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CURVATURE AND CHARACTERISTIC CLASSES OF COMPACT PSEUDO-RIEMANNIAN MANIFOLDS[1]

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Introduction. In the last three decades various authors have studied the relationships between curvatures and certain topological invariants such as characteristic classes of a compact Riemannian manifold. One of the earliest results was the Gauss-Bonnet formula [1], [6], which expresses the Euler-Poincaré characteristic of a compact orientable Riemannian manifold of even dimension n as the integral, over the manifold, of the nth sectional curvature or the Lipschitz-Killing curvature times the volume element of the manifold.

Later, Chern [8] obtained curvature conditions respectively for determining the sign of the Euler-Poincaré characteristic and for the vanishing of the Pontrjagin classes of a compact orientable Riemannian manifold. Recently, Thorpe [11] extended a special case of Chern's conditions by using higher order sectional curvatures, which are weaker invariants of the Riemannian structure than the usual sectional curvature, and Cheung and Hsiung [5] jointly further extended the conditions of both Chern and Thorpe.

On the other hand, Avez [3] and Chern [9] used different methods to show that the Gauss-Bonnet formula is also true up to a sign on a compact orientable pseudo-Riemannian manifold. Very recently, from general remarks on connexions and characteristic homomorphisms of Weil [7, pp. 57-58], Borel [4] elegantly deduced this fact and expressed the Pontrjagin classes of a compact orientable pseudo-Riemannian manifold in terms of the curvature 2-forms.

The purpose of this paper is to give an independent proof of Borel's result on the Pontrjagin classes and to extend the above mentioned joint work of Cheung and Hsiung to a compact orientable pseudo-Riemannian manifold.

§1 contains some fundamental formulas for a pseudo-Riemannian manifold such as the equations of structure, and the formulas for the higher order sectional curvatures and related differential forms. §2 is devoted to expressing the Euler-Poincaré characteristic and the Pontrjagin classes of a compact orientable pseudo-Riemannian manifold in terms of the curvature 2-forms in the sense of de Rham's theorem. In §3, we extend the above mentioned joint results of Cheung and Hsiung

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to the pseudo-Riemannian case by first establishing several lemmas and then deducing the proofs of the two main theorems of this section.

1. **Fundamental formulas.** Let M^n be a compact orientable manifold of dimension n ($\geqq 2$) with a pseudo-Riemannian metric $g_{\alpha\beta}(u^\lambda)$ of signature r, the number of positive eigenvalues of the matrix of the metric, where $u^1, \cdots, u^n$ are local coordinates of an arbitrary point $x \in M^n$; throughout this paper all Greek and Latin indices take the values $1, \cdots, n$ unless stated otherwise. At any point $x \in M^n$ and over a neighborhood U of x we consider the spaces V_x and V_x^* of tangent vectors and covectors respectively, and the family $xe_1 \cdots e_n$ of orthonormal frames and linear differential forms $\omega^1, \cdots, \omega^n$ with respect to an orthonormal basis in V_x and its dual basis in V_x^*; that is,

$$\begin{aligned} \langle e_i, \omega^j\rangle = \delta_{ij} &= 1, \quad \text{if } i = j, \\ &= 0, \quad \text{if } i \neq j. \end{aligned} \tag{1.1}$$

The pseudo-Riemannian metric of M^n is of the form

$$ds^2 = \sum_{i=1}^{r} (\omega^i)^2 - \sum_{i=r+1}^{n} (\omega^i)^2. \tag{1.2}$$

For indices we use $I(p)$ to indicate the ordered set of p integers $i_1, \cdots, i_p$ among $1, \cdots, n$. When more than one set of indices is needed at one time, we shall use other capital letters in addition to I. The equations of structure of the pseudo-Riemannian metric are

$$\begin{aligned} d\omega^i &= \sum_j \omega^j \wedge \omega^i_j, \qquad \omega^i_j + \omega^j_i = 0, \\ d\omega^i_j &= \sum_k \omega^k_j \wedge \omega^i_k + \Omega^i_j, \qquad \Omega^i_j + \Omega^j_i = 0, \end{aligned} \tag{1.3}$$

where the components for the curvature 2-form Ω_{ij} satisfy

$$\begin{aligned} \Omega^i_j &= \Omega_{ij} \qquad (1 \leqq i \leqq r), \\ \Omega^i_j &= -\Omega_{ij} \quad (r < i \leqq n), \\ \Omega_{ij} &= \Omega_{ji} \qquad (1 \leqq i, j \leqq n). \end{aligned} \tag{1.4}$$

