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UNIFORM CONVERGENCE OF FOURIER SERIES ON GROUPS. II

C. W. ONNEWEER

1. **Introduction.** In [3] A. M. Garsia and S. Sawyer proved the following. Let f be a real-valued continuous function of period 2π and normalized so that the range of f is precisely $[0, 1]$. For each $y \in [0, 1]$ set $E_y = \{x \in [0, 2\pi];\ f(x) > y\}$ and let $N(E_y)$ be the number of disjoint open intervals in the open set E_y. Then $\int_0^1 \log N(E_y)dy < \infty$ implies that the Fourier series of f converges uniformly. In [6] N. Ja. Vilenkin considered certain 0-dimensional, compact, metrizable, abelian groups G and their charactergroups X, which are discrete, countable, abelian torsion groups [4, (24, 15) and (24, 26)]. He defined an enumeration for the elements of X and developed part of the Fourier theory for functions on G. In this paper we will show that a modified version of Garsia and Sawyer's result holds for functions on a large class of the groups as described by Vilenkin.

2. **The groups G and X.** Let G and X be as in the introduction. Vilenkin [6] proved that there exists an increasing sequence of finite subgroups $\{X_n\}$ in X such that
(i) $X_0 = \{\chi_0\}$, where $\chi_0(x) = 1$ for all x in G,
(ii) each X_n/X_{n-1} is of prime order p_n, and
(iii) $\bigcup_{n=0}^{\infty} X_n = X$.
Furthermore, the subgroups X_n can be chosen in such a way that there exists a sequence $\{\varphi_n\}$ of characters on G satisfying $\varphi_n \in X_{n+1} \setminus X_n$ and $\varphi_n^{p_{n+1}} \in X_n$. Also, we can enumerate the elements of X as follows. Let $m_0 = 1$ and $m_n = p_n m_{n-1}$. If k is a natural number and $k = \sum_{i=0}^{s} a_i m_i$ with $0 \leqq a_i < p_{i+1}$ for $0 \leqq i \leqq s$, then $\chi_k = \varphi_0^{a_0} \cdot \cdots \cdot \varphi_s^{a_s}$. This implies that $X_n = \{\chi_i \mid 0 \leqq i < m_n\}$.

Next, let G_n be the annihilator of X_n, i.e.,

$$G_n = \{x \in G; \chi_k(x) = 1 \text{ for } 0 \leqq k < m_n\}.$$

Then, obviously, $G = G_0 \supset G_1 \supset G_2 \supset \cdots$, $\bigcap_{n=0}^{\infty} G_n = \{0\}$ and the G_n form a basis for the neighborhoods of zero in G. In [6, 3.2] Vilenkin showed that for each n there is an $x_n \in G_n \setminus G_{n+1}$ such that $\chi_{m_n}(x_n) = \exp(2\pi i/p_{n+1})$. He also observed that each $x \in G$ has a

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unique representation $x = \sum_{i=0}^{\infty} b_i x_i$ with $0 \leqq b_i < p_{i+1}$. This enables us to order G by means of the lexicographical ordering of the sequences $\{b_n\}$. Furthermore,

$$G_n = \left\{ x \in G;\, x = \sum_{i=0}^{\infty} b_i x_i \text{ with } b_0 = \cdots = b_{n-1} = 0 \right\}.$$

Consequently, each coset of G_n has a representation of the form $z + G_n$, where $z = \sum_{i=0}^{n-1} b_i x_i$ for some choice of the b_i, $0 \leqq b_i < p_{i+1}$. We will denote these z, ordered lexicographically, by $z_\alpha^{(n)}$, $0 \leqq \alpha < m_n$. At times we will denote $z_\alpha^{(n)} + G_n$ simply by $z_\alpha + G_n$.

REMARK 1. Examples of groups G and X as described above are

(a) $G = \prod_{n=1}^{\infty} (Z(2))_n$; then X is the group of Walsh functions, see [**2**].

(b) $G = \prod_{n=1}^{\infty} Z(p_n)$, where $\{p_n\}$ is some sequence of prime numbers; in case $p_n = p$ for all n, the elements of X are the generalized Walsh funtctions, see [**1**].

(c) G is the group of p-adic integers; then $X = Z(p^{\infty})$, see [**4**, §10 and (25.2)].

