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REAL PLACES AND ORDERED FIELDS

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Let $\tau : F \to \boldsymbol{R} \cup \{\infty\}$ be a place from a field F into the real numbers $\boldsymbol{R}$. We say an ordering P of F (i.e. the set of nonnegative elements of an order making F an ordered field) is *associated* with τ if and only if $\tau(P) \geqq 0$. This definition is closely related to one of Lang's (see the appendix below). In this note we relate the set $\mathrm{Ord}(\tau)$ of orderings of F associated with τ to the value group of the valuation canonically associated with τ. Precisely, $\mathrm{Ord}(\tau)$ is bijective with the dual of the square factor group of the value group. In particular, $\mathrm{Ord}(\tau)$ is finite if and only if the square factor group is finite, in which case they have the same number of elements. (See [2] for orderings, places and valuations.)

The next lemma is essentially an interpretation of some results of Lang [3]. Its proof gives the "usual" construction for the real-valued place associated with a given ordering. Notice that an ordering P is associated with τ if and only if P contains every element of F which τ maps to a positive real number.

LEMMA. *Each ordering is associated with a unique real-valued place, and each real-valued place has associated with it at least one ordering.*

PROOF. Let P be an ordering of F. The set of elements of F not infinitely large over the rational numbers forms a valuation ring of F [2, p. 272]; call the valuation ring A. Let σ_0 be the surjective place canonically associated with A (e.g. [2, p. 298]). Give $\sigma_0(A)$ the ordering $\sigma_0(A \cap P)$. Since this ordering is Archimedean, there is an order embedding σ_1 of $\sigma_0(A)$ into $\boldsymbol{R}$. Then $\sigma = \sigma_1\sigma_0$ is a real-valued place associated with P.

Now suppose σ' is any real-valued place associated with P. Let $a \in F$. If $\sigma'(a) < \infty$, then for some natural number n, $0 < \sigma'(n \pm a) < \infty$, whence $n \pm a \in P$ (since P is associated with σ'), so $a \in A$. On the other hand, if $\sigma'(a) = \infty$, then for each natural number n,

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$\infty > \sigma'(n^{-1} \pm a^{-1}) > 0$, whence $n^{-1} \pm a^{-1} \in P$, so $a \notin A$. Thus A is the valuation ring canonically associated with σ'.

Consequently, the correspondence $\sigma(a) \leftrightarrow \sigma'(a)$ $(a \in A)$ is an isomorphism of $\sigma(A)$ and $\sigma'(A)$; it is an order isomorphism since the orderings are $\sigma(A \cap P)$ and $\sigma'(A \cap P)$ respectively. That $\sigma = \sigma'$ now follows from the uniqueness of the order completion of an Archimedean ordered field.

That each real-valued place has associated with it an ordering follows from Theorem 6 of [3]. The lemma is proved.

Let $P, P' \in \mathrm{Ord}(\tau)$. Let v be the valuation canonically associated with A [2, p. 297]. Let Z_2 denote the additive group of integers modulo two, and let $F^\times$ denote the multiplicative group of nonzero elements of F. There is a unique group homomorphism

$$\langle P, P' \rangle : v(F^\times) \to Z_2$$

with kernel $v(P \cap P' \cap F^\times)$ (one checks directly that the product of two elements of $v(F^\times)$ each not in $v(P \cap P' \cap F^\times)$ is itself in $v(P \cap P' \cap F^\times)$). Our main result is

PROPOSITION. *Let $T \in \mathrm{Ord}(\tau)$. The induced mapping*

$$\langle\ \cdot, T \rangle : \mathrm{Ord}(\tau) \to \mathrm{Hom}(v(F^\times), Z_2)$$

is a bijection.

REMARK. We may identify $\mathrm{Hom}(v(F^\times), Z_2)$ with the character group of $v(F^\times)/v(F^\times)^2$, namely

$$\mathrm{Hom}(v(F^\times)/v(F^\times)^2, Z_2).$$

Hence $\mathrm{Hom}(v(F^\times), Z_2)$ canonically admits the structure of a compact abelian group [4, p. 237]. Give $\mathrm{Ord}(\tau)$ the coarsest topology such that for each $a \in F$, the set of $P \in \mathrm{Ord}(\tau)$ containing a is open (e.g. see [1, p. 10]). One can easily show that the bijection of the proposition above is a homeomorphism.