Then, for any even $p \leqq n$ and distinct set of integers $i_1, \cdots, i_p$ we define the p-form

$$\Theta_{I(p)} = \frac{1}{p!} \sum_{J(p)} (-1)^{c(J)} \delta^{J(p)}_{I(p)} \Omega_{j_1 j_2} \wedge \cdots \wedge \Omega_{j_{p-1} j_p}, \tag{1.5}$$

where $c(J)$ denotes the number of the curvature 2-forms Ω_{jk} with $j > r$ for each combination $(j_1 j_2, \cdots, j_{p-1} j_p)$, and

$$\begin{aligned}\delta^{J(p)}_{I(p)} &= +1, \quad \text{if } J(p) \text{ is an even permutation of } I(p),\\ &= -1, \quad \text{if } J(p) \text{ is an odd permutation of } I(p),\\ &= 0, \quad \text{otherwise.}\end{aligned} \tag{1.6}$$

In terms of a natural orthonormal basis in local coordinates $u^1, \cdots, u^n$ in the neighborhood U we have

$$\Omega_{ij} = \tfrac{1}{2} R_{ijk\ell}\, \omega^k \wedge \omega^\ell, \tag{1.7}$$

where repeated indices imply summation over their ranges, and $R_{ijk\ell}$ is the Riemann-Christoffel tensor. Thus (1.5) can be written as

$$\Theta_{I(p)} = \frac{1}{2^{p/2}p!} \sum_{J(p),H(p)} \delta^{J(p)}_{I(p)} R^{j_1}{}_{j_2h_1h_2} \cdots R^{j_{p-1}}{}_{j_ph_{p-1}h_p} dV_{H(p)}, \tag{1.8}$$

where

$$dV_{H(p)} = \omega^{h_1} \wedge \cdots \wedge \omega^{h_p} = du^{h_1} \wedge \cdots \wedge du^{h_p} \tag{1.9}$$

is the volume element of the p-dimensional submanifold of M^n with local coordinates $u^{h_1}, \cdots, u^{h_p}$.

Let P be any p-dimensional plane in the tangent space V_x of the manifold M^n at a point x. Then the Lipschitz-Killing curvature at x of the p-dimensional geodesic submanifold of M^n tangent to P at x is called the pth sectional curvature of M^n at x with respect to P, and is given (see, for instance, [2, p. 257]) in terms of any orthonormal basis $e_{i_1} \cdots e_{i_p}$ of P by

$$K_{I(p)}(P) = \frac{(-1)^{p/2}}{2^{p/2}p!} \sum_{J(p)} \delta^{J(p)}_{I(p)} \delta^{H(p)}_{I(p)} R^{j_1}{}_{j_2h_1h_2} \cdots R^{j_{p-1}}{}_{j_ph_{p-1}h_p}. \tag{1.10}$$

2. **Characteristic classes.** Let M^n be a connected manifold of dimension n ($\geqq 2$) endowed with a pseudo-Riemannian metric $g_{\alpha\beta}$ of signature r. Consider any Riemannian metric $h_{\alpha\beta}$ on M^n. In the tangent space M_x of M^n at each point x, $g_{\alpha\beta}$ defines the field T of symmetric linear transformations by means of $h_{\alpha\beta}$. Now, consider the decomposition

$$M_x = \sum_{i;\lambda_i>0} W_i + \sum_{j;\lambda_j<0} W_j, \tag{2.1}$$

where

$$\begin{aligned} W_i &= \{X \in M_x \mid T_x(X) = \lambda_i X\}\\ &= \{X \in M_x \mid g_x(X, Y) = \lambda_i h_x(X, Y), \text{ for all } Y \in M_x\}. \end{aligned} \tag{2.2}$$