3. **On Fourier series of functions on G and Dirichlet kernels.** Let dx denote the normalized Haar measure on G. If $f \in L_1(G)$ then the Fourier series of f is the series

$$\sum_{i=0}^{\infty} c_i \chi_i(x) \qquad \text{where } c_i = \int_G f(t)\overline{\chi_i(t)}dt.$$

For its partial sums we have

$$S_n(x;f) = \sum_{i=0}^{n-1} c_i \chi_i(x) = \int_G f(x-t) D_n(t) dt, \tag{1}$$

where $D_n(t) = \sum_{i=0}^{n-1} \chi_i(t)$. $D_n(t)$ is called the Dirichlet kernel of order n. We will need the following properties of these Dirichlet kernels.

LEMMA 1. *For each n,*

(a) *if $x \in G_n$ then $D_{m_n}(x) = m_n$,*

(b) *if $x \notin G_n$ then $D_{m_n}(x) = 0$,*

(c) *if $x \notin G_n$ then $|D_k(x)| \leqq m_n$ for all k,*

(d) *if $x \notin G_n$ and $k \geqq m_n$ then $\int_{x+G_n} D_k(t)dt = 0$.*

PROOF. For (a) and (b), see [**6**, 2.2]. For (c), see [**6**, 3.61]. In order to prove (d) we observe that, according to [**6**, 3.22], if $k \geqq m_n$ and $x \notin G_n$ then

$$\int_{x+G_n} \chi_k(t)dt = 0.$$

This in addition to (a) shows that

$$\int_{x+G_n} D_k(t)dt = \int_{x+G_n} D_{m_n}(t)dt + \sum_{i=m_n}^{k-1} \int_{x+G_n} \chi_i(t)dt = 0.$$

DEFINITION 1. G satisfies property (P) if $\sup p_n = p < \infty$.

LEMMA 2. *Let G satisfy property* (P). *Then for all k, n and α, $0 < \alpha < m_n$, we have*

$$|D_k(z_\alpha^{(n)})| < (p + 1)\, m_n/\alpha.$$

PROOF. For each $z_\alpha^{(n)}$ there exists an ℓ with $0 \leqq \ell < n$ such that $z_\alpha^{(n)} \in G_\ell \backslash G_{\ell+1}$. Consequently,

$$z_\alpha^{(n)} = \sum_{i=\ell}^{n-1} b_i x_i, \qquad \text{with } b_\ell \neq 0 \text{ and } 0 \leqq b_i < p_{i+1}.$$

Also,

$$\alpha = b_\ell\, p_{\ell+2} \cdot \ldots \cdot p_n + b_{\ell+1} p_{\ell+3} \cdot \ldots \cdot p_n + \ldots + b_{n-2} p_n + b_{n-1}.$$

Therefore,

$$m_{\ell+1} \frac{\alpha}{m_n} = b_\ell + \frac{b_{\ell+1}}{p_{\ell+2}} + \cdots + \frac{b_{n-2}}{p_{\ell+2} \cdot \ldots \cdot p_{n-1}} + \frac{b_{n-1}}{p_{\ell+2} \cdot \ldots \cdot p_n}$$

$$< b_\ell + 2 \leqq p + 1.$$

Hence, Lemma 1(c) implies that

$$|D_k(z_\alpha^{(n)})| \leqq m_{\ell+1} < (p + 1)\, m_n/\alpha.$$

REMARK 2. Lemma 2 is a generalization of Lemma 1 in [2].

4. **The main theorem.** Before stating our main result we first formulate the analogue on G of the well-known fact that each open subset of the set of real numbers R is the union of at most countably many open intervals.

DEFINITION 2. A subset I in G is called an interval of G if for all $a, b \in I$ with $a < b$ and all $x \in G$ such that $a < x < b$ we have $x \in I$. Here $<$ refers to the ordering of the elements of G as defined in §2.

LEMMA 3. *If E is an open subset in G, then E is the union of at most countably many disjoint open intervals of G, which are separated from each other by elements of $G \backslash E$. We will denote the number of such intervals by $N(E)$.*

Since the proof of this lemma is similar to the proof of the classical case, i.e. for R, we omit it. The role of the rational numbers in R is taken over by the elements $z_\alpha^{(n)}$ of G, which, from now on, we will call the rational elements of G.

THEOREM 1. *Let G satisfy property* (P). *Let f be a continuous function on G with minimum value* 0 *and maximum value* 1. *For $y \in [0,1]$ set*

$$E_y = \{x \in G; f(x) > y\}.$$

Then

$$\int_0^1 \log N(E_y)dy < \infty$$

implies that the Fourier series of f converges uniformly on G.

The proof of the theorem will be preceded by a number of lemmas, many of which are similar to results in [3].

