ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 4, Fall 1971

A NOTE ON THE INTERSECTION OF THE POWERS OF THE JACOBSON RADICAL

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1. **Introduction and preliminaries.** All rings will be assumed to have identity. If R is a ring, $J = J(R)$ will denote its Jacobson radical. The purpose of this note is to establish conditions on R such that $\bigcap_{i=1}^{\infty} J^i = 0$. In particular, we show that if R is a right Noetherian J-prime ring such that every ideal of R is a principal right ideal, and in addition, J is a principal left ideal, then J is the nilpotent radical of R or $\bigcap_{i=1}^{\infty} J^i = 0$. Further, we show that $\bigcap_{i=1}^{\infty} J^i = 0$ if R is a right Noetherian ring, J is a principal right ideal, and $\bigcap_{i=1}^{\infty} J^i$ is a finitely generated left ideal of R. The methods of J. C. Robson [5] are used throughout, and Theorems 3.5 and 5.3 of Robson's paper are generalized.

A ring is called an *ipri-ring* (*ipli-ring*) if every ideal is a principal right (left) ideal [5, p. 127]. Condition (α) is said to hold in R if ab being regular in R is equivalent to both a and b being regular in R. Combining [1, Theorems 4.1 and 4.4, pp. 212-213] and [4, Corollary 2.6, p. 603] one sees that if R is a semiprime right Noetherian ring, then (α) holds in R. A ring R is said to be *J-prime* (*J-simple*) if R/J is a prime (simple) ring. The nilpotent radical of a ring is denoted by W and *W-simple* is defined similarly. The symbol $\subset$ will denote proper containment.

A result important to our work is the following lemma [3, p. 200]:

LEMMA 1.1. *For any ring R, if G is a nonzero ideal of R finitely generated as a right (left) ideal of R and $G \subseteq J = J(R)$, then $GJ \subset G$ ($JG \subset G$).*

LEMMA 1.2. *Let R be a right Noetherian J-prime ipri-ring. If T is an ideal of R such that $T \not\subseteq J$, then $J \subset T$.*

PROOF. Let $B = T + J = bR$ and $J = aR$. Assume $J \subset B$. Then the image of B in R/J is a nonzero ideal and hence the image of b is regular since R/J is a prime right Noetherian ring [5]. Since $J \subset bR$, we have $J = bJ$. Hence $J \subset T + J^2$ and there exist $t \in T$ and $r \in R$ such that $a(1 - ar) = t$. But $1 - ar$ is a unit in R so $a \in T$. Thus $J \subset T$.

Received by the editors October 17, 1969, and, in revised form, May 23, 1970.
AMS 1969 *subject classifications.* Primary 1620; Secondary 1625.

Corollary 1.3. *Let R be as in Lemma* 1.2. *If A, B are ideals of R such that $AB = 0$ and $B \neq 0$, then $A \subseteq J$.*

2. Results concerning $\bigcap_{i=1}^{\infty} J^i$.

Theorem 2.1. *Let R be a right Noetherian J-prime ipri-ring. Assume that $J = aR = Rb$. Then either $J = W$ or*

(i) $\bigcap_{i=1}^{\infty} J^i = 0$,
(ii) *R is a prime ring,*
(iii) $J = aR = Ra = bR = Rb$,
(iv) *J^k, $k = 1, 2, \cdots$, are the only proper ideals of R, and*
(v) *R is also an ipli-ring.*

Proof. Assume $J \neq W$. Let $\mathcal{S}$ be the set of all ideals B of R such that $a^k \notin B$ for all k. Let C be a maximal element of $\mathcal{S}$ containing $G = \bigcap_{i=1}^{\infty} J^i$. By Lemma 1.2, $C \subset J$. Since C is a prime ideal of R, the image of a in R/C is regular and $C = aC$. Thus $C = G$ and, therefore, $\bar{R} = R/G$ is a prime ring. Now $\bar{J} = \overline{aR} = \overline{Rb}$ is a proper ideal of $\bar{R}$. Hence $\bar{a}$ is regular in $\bar{R}$ and $\bar{a} = \overline{rb}$ for some $\bar{r} \in \bar{R}$, so $\bar{b}$ is regular in $\bar{R}$ since condition (α) holds in $\bar{R}$. Thus $G \subseteq Rb$ implies $G = Gb$ and consequently $G = GJ$. Hence by Lemma 1.1, $G = 0$. Thus (i) and (ii) have been proved.