Put $r = \sum_{i;\lambda_i>0} \dim W_i$. Since g and therefore T are nonsingular, r is constant and we have

$$n - r = \sum_{j;\,\lambda_j < 0} \dim W_j\,.$$

Suppose that

$$W^r{}_n = \sum_{i;\,\lambda_i > 0} W_i, \qquad W^{n-r}{}_x = \sum_{j;\,\lambda_j < 0} W_j. \tag{2.3}$$

Then the distributions $W^r : x \to W^r{}_x$ and $W^{n-r} : x \to W^{n-r}{}_x$ are differentiable, and this leads us to define

$$g_x | W^r{}_x = a, \qquad -\, g_x | W^{n-r}{}_x = b, \tag{2.4}$$

where the distributions $a : x \to a_x$ and $b : x \to b_x$ are differentiable. Now, define

$$\bar{g} = a + b\,. \tag{2.5}$$

Then $\bar{g}$ is positive definite, and with respect to this $\bar{g}$, g has eigenvalue 1 of multiplicity r and eigenvalue -1 of multiplicity $n-r$, that is, for each $x \in M^n$, $M_x = W^r{}_x + W^{n-r}{}_x$ is the eigenspace-decomposition with respect to $\bar{g}$. By considering the tensor

$$\ell(t) = \bar{g} + tb, \tag{2.6}$$

or

$$\ell(t) = a + (1 + t)b, \tag{2.7}$$

where t is a real parameter, we see that ℓ defines a nonsingular pseudo-Riemannian metric on M^n for $t \neq -1$, which is Riemannian for $t > -1$ and is of signature r for $t < -1$; in particular, $\ell(-2) = g$.

The inverse tensor of (2.6) is given by

$$\ell^{\alpha\beta}(t) = \bar{g}^{\alpha\beta} - \frac{t}{1 + t} b^{\alpha\beta}\,. \tag{2.8}$$

Let $\Gamma^\gamma{}_{\alpha\beta}(t)$ be the Christoffel symbols with respect to $\ell_{\alpha\beta}(t)$. Then $C^\gamma{}_{\alpha\beta}(t) = \Gamma^\gamma{}_{\alpha\beta}(t) - \Gamma^\gamma{}_{\alpha\beta}(0)$ defines a tensor, where $\Gamma^\gamma{}_{\alpha\beta}(0)$ are the Christoffel symbols with respect to $\ell_{\alpha\beta}(0) = \bar{g}_{\alpha\beta}$.

Let $\nabla^{(t)}$ and $\nabla^{(0)}$ be the covariant derivation operators in the Riemannian connexion associated with $\ell_{\alpha\beta}(t)$ and $\bar{g}_{\alpha\beta} = \ell_{\alpha\beta}(0)$, respectively. By using normal coordinates at a point x on M^n with respect to $\bar{g}_{\alpha\beta}$ so that $\Gamma^\gamma{}_{\alpha\beta}(0) = 0$ at the point x, we have

$$\begin{aligned} 0 &= \nabla_\alpha{}^{(t)} \ell_{\beta\sigma}(t) \\ &= \nabla_\alpha{}^{(0)} \ell_{\beta\sigma}(t) - \ell_{\gamma\sigma}(t) C^\gamma{}_{\beta\alpha}(t) - \ell_{\beta\gamma}(t) C^\gamma{}_{\sigma\alpha}(t). \end{aligned} \tag{2.9}$$

The cyclical permutation of the indices α, β, σ in (2.9) yields

$$0 = \nabla_\sigma{}^{(0)}\mathfrak{L}_{\alpha\beta}(t) - \mathfrak{L}_{\gamma\beta}(t)C^\gamma{}_{\alpha\sigma}(t) - \mathfrak{L}_{\alpha\gamma}(t)C^\gamma{}_{\beta\sigma}(t), \tag{2.10}$$

$$0 = \nabla_\beta{}^{(0)}\mathfrak{L}_{\sigma\alpha}(t) - \mathfrak{L}_{\gamma\alpha}(t)C^\gamma{}_{\sigma\beta}(t) - \mathfrak{L}_{\sigma\gamma}(t)C^\gamma{}_{\alpha\beta}(t). \tag{2.11}$$