LEMMA 4. *Let G satisfy condition* (P). *Let E be an open subset of G for which $N(E) < \infty$ and let ψ_E be the characteristic function of E. Let*

$$S^*(x; \psi_E) = \sup_n |S_n(x; \psi_E)|.$$

Then there exist constants A and B, independent of E, such that for all $x \in G$

$$S^*(x; \psi_E) \leqq A + B \log N(E). \tag{2}$$

PROOF. Let $E = I_1 \cup I_2 \cup \cdots \cup I_{N(E)}$, where the I_j are the disjoint open intervals as in Lemma 2 and set $\psi_{I_j}(x) = \psi_j(x)$, $j = 1, 2, \cdots, N(E)$. For any given k choose n so that $m_{n-1} \leqq k < m_n$. Then, according to (1)

$$\begin{aligned} S_k(x; \psi_E) &= \int_G \psi_E(x-t)D_k(t)dt \\ &= \int_{G_{n-1}} \psi_E(x-t)D_k(t)dt + \sum_{i=1}^{N(E)} \sum_{\alpha=1}^{m_{n-1}-1} \int_{z_\alpha + G_{n-1}} \psi_i(x-t)D_k(t)dt \\ &= B_1 + B_2. \end{aligned}$$

It is obvious that

$$|B_1| \leqq \int_{G_{n-1}} |D_k(t)|dt \leqq m_{n-1}^{-1}\, k \leqq p. \tag{3}$$

In order to find an estimate for B_2 we observe that if $z \notin G_{n-1}$ and if $\psi_i(x-t)$ is constant on $z + G_{n-1}$ then Lemma 1(d) implies that

$$\int_{z+G_{n-1}} \psi_i(x-t)D_k(t)dt = 0.$$

Therefore, for each interval I_i, $i = 1, 2, \cdots, N(E)$, at most two cosets of G_{n-1} will contribute to B_2, say $z_{\alpha(i,1)} + G_{n-1}$ and $z_{\alpha(i,2)} + G_{n-1}$. Hence

$$|B_2| \leqq \sum_{i=1}^{N(E)} \sum_{j=1}^{2} \int_{z_{\alpha(i,j)} + G_{n-1}} |D_k(t)|dt,$$

where, if $\alpha(i,j) = \alpha(k,\ell)$ for some $i \neq k$ or $j \neq \ell$, we count such a term only once in this sum. Using this summation convention again we obtain from Lemma 2,

$$\begin{aligned} (4)\qquad |B_2| &\leqq \sum_{i=1}^{N(E)} \sum_{j=1}^{2} \sum_{k=0}^{p_n-1} \int_{z^{(n-1)}_{\alpha(i,j)}+kx_{n-1}+G_n} |D_k(t)|\,dt \\ &\leqq m_n^{-1}(p+1)m_n \sum_{\alpha=1}^{2p_nN(E)} \alpha^{-1} \leqq (p+1)C\log 2p_nN(E) \end{aligned}$$

for some constant C. Combining (3) and (4) we easily obtain (2).

In [**6**, 3.2] Vilenkin defined the concept of bounded variation for functions on G in the usual way. In the following we derive a characterization for functions of bounded variation.

DEFINITION 3. For a real-valued function f in $L_1(G)$ and any n let

$$F_n(f) = m_n \int_{G_n} \sum_{\alpha=0}^{m_n-2} |f(t + z_\alpha^{(n)}) - f(t + z_{\alpha+1}^{(n)})|dt.$$

LEMMA 5. *Let f be a real-valued function in $L_1(G)$. Then $F_n(f) = O(1)$ as $n \to \infty$ if and only if f is equivalent to a function of bounded variation on G. Moreover, if f is continuous on G and $F_n(f) = O(1)$ as $n \to \infty$, then f is of bounded variation on G.*

PROOF. (i) Assume $F_n(f) = O(1)$ as $n \to \infty$. For each n and each $x \in G$ set

$$\tilde{f}_n(x) = m_n \int_{x+G_n} f(t)dt.$$

Then, according to [**6**, 3.32], $\lim_{n\to\infty} \tilde{f}_n(x) = f(x)$ a.e. on G, say for all $x \notin H$, where H is a set of measure zero. Also, if f is continuous then H is empty. Let $x_1 < x_2 < \cdots < x_n$ be elements of $G\backslash H$. Choose q so large that the cosets $x_i + G_q$ are mutually disjoint and let $x_i + G_q = z_{\alpha(i)} + G_q$. Then $\alpha(1) < \alpha(2) < \cdots < \alpha(n)$ and consequently

$$\sum_{i=1}^{n-1} |\tilde{f}_q(x_i) - \tilde{f}_q(x_{i+1})| \leqq \sum_{i=1}^{n-1} m_q \int_{G_q} |f(t + z_{\alpha(i)}^{(q)}) - f(t + z_{\alpha(i+1)}^{(q)})| dt$$

$$\leqq \sum_{\alpha=0}^{m_n-2} m_q \int_{G_q} |f(t + z_\alpha{}^{(q)} - f(t + z_{\alpha+1}^{(q)})| dt = O(1).$$

Therefore, f is of bounded variation on $G \backslash H$. A standard argument completes the proof.