To prove (iii) note that $a = ub$ and $b = av$ for some u, $v \in R$. Then $a = uav$ and since $ua \in J$, $a = awv$ for some $w \in R$. Since R is a prime ring, $1 = wv$ and since condition (α) holds in R, w and v are regular in R. This means v is a unit in R so $bR = avR = aR$. The proof of the other part is similar.

Let $M = xR$ be an ideal of R properly containing J. By passing to R/J we see that $J = xJ$. Thus $a = xra$ for some $r \in R$. Condition (α) and the regularity of a imply that x is a unit in R. Hence J is the unique maximal ideal of R. Now let T be any nonzero ideal of R. Pick n so that $T \subseteq J^n$, $T \nsubseteq J^{n+1}$. Then $S = \{x \in R \mid a^n x \in T\}$ is an ideal of R not contained in J; hence $S = R$. This proves that $T = J^n$ and completes the proof.

J. C. Robson proves a theorem [5, p. 133] similar to the above theorem under the assumptions that R is Noetherian (on both sides) and W-simple where W is a principal left ideal and a principal right ideal.

We immediately get the following

Corollary 2.2. *If R is a right Noetherian prime ipri-ring such that $J = Rb$ for some $0 \neq b \in R$, then J is a prime ideal if and only if J is maximal.*

Theorem 2.3. *Let R be a right Noetherian ring with $J = aR$ for* *some* $a \in R$. *If* $G = \bigcap_{i=1}^{\infty} J^i$ *is a finitely generated left ideal of* R, *then* $G = 0$.

PROOF. Since $G = \bigcap_{i=1}^{\infty} a^i R$, $x \in G$, implies that $x = ar_1 = a^2 r_2 = \cdots$, for $r_1, r_2, \cdots$ in R. Since R is right Noetherian, there exists an integer k such that $r_{k+1} \in r_1 R + r_2 R + \cdots + r_k R$. Hence $r_{k+1} = r_1 s_1 + r_2 s_2 + \cdots + r_k s_k$ for some $s_1, s_2, \cdots, s_k \in R$. Thus $x = a^{k+1} r_{k+1} = a^k(ar_1)s_1 + \cdots + a(a^k r_k)s_k \in JG$. Hence $G = JG$ and since G is a finitely generated left ideal of R, $G = 0$ by Lemma 1.1.

To see that the finite generation of G as a left ideal is necessary in the above theorem consider the following example [2, pp. 35-36]. Let A be the ring of rationals with odd denominators. Let R be the ring of all matrices of the form

$$\begin{pmatrix} a & \alpha \\ 0 & \beta \end{pmatrix}$$

where $a \in A$ and α, β are rationals. Then R is a right Noetherian ring such that $J = J(R)$ is a prinicpal right ideal. However $\bigcap_{i=1}^{\infty} J^i \neq 0$.

COROLLARY 2.4. *If* R *is a Noetherian ring with* $J = aR$ *for some* $a \in R$, *then* $\bigcap_{i=1}^{\infty} J^i = 0$.

If in Corollary 2.4 we assume in addition that $J = Rb$ for some $b \in R$ and that R is J-simple, then using [5, Theorem 5.3, p. 133] in the case $J = W$ and Corollary 2.4, otherwise, we can show that $aR = Ra = Rb = bR$, that J^k, $k = 1, 2, \cdots$, are the only proper ideals of R, that R is W-simple or a prime ring, and that R is an ipri- and ipli-ring.