Subtracting (2.10) from the sum of (2.9) and (2.11) we thus obtain

$$C^\gamma{}_{\alpha\beta}(t) = \tfrac{1}{2}\mathfrak{L}^{\gamma\sigma}(t)(\nabla_\alpha{}^{(0)}\mathfrak{L}_{\sigma\beta}(t) + \nabla_\beta{}^{(0)}\mathfrak{L}_{\sigma\alpha}(t) - \nabla_\sigma{}^{(0)}\mathfrak{L}_{\alpha\beta}(t)). \tag{2.12}$$

Now, for a contravariant vector v^μ on M^n we have

$$\begin{aligned}
\nabla_\alpha{}^{(t)}v^\mu &= \nabla_\alpha{}^{(0)}v^\mu + C^\mu{}_{\alpha\beta}v^\beta,\\
\nabla_\lambda{}^{(t)}\nabla_\alpha{}^{(t)}v^\mu &= \nabla_\lambda{}^{(t)}(\nabla_\alpha{}^{(0)}v^\mu) + \nabla_\lambda{}^{(t)}(C^\mu{}_{\alpha\beta}v^\beta)\\
&= (\nabla_\lambda{}^{(0)}\nabla_\alpha{}^{(0)}v^\mu + C^\mu{}_{\alpha\rho}\,\nabla_\alpha{}^{(0)}v^\rho - C^\rho{}_{\lambda\alpha}\,\nabla_\rho{}^{(0)}v^\mu)\\
&\quad + [(\nabla_\lambda{}^{(0)}C^\mu{}_{\alpha\beta})v^\beta + C^\mu{}_{\alpha\beta}\,\nabla_\lambda{}^{(0)}v^\beta\\
&\qquad + C^\mu{}_{\lambda\rho}C^\rho{}_{\alpha\beta}\,v^\beta - C^\rho{}_{\lambda\alpha}C^\mu{}_{\rho\beta}v^\beta],
\end{aligned}$$

and therefore

$$\begin{aligned}
(\nabla_\lambda{}^{(t)}\nabla_\alpha{}^{(t)} - \nabla_\alpha{}^{(t)}\nabla_\lambda{}^{(t)})v^\mu &= (\nabla_\lambda{}^{(0)}\nabla_\alpha{}^{(0)} - \nabla_\alpha{}^{(0)}\nabla_\lambda{}^{(0)})v^\mu\\
&\quad + [(\nabla_\lambda{}^{(0)}C^\mu{}_{\alpha\beta}) - (\nabla_\alpha{}^{(0)}C^\mu{}_{\lambda\beta})\\
&\qquad + C^\mu{}_{\lambda\rho}C^\rho{}_{\alpha\beta} - C^\mu{}_{\alpha\rho}C^\rho{}_{\lambda\beta}]\,v^\beta.
\end{aligned}$$

Thus the Ricci identity gives

$$\begin{aligned}
R^\mu{}_{\beta\alpha\lambda}(t) &= R^\mu{}_{\beta\alpha\lambda}(0) + \nabla_\lambda{}^{(0)}C^\mu{}_{\alpha\beta}(t) - \nabla_\alpha{}^{(0)}C^\mu{}_{\lambda\beta}(t)\\
&\quad + C^\mu{}_{\lambda\rho}(t)C^\rho{}_{\alpha\beta}(t) - C^\mu{}_{\alpha\rho}(t)C^\rho{}_{\lambda\beta}(t),
\end{aligned} \tag{2.13}$$

where $R^\mu{}_{\beta\alpha\lambda}(t)$ and $R^\mu{}_{\beta\alpha\lambda}(0)$ are the Riemann-Christoffel tensors with respect to $\Gamma^\gamma{}_{\alpha\beta}(t)$ and $\Gamma^\gamma{}_{\alpha\beta}(0)$ respectively.