Any finite abelian group is isomorphic to its character group [2, p. 50]. Thus the above proposition implies that $\mathrm{Ord}(\tau)$ is finite if and only if

$$v(F^\times)/v(F^\times)^2 \qquad (\cong F^\times/\tau^{-1}(\mathbf{R}^\times) \cdot F^{\times 2})$$

is finite, in which case they have the same number of elements (the above isomorphism follows from elementary valuation theory). For example, the field of formal series $\mathbf{R}((x))$ admits exactly two orderings, since any ordering is associated with the canonical real-valued place

$$f(x) \mapsto f(0) \in \mathbf{R} \cup \{\infty\} \qquad (f(x) \in \mathbf{R}((x))).$$

We now prove the proposition. Just suppose $\langle P, T\rangle = \langle P', T\rangle$ for some $P, P' \in \mathrm{Ord}(\tau)$. We claim $P = P'$. By symmetry it suffices to show for any $a \in P \cap F^\times$, that $a \in P'$. First suppose $a \in T$. Then

$$\langle P, T\rangle(v(a)) = \langle P', T\rangle(v(a)) = 0$$

so for some $b \in P' \cap T \cap F^\times$, $\tau(a/b)$ is finite and nonzero. Indeed, since T is associated with τ, $\tau(a/b)$ is positive. Since P' is associated with τ, $a/b \in P'$, so $a \in P'$. Now suppose $a \notin T$. If $a \notin P'$, then repeating the above argument with a, P, P' replaced by $-a$, P', P, respectively, we obtain $-a \in P$, a contradiction. We have proved the injectivity of $\langle \cdot, T\rangle$.

$\mathrm{Hom}(v(F^\times), Z_2)$ is naturally bijective with the set of subgroups of $v(F^\times)$ of index at most two (any homomorphism into Z_2 is determined by its kernel). Hence to prove surjectivity we must show that for any subgroup G of $v(F^\times)$ of index at most two, there exists $P \in \mathrm{Ord}(\tau)$ with $v(P \cap T \cap F^\times) = G$. We may suppose $G \neq v(F^\times)$; otherwise take $P = T$. Since $v(T) = v(F)$, we can find $a \in T \cap F^\times$ and a subset B of $T \cap F^\times$ such that $v(a) \notin G$ and the natural map $F^\times \to v(F^\times)/v(F^\times)^2$ carries B bijectively onto a basis of $G/v(F^\times)^2$ (one checks $G \supseteq v(F^\times)^2$). τ has an extension to a (not necessarily real-valued) place τ' on $F[(-a)^{1/2}][\{b^{1/2} | b \in B\}]$. Let v' be the valuation associated with τ'. If B_0 is a finite subset of $B \cup \{-a\}$, say with n elements, then the degree of the extension

$$F[\{b^{1/2} | b \in B_0\}]/F$$

is at most 2^n. By construction, the group index of $v(F^\times)$ in $v'(F[\{b^{1/2} | b \in B_0\}]^\times)$ is at least 2^n. Thus (cf. [1, p. 300])

$$\tau(F) = \tau'(F[\{b^{1/2} | b \in B_0\}]).$$

Since B_0 was arbitrary, we conclude that τ' is real-valued. By the lemma there is an ordering P' associated with τ'; set $P = P' \cap F$. P is clearly associated with τ. By construction $P \neq T$ (since $a \notin P$) and $v(P \cap T \cap F^\times) \supseteq G$ (since $P \cap T \cap F^\times \supseteq B$). But this containment must be equality since otherwise $v(P \cap T \cap F^\times) = v(F^\times)$, i.e. $\langle P, T\rangle = \langle T, T\rangle$, which contradicts our injectivity proof.

This completes the proof of the proposition.

We mention a relative form of the above proposition. Let $T \in \mathrm{Ord}(\tau)$. Let E be a subfield of F, and let $T_0 = T \cap E$ be the induced ordering on E. Then $\langle \cdot, T\rangle$ induces a bijection from the set of $P \in \mathrm{Ord}(\tau)$ containing T_0 onto $\mathrm{Hom}(v(F^\times)/v(E^\times), Z_2)$ (since, as is easily checked, $P \supseteq T_0$ if and only if $\langle P, T\rangle$ maps $v(E^\times)$ to zero).

Appendix. We sketch here first a generalization of the above remarks, and then the relation between our generalized concepts and those of Lang [3].

All our definitions make sense with $\boldsymbol{R}$ replaced by an arbitrary ordered field and the proposition remains true in this generalized context (proofs generalize immediately). The only modification needed is in the uniqueness part of the lemma; the situation is the following. Let $P \in \mathrm{Ord}(\tau)$ (τ and v as above). For each isolated (= "convex") subgroup I of $v(F^{\times})$, the associated valuation $F \to \{0\} \cup (v(F^{\times})/I)$ induces canonically a place $\tau(I)$ on F. Giving the residue class field of $\tau(I)$ the ordering induced by P, we have that P is associated with $\tau(I)$ in the above generalized sense. Moreover, all surjective places with which P is associated are obtained in this way (basically because τ, by its uniqueness, must factor through such places).

Let I be as above. It is not hard to prove that the subfields F_0 of F such that I is the isolated subgroup of $v(F^{\times})$ generated by $v(F_0^{\times})$ (and surely there exist such) are exactly those for which $\tau(I)$ is the canonical place of F with respect to F_0 induced by P, in the sense of Lang [2]. Moreover, all such "canonical places" are obtained in this way.

NOTE. An interesting generalization of the above lemma to D. K. Harrison's "infinite primes" can be found in [5].

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