THEOREM 2.5. *If* R *is a nonsemisimple right Noetherian ipri-*J*-prime ring, then* J *is nilpotent if and only if* J *does not properly contain a prime ideal of* R.

PROOF. If J is nilpotent, the result is trivial. Suppose J does not properly contain a prime ideal of R and suppose that J is not nilpotent. Then $J = xR$ for some nonnilpotent element x of R. Let I be a nonzero prime ideal of R maximal with respect to the exclusion of powers of x. Then $I \not\supseteq J$ and so by Lemma 1.2, $J \subset I$ which is a contradiction.

One can see that the assumption that R is a J-prime ipri-ring in Theorem 2.5 is necessary by reexamining the example cited after Theorem 2.3. The Jacobson radical of R is the set of matrices of the form

$$\begin{pmatrix} a & \alpha \\ 0 & 0 \end{pmatrix}$$

where a is in the Jacobson radical of A and α is a rational. Hence R is not semisimple. Moreover $J(R)$ does not properly contain a prime ideal of R and $J(R)$ is not a prime ideal, but yet $J(R)$ is not nilpotent.

COROLLARY 2.6. *If R is a right Noetherian ipri- J-prime ring such that $\bigcap_{i=1}^{\infty} J^i = 0$, then either R is a prime ring or else J is nilpotent.*

PROOF. If $J = 0$, then R is a prime ring. Suppose R is not a prime ring and let I be a prime ideal of R with $I \subset J$. Then $\bar{J} = \bar{x}\bar{R}$, for some $x \in R$, is a nonzero ideal in the right Noetherian prime ring $\bar{R} = R/I$. Hence $I = xI = \cdots$ and so $I \subset \bigcap_{i=1}^{\infty} J^i = 0$. This contradiction shows that no prime ideal is properly contained in J. Hence J is nilpotent.

3. Noetherian ipri-rings.

THEOREM 3.1. *Let R be a Noetherian J-prime ipri-ring. Then R is J-simple or a prime ring.*

PROOF. If $J = 0$, then R is a prime ring. Assume $J \neq 0$ and let $\bar{A} = \bar{a}\bar{R}$ be a nonzero ideal of the prime ring $\bar{R} = R/J$. Using an argument similar to that of Lemma 1.2, we get $J = aJ = a^2J = \cdots$ and by [5, Corollary 3.2, p. 129] $(1 - au) \bigcap_{i=1}^{\infty} a^iR = 0$ for some $u \in R$. But $J \subseteq \bigcap_{i=1}^{\infty} a^iR$ so $(1 - au)J = 0$. Hence the fact that $R(1 - au) \cdot RJ = 0$ and Corollary 1.3 show that $1 - au \in J$. Therefore a is a unit in R. Thus $\bar{A} = \bar{R}$.

If we combine Theorem 3.1 and Corollary 2.4 we obtain

THEOREM 3.2. *If R is a Noetherian J-prime ipri-ring, then R is W-simple or is a prime ring.*

THEOREM 3.3. *Let R be a Noetherian J-simple ring such that $J = aR = Rb$. Let R^* be the associated graded ring of R with respect to J* [5, p. 137]. *Then R^* is a Hilbert polynomial ring over R/J of index n where n is the index of nilpotency of W or R^* is a Hilbert polynomial ring over R/J* [5, p. 134]. *In the former case, R^* is a Noetherian W-simple ipri- and ipli-ring, and in the latter case, R^* is a Noetherian, prime, ipri- and ipli-ring.*

PROOF. Apply the results of the remark following Corollary 2.4 to assert that R is an ipri- and ipli-ring and R is either W-simple or is a prime ring. Then Theorems 7.1 and 7.4 of [5, pp. 137-138] give the result.

The authors have been unable to decide whether, in the second case of the last theorem, R^* is necessarily semisimple.

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