Let $dV(t)$ and $dV(0)$ be the volume elements of M^n associated with $\mathfrak{L}_{\alpha\beta}(t)$ and $\mathfrak{L}_{\alpha\beta}(0)$ respectively. By using equation (2.7) and orthonormal local coordinates $u^1, \cdots, u^n$, we readily obtain

$$\begin{aligned}
dV(0) &= |\det(\mathfrak{L}_{\alpha\beta}(0))|^{1/2}\,du^1 \wedge \cdots \wedge du^n\\
&= |\det(a)\det(b)|^{1/2}du^1 \wedge \cdots \wedge du^n,
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
dV(t) &= |\det(\mathfrak{L}_{\alpha\beta}(t))|^{1/2}\,du^1 \wedge \cdots \wedge du^n\\
&= |1 + t|^{(n-r)/2}dV(0).
\end{aligned} \tag{2.15}$$

More generally, the volume elements $dV_{H(4k)}(t)$ and $dV_{H(4k)}(0)$, respectively, associated with $\mathfrak{L}_{h_ih_j}(t)$ and $\mathfrak{L}_{h_ih_j}(0)$, $i, j = 1, \cdots, 4k$, of the $4k$-dimensional submanifold of M^n, $4k \leqq n$, with the local coordinates $(u^{h_1}, \cdots, u^{h_{4k}})$ are related by

$$\begin{aligned} dV_{H(4k)}(t) &= |\det(\mathfrak{L}_{h_i h_j}(t))|^{1/2} du^{h_1} \wedge \cdots \wedge du^{h_{4k}} \\ &= |1+t|^{s/2} dV_{H(4k)}(0), \end{aligned} \tag{2.16}$$

where s h's are greater than r, and the remaining h's are less than or equal to r.

THEOREM 2.1. *Let M^n be a compact orientable manifold of even dimension n with a pseudo-Riemannian metric of signature r. Then the Euler-Poincaré characteristic $\chi(M^n)$ of the manifold M^n is given by*

$$\chi(M^n) = \frac{(-1)^{[r/2]} n!}{2^n \pi^{n/2} (n/2)!} \int_{M^n} \Theta_{1\cdots n}, \tag{2.17}$$

where $\Theta_{1\cdots n}$ is given by (1.5).

Theorem 2.1 is due to Allendorfer and Weil [**1**] and Chern [**6**] for the Riemannian case, and due to Avez [**3**] and Chern [**9**] for the pseudo-Riemannian case. Our proof is essentially the same as that of Avez.

PROOF. By means of (1.8) we have

$$\begin{aligned} \Theta_{1\cdots n} = \frac{1}{2^{n/2} n!} \sum_{J(n),H(n)} & \delta^{J(n)}_{1\cdots n} R^{j_1}{}_{j_2 h_1 h_2} \\ & \cdots R^{j_{n-1}}{}_{j_n h_{n-1} h_n} dV_{H(n)}. \end{aligned} \tag{2.18}$$

Now let $\Theta_{1\cdots n}(t)$ be the form $\Theta_{1\cdots n}$ associated with $\mathfrak{L}_{\alpha\beta}(t)$ given by (2.6). Then

$$\begin{aligned} \Theta_{1\cdots n}(t) &= \frac{1}{n!} \sum_{J(n)} (-1)^{c(J)} \delta^{J(n)}_{1\cdots n} \Omega_{j_1 j_2}(t) \cdots \Omega_{j_{n-1} j_n}(t) \\ &= \frac{1}{2^{n/2} n!} \sum_{J(n),H(n)} \delta^{J(n)}_{1\cdots n} R^{j_1}{}_{j_2 h_1 h_2}(t) \\ &\qquad \cdots R^{j_{n-1}}{}_{j_n h_{n-1} h_n}(t) dV_{H(n)}(t), \end{aligned}$$

and from (2.13), (2.12), (2.8), (2.7), (2.15) it follows that

$$\Theta_{1\cdots n}(t) = |1+t|^{(n-r)/2} \frac{P(t)}{(1+t)^n}, \tag{2.19}$$

where $P(t)$ is a polynomial in t. Thus

$$\begin{aligned} f(t) &\equiv \frac{(-1)^{n/2} n!}{2^n \pi^{n/2} (n/2)!} \int_{M^n} \Theta_{1\cdots n}(t) \\ &= |1+t|^{(n-r)/2} \frac{Q(t)}{(1+t)^n}, \end{aligned} \tag{2.20}$$

where $Q(t)$ is a polynomial in t.