(ii) Let g be of bounded variation and $g(x) = f(x)$ a.e. Then by [**6**, 3.22] there are two monotone increasing functions g_1, g_2 on G such that $g(x) = g_1(x) - g_2(x)$ for all $x \in G$. For each n we have $F_n(f) = F_n(g) \leqq F_n(g_1) + F_n(g_2)$. From the monotonicity of g_1 and g_2 it follows that

$$F_n(f) \leqq \sum_{i=1}^{2} m_n \int_{G_n} \sum_{\alpha=0}^{m_n-2} (g_i(t + z_\alpha{}^{(n)}) - g_i(t + z_{\alpha+1}^{(n)}))dt$$

$$= \sum_{i=1}^{2} m_n \int_{G_n} (g_i(t + z_{m_n-1}^{(n)}) - g_i(t + z_0{}^{(n)}))dt$$

$$\leqq \sum_{i=1}^{2} m_n m_n{}^{-1}\, 2M_i,$$

where $M_i = \text{lub}\,\{g_i(x);\ x \in G\}$. So $F_n(f)$ is bounded uniformly in n.

REMARK 3. A straightforward computation shows that for each $f \in L_1(G)$, $F_n(f)$ is an increasing function of n, so that $F_n(f) = O(1)$ as $n \to \infty$ if and only if $\lim_{n\to\infty} F_n(f)$ exists and is finite.

LEMMA 6. *Let G, f, E_y and $N(E_y)$ be as in Theorem* 1 *and let $\hat{N}(E_y)$ be defined by $\hat{N}(E_y) = \frac{1}{2}\lim_{n\to\infty} F_n(\psi_{E_y})$. Then $\hat{N}(E_y) \leqq N(E_y)$ except for at most countably many $y \in [0, 1]$.*

PROOF. Since the proof is similar to the proof of Lemma 2.1 in [**3**] we will only give an outline here. Let Γ be the set of all $y \in [0, 1]$ such that each two open intervals of E_y are separated by a coset of some G_n on which $f(x) < y$, and if $0 \notin E_y$ or $e \notin E_y$, where $0 = \sum_{i=0}^{\infty} 0x_i$ and $e = \sum_{i=0}^{\infty}(p_{i+1} - 1)x_i$, then $f(x) < y$ on the coset of some G_n containing 0 or e. Let Φ be the set of all $y \in [0, 1]$ such that y is a relative maximum or relative minimum of f on G. Then the following holds:

(a) $[0, 1]\backslash\Gamma \subset \Phi$,

(b) Φ is at most countable.

Now assume that $\tilde{y} \in \Gamma$ and that $N(E_{\tilde{y}}) = \infty$. Then $\psi_{E_{\tilde{y}}}$ is not equivalent to a function of bounded variation. Lemma 5 implies that $\hat{N}(E_{\tilde{y}}) = \infty$. Next, assume $\tilde{y} \in [0, 1]\backslash\Phi$ and $N(E_{\tilde{y}}) = s < \infty$. We also assume that $0, e \in E_{\tilde{y}}$; if this is not the case then the following argument requires some obvious modifications. There exist s open intervals $I_1, \cdots, I_s$ separated by subsets $J_1, \cdots, J_{s-1}$ in $G\backslash E_{\tilde{y}}$, so that $I_1 < J_1 < I_2 < \cdots < J_{s-1} < I_s$ and, moreover, each of these sets will contain at least one coset of G_m for some sufficiently large m. Then it is easy to see that $F_m(\psi_{E_{\tilde{y}}})$ is equal to the number of changes from a set I to a set J or conversely, from which it follows that $\hat{N}(E_{\tilde{y}}) = N(E_{\tilde{y}}) - 1$.

Lemma 7. *Let f be as in Theorem* 1. *For each k let*

$$H_k = \{y \in [0, 1]\,;\, N(E_{\tilde{y}}) = k\}$$

and

$$f_k(x) = \int_{H_k} \psi_{E_y}(x)dy.$$

Then each $f_k(x)$ is a continuous function of bounded variation on G.

