroup of which is subnormal, has an abelian subgroup A such that $\exp^2|A| \geqq |G|$. Since SI^*-groups are FCI^*-groups, this would certainly be the case if G were always an SI^*-group; but this is an open problem, so we settle for

THEOREM 6. *If an infinite group G has an abelian subgroup A* such that $A \lhd^n G$ *and such that* $|C(A)| \leqq \text{sym } |A|$, *then* $\exp^n|A| \geqq |G|$.

PROOF. Let $A = A_0 \lhd A_1 \lhd \cdots \lhd A_n = G$ be a finite normal series from A to G. For each integer $k < m$,

$$\begin{aligned} |A_{k+1}| \leqq |N(A_k)| &= [N(A_k) : C(A_k)]\ |C(A_k)| \\ &\leqq \text{sym } |A_k| \cdot |C(A)| \leqq \text{sym } |A_k|. \end{aligned}$$

Proceeding by induction, we get

$$|G| = |A_n| \leqq \text{sym}^n|A|.$$

Again, A cannot be finite, so $\text{sym}^n|A| = \exp^n|A|$.

THEOREM 7. *If H_1 is a normal mC-subgroup of an infinite mCI^*-group G, then G has a ZmC-subgroup H containing H_1 such that* $\exp |H| \geqq |G|$. *If A_1 is a normal abelian subgroup of an infinite ZmC-group H, then H has an abelian subgroup A containing A_1 such that $m^{|A|} \geqq |H|$. Thus, if A_1 is a normal abelian subgroup of an infinite mCI^*-group G, then G has an abelian subgroup A containing A_1 such that* $\exp m^{|A|} \geqq |G|$.

PROOF. The arguments used above are easily applied here. Details are omitted.

REMARK. If G is any group, the *Fitting subgroup* of G, $\nu(G)$, is defined to be the product of all of the normal nilpotent subgroups of G. If G coincides with $\nu(G)$, G is said to be a *Fitting group*. If G is an SI^*-group, then $C(\nu(G))$ is contained in $\nu(G)$. (For details see [7, p. 16].) Thus, *if G is an infinite SI^*-group*, $\exp |\nu(G)| \geqq |G|$.

Consequently, if we knew that every infinite Fitting group N had an abelian subgroup A such that $\exp |A| \geqq |N|$, then we could easily deduce that every infinite SI^*-group G has an abelian subgroup A with $\exp^2 |A| \geqq |G|$. We show

THEOREM 8. *If one assumes the generalized continuum hypothesis, then every infinite Fitting group G has an abelian subgroup A such that* $\exp |A| \geqq |G|$.

PROOF. Since Fitting groups are SI^*-groups, by Theorem 5, G has an abelian subgroup A with $\exp^2 |A| \geqq |G|$.

Let $\aleph_\alpha$ be the order of G. If α is of the second kind, then we must have $|A| = |G|$. If α is of the first kind, but $\alpha - 1$ is of the second kind, then we cannot have $\exp |A| < |G|$. For, if $|A| < \aleph_{\alpha-1}$, then $\exp^2 |A| < \aleph_{\alpha-1}$, while if $|A| = \aleph_{\alpha-1}$, then $\exp |A| = \aleph_\alpha = |G|$.

So we may assume that neither α nor $\alpha - 1$ is of the second kind. If $|x^G| = |G|$ for some $x \in G$, then by Corollary 1, x^G has an abelian subgroup B with $\exp |B| \geqq |x^G| = |G|$. Thus we may assume that $|x^G| < |G|$ for all x in G. But then G is an $\aleph_{\alpha-1}C$-group; and so by Theorem 7, G has an abelian subgroup B such that $\aleph_{\alpha-1}^{|B|} \geqq \aleph_\alpha$. If $|B| \leqq \aleph_{\alpha-2}$, then $\aleph_{\alpha-1}^{|B|} \leqq \aleph_{\alpha-1}^{\aleph_{\alpha-2}} = \aleph_{\alpha-1}$, a contradiction. Thus $|B| \geqq \aleph_{\alpha-1}$; but then $\exp |B| \geqq \exp \aleph_{\alpha-1} = \aleph_\alpha$.

REMARK. If G is any group, the *Gruenberg* (respectively, *Baer*) *radical* of G, $\rho(G)$, is the group generated by all the ascendant (respectively, subnormal) abelian subgroups of G [7, p. 100]. One can prove that if G is an SN^*-group (respectively, SJ^*-group), then $C(\rho(G))$ is contained in $\rho(G)$. (See, for example, [6, p. 352].) Thus, if G is an infinite SN^*-group (respectively, SJ^*-group), then $\exp |\rho(G)| \geqq |G|$. We note, however, that the Kovacs-Neumann example [7, p. 110] is an infinite locally nilpotent p-group with trivial Gruenberg radical. If we generalize this example by wreathing together ω_ω copies (instead of only ω_1 copies) of Z_p, and if we assume the generalized continuum hypothesis, we then discover that *there exists a locally finite p-group G in which each abelian subgroup A has* $\exp^n |A| < |G|$ *for all positive integers n.* For details, see [7, p. 111].

There is an infinite two-step nilpotent group G which has a maximal normal abelian subgroup A such that $\exp |A| = |G|$ [8, 9.2.17]. This shows that the bounds given in Corollary 1 and Theorem 3 are the best possible. The author does not know whether any of the other results are the best possible. There seems to be no known counterexample to the following question: *Does every locally nilpotent group of order* $>$ exp m *have an abelian subgroup of order* $> m$?

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