Now, suppose that r is even. Since $\mathfrak{L}_{\alpha\beta}(t)$ for $t > -1$ defines a Riemannian metric on M^n, it is known [1], [6] that $f(t)$ for $t > -1$ is the Euler-Poincaré characteristic $\chi(M^n)$ of M^n. Thus from (2.20) we have

$$Q(t) = (1 + t)^{(n+r)/2}\chi(M^n) \tag{2.21}$$

for $t > -1$ and therefore for all t. Substitution of (2.21) in (2.20) thus gives

$$f(t) = \chi(M^n)\frac{|1 + t|^{(n-r)/2}}{(1 + t)^{(n-r)/2}}, \quad \text{for all } t. \tag{2.22}$$

For $t < -1$ we have $1 + t < 0$ so that

$$|1 + t|^{(n-r)/2} = (-1)^{(n-r)/2}(1 + t)^{(n-r)/2},$$

and hence

$$f(t) = (-1)^{(n-r)/2}\chi(M^n),$$

which proves our formula (2.17) for even r.

Finally, suppose that r is odd such that $r = 2r' + 1$. Then we have, for $t > -1$,

$$\chi(M^n) = \frac{Q(t)}{(1 + t)^{r'+n/2}} \cdot \frac{1}{(1 + t)^{1/2}}, \tag{2.23}$$

which implies that $Q(t) = 0$ for $t > -1$ and therefore for all t. Hence $\chi(M^n) = 0 = f(t)$ for all t, which shows that our formula (2.17) is also true for odd r, and completes the proof of Theorem 2.1.

Theorem 2.2. *Let M^n be a compact orientable n-dimensional manifold with a pseudo-Reimannian metric of signature r. Then the differential form*

$$\Psi_{4k} \equiv \frac{[(2k)!]^2}{(2^k k!)^2(2\pi)^{2k}} \sum_{I(2k)} \Theta_{I(2k)} \wedge \Theta_{I(2k)} \tag{2.24}$$

defines the kth Pontrjagin class P_k of the manifold M^n in the sense of de Rham's theorem.

Theorem 2.2 is due to Chern [8] for the Riemannian case and due to Borel [4] for the pseudo-Riemannian case. However the proof given below is different from that of Borel.

Proof. By means of (1.8) we can rewrite (2.24) as

$$\begin{aligned}\Psi_{4k} = \frac{1}{(2^{2k}k!)^2(2\pi)^{2k}} \sum_{I,H,J,\bar{J}} \delta^{J(2k)}_{I(2k)} R^{j_1}{}_{j_2h_1h_2} \\ \cdots R^{j_{2k-1}}{}_{j_{2k}h_{2k-1}h_{2k}} \\ \cdot \delta^{\bar{J}(2k)}_{I(2k)} R^{\bar{j}_1}{}_{\bar{j}_2h_{2k+1}h_{2k+2}} \cdots R^{\bar{j}_{2k-1}}{}_{\bar{j}_{2k}h_{4k-1}h_{4k}} dV_{H(4k)}.\end{aligned} \tag{2.25}$$

Now let $\Psi_{4k}(t)$ be the form Ψ_{4k} associated with $\mathfrak{L}_{\alpha\beta}(t)$. Then