Proof. The continuity of $f_k(x)$ follows from

$$f_k(x) = \int_0^1 \psi_{H_k}(y)\psi_{E_y}(x)dy = \int_0^{f(x)} \psi_{H_k}(y)dy. \tag{5}$$

Next we determine $F_n(f_k)$.

$$\begin{aligned} F_n(f_k) & \\ &= m_n \int_{G_n} \sum_{\alpha=0}^{m_n-2} \left| \int_0^1 \psi_{H_k}(y)(\psi_{E_y}(t + z_\alpha^{(n)}) - \psi_{E_y}(t + z_{\alpha+1}^{(n)}))dy \right| dt \\ &\leqq m_n \sum_{\alpha=0}^{m_n-2} \int_0^1 \int_{G_n} |\psi_{H_k}(y)(\psi_{E_y}(t + z_\alpha^{(n)}) - \psi_{E_y}(t + z_{\alpha+1}^{(n)}))| dt\, dy \\ &= m_n \int_0^1 \psi_{H_k}(y) \int_{G_n} \sum_{\alpha=0}^{m_n-2} |\psi_{E_y}(t + z_\alpha^{(n)}) - \psi_{E_y}(t + z_{\alpha+1}^{(n)})| dt\, dy. \end{aligned}$$

Since $F_n(\psi_{E_y})$ increases with n and $\lim_{n\to\infty} F_n(\psi_{E_y}) = 2\hat{N}(E_y)$, we have

$$\begin{aligned} F_n(f_k) &\leqq 2 \int_0^1 \psi_{H_k}(y)\hat{N}(E_y)dy \\ &\leqq 2 \int_0^1 \psi_{H_k}(y)N(E_y)dy < \infty, \end{aligned}$$

because $N(E_y) = k$ on H_k. An application of Lemma 5 shows that f_k is of bounded variation.

As an immediate consequence of Lemma 7 we have

COROLLARY 1. *If f is continuous on G and if $N(E_y) < \infty$ a.e. on $[0, 1]$ then there exists a sequence of continuous functions of bounded variation $\{f_k\}$ on G such that $f(x) = \sum_{k=1}^{\infty} f_k(x)$ uniformly in $x \in G$.*

PROOF OF THEOREM 1. According to Corollary 1 for each $x \in G$ and each m we have

$$S_m(x; f) = \sum_{k=1}^{\infty} S_m(x; f_k). \tag{6}$$

In [5, Corollary 3] it was shown that the Fourier series of a continuous function of bounded variation on G converges uniformly. Consequently, for each N,

$$\lim_{m\to\infty} \sum_{k=1}^{N} S_m(x; f_k) = \sum_{k=1}^{N} f_k(x), \tag{7}$$

uniformly in $x \in G$. Furthermore, using (5) we see that for all $x \in G$ and all m

$$\begin{aligned}\sum_{k=N+1}^{\infty} |S_m(x; f_k)| &\leqq \sum_{k=N+1}^{\infty} \int_{H_k} |S_m(x; \psi_{E_y})|dy \\ &\leqq \sum_{k=N+1}^{\infty} \int_{H_k} S_m^*(x; \psi_{E_y})dy.\end{aligned}$$

Applying Lemma 4 we obtain

$$\begin{aligned}\sum_{k=N+1}^{\infty} |S_m(x; f_k)| &\leqq \sum_{k=N+1}^{\infty} \int_{H_k} (A + B \log N(E_y))dy \\ &= A \sum_{k=N+1}^{\infty} \mu(H_k) + B \sum_{k=N+1}^{\infty} \mu(H_k) \log k.\end{aligned}$$

Since $\sum_{k=1}^{\infty} \mu(H_k) \log k = \int_0^1 \log N(E_y)dy < \infty$, we have, given $\epsilon > 0$, for sufficiently large N,

$$\sum_{k=N+1}^{\infty} |S_m(x; f_k)| < \epsilon. \tag{8}$$

Combining (6), (7) and (8) completes the proof of the theorem.

REMARK 4. Lemmas 5 and 6 and a simple computation show that if f is a continuous real-valued function on G with range in $[0, 1]$ then f is of bounded variation if and only if $\int_0^1 N(E_y)dy < \infty$. Also, as we observed earlier, the Fourier series of a continuous function of bounded variation converges uniformly on G. Consequently, Theorem 1 can be regarded as an improvement of this result.

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UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87106