$$\begin{aligned}\Psi_{4k}(t) &= \frac{[(2k)!]^2}{(2^kk!)(2\pi)^{2k}} \sum_{I(2k)} \Theta_{I(2k)}(t) \wedge \Theta_{I(2k)}(t) \\ &= \frac{1}{(2^{2k}k!)^2(2\pi)^{2k}} \sum_{I,H,J,\bar{J}} \delta^{J(2k)}_{I(2k)} R^{j_1}{}_{j_2h_1h_2}(t) \\ &\qquad \cdots R^{j_{2k-1}}{}_{j_{2k}h_{2k-1}h_{2k}}(t) \\ &\qquad \cdot \delta^{\bar{J}(2k)}_{I(2k)} R^{\bar{j}_1}{}_{\bar{j}_2h_{2k+1}h_{2k+2}}(t) \\ &\qquad \cdots R^{\bar{j}_{2k-1}}{}_{\bar{j}_{2k}h_{4k-1}h_{4k}}(t) dV_{H(4k)}(t),\end{aligned} \tag{2.26}$$

and from (2.13), (2.12), (2.8), (2.7), (2.16), (2.25) it follows easily that $\Psi_{4k}(t)$ can be expressed in the following general form:

$$\psi_{4k}(t) = (1+t)^{-4k} \sum_{i=0}^{n-r} |1+t|^{1/2} Q_i(t), \tag{2.27}$$

or

$$\Psi_{4k}(t) = (1+t)^{-4k}(E(t) + |1+t|^{1/2}F(t)), \tag{2.28}$$

where $Q_i(t)$, $E(t)$ and $F(t)$ are polynomials in t. It is known [8] that $\Psi_{4k}(t)$ for $t > -1$ defines the kth Pontrjagin class P_k of M^n with real coefficients, so that

$$P_k = \Psi_{4k}(t) + B_{4k} \tag{2.29}$$

for $t > -1$, where B_{4k} is the group of the exact $4k$-forms of the manifold M^n, which is obviously independent of t. Substitution of (2.28) in (2.29) thus gives

$$(1+t)^{4k}P_k = E(t) + |1+t|^{1/2}F(t) + (1+t)^{4k}B_{4k}, \quad \text{for } t > -1. \tag{2.30}$$

Since $E(t)$ and $F(t)$ are polynomials in t, (2.30) implies that $F(t) = 0$ for $t > -1$ and therefore for all t. Hence from (2.28), (2.30) we see that (2.29) holds for all t, so that $\Psi_{4k}(t)$ defines the kth Pontrjagin class P_k for all t, and in particular the case where $t = -2$ gives our Theorem 2.2.

3. **Relationships between curvatures and characteristic classes.** Let M^n be a connected manifold of dimension n ($\geqq 2$) with a pseudo-

Riemannian metric $g_{\alpha\beta}$ of signature r, p be an even positive integer $\leqq n$, and $a_{ih} = a_{hi}$ $(i, h = 1, \cdots, n)$ be given smooth real-valued functions on M^n. Denote

$$A_{I(p),H(p)} = \det(a_{i_\alpha h_\beta}) \qquad (\alpha, \beta = 1, \cdots, p), \tag{3.1}$$

where the rows and columns of $\det(a_{i_\alpha h_\beta})$ are arranged in the natural order of α and β, respectively. Consider the following curvature condition at $x \in M^n$:

$$\sum_{J(p)} \delta^{J(p)}_{I(p)} R^{j_1}{}_{j_2 h_1 h_2} \cdots R^{j_{p-1}}{}_{j_p h_{p-1} h_p} = 2^{p/2} \kappa_p A_{I(p),H(p)}, \tag{3.2}$$

for all $I(p)$, $H(p) \in (1, \cdots, n)$, where κ_p is a smooth real-valued function on M^n at x. In the Riemannian case, this condition was first used by Chern [8] for $p = 2$, by Thorpe [**11**] for a general p but $a_{ih} = \delta_{ih}$ (in this case (3.2) implies that the Lipschitz-Killing curvature $K_{I(p)}(P)$ is constant at x for every P and all $I(p)$), and then jointly by Cheung and Hsiung [5] for general p and a_{ih}. Furthermore, it is easy to see another geometric significance of the condition (3.2) for the Riemannian case, namely, if M^n is a hypersurface of a Euclidean space, then the symmetric tensor a_{ih} may be taken to be the second fundamental form of M^n.

From (3.2), (1.8), (1.9), (1.10) follows immediately

LEMMA 3.1. *For a fixed set of indices $I(p)$, condition* (3.2) *implies*

$$\Theta_{I(p)} = \frac{1}{p!} \kappa_p A_{I(p),H(p)} \omega^{H(p)}, \tag{3.3}$$

$$K_{I(p)} = (-1)^{p/2} \kappa_p A_{I(p),H(p)}, \tag{3.4}$$

and also equation (3.3) *implies condition* (3.2), *where*

$$\omega^{H(p)} = \omega^{h_1} \wedge \cdots \wedge \omega^{h_p}. \tag{3.5}$$

In particular, when $a_{ij} = \delta_{ij}$, then

$$A_{I(p),H(p)} = \Delta_{I(p),H(p)}, \tag{3.6}$$

where

$$\Delta_{I(p),H(p)} = \det(\delta_{i_\alpha h_\beta}) \qquad (\alpha, \beta = 1, \cdots, p). \tag{3.7}$$

Therefore (3.3), (3.4) are reduced to

$$\Theta_{I(p)} = \kappa_p \omega^{I(p)}, \tag{3.8}$$

$$K_{I(p)} = (-1)^{p/2} \kappa_p. \tag{3.9}$$

Thus, from (3.9) we have

LEMMA 3.2. *Condition* (3.2) *with* $a_{ij} = \delta_{ij}$ *implies that the pth sectional curvature* $K_{I(p)}(P)$ *at the point* x *of the manifold* M^n *is constant, that is, independent of the p-dimensional plane* P *at the point* x.

On the other hand, from (1.8), (1.9), (1.10) it follows immediately that (3.8) implies (3.9). The converse is also true (can be proved in exactly the same way as given by Thorpe [**11**] for the Riemannian case), so that we can state, altogether,

LEMMA 3.3. *Equations* (3.7) *and* (3.8) *are equivalent.*

For the converse of Lemma 3.2, we notice that [**10**, p. 238]

$$\delta^{H(p)}_{I(p)} = \Delta_{I(p),H(p)}, \tag{3.10}$$

so that (3.8) can be written as

$$\begin{aligned} \Theta_{I(p)} &= \frac{1}{p!}\kappa_p \sum_{H(p)} \delta^{H(p)}_{I(p)} \omega^{H(p)} \\ &= \frac{1}{p!}\kappa_p \Delta_{I(p),H(p)} \omega^{H(p)}. \end{aligned} \tag{3.11}$$

A comparison of (3.11) with (1.8), (1.9) yields immediately condition (3.2) with $a_{ij} = \delta_{ij}$. By combining this result with Lemma 3.2 and using Lemma 3.3 we hence obtain

LEMMA 3.4. *The pth sectional curvature* $K_{I(p)}$ *of the manifold* M^n *at a point* x *is constant if and only if condition* (3.2) *with* $a_{ij} = \delta_{ij}$ *holds.*

LEMMA 3.5. *On a pseudo-Riemannian manifold* M^n *of dimension* n, *if condition* (3.2) *holds for some even* p *and* q *with* $p + q \leqq n$, *then*

$$\Theta_{I(p+q)} = \frac{1}{(p+q)!}\kappa_p\kappa_q A_{I(p+q),H(p+q)}\omega^{H(p+q)}, \tag{3.12}$$

so that (3.2) *also holds for* $p + q$ *with* $\kappa_{p+q} = \kappa_p\kappa_q$.

PROOF OF LEMMA 3.5. Let the set $I(p + q)$ have distinct elements, and $(I_1(p), I_2(q))$ be a partition of $I(p + q)$, where $I_1(p) = (i_{11}, \cdots, i_{1p})$ and $I_2(q) = (i_{21}, \cdots, i_{2q})$. Then, from (1.5),

$$\begin{aligned} \Theta_{I(p+q)} = \frac{1}{(p+q)!} \sum_{(I_1,I_2)} & (-1)^{c(I_1)+c(I_2)} \delta^{I_1(p)I_2(q)}_{I(p+q)} \Omega_{i_{11}i_{12}} \\ & \wedge \cdots \wedge \Omega_{i_{1,p-1}i_{1p}} \wedge \Omega_{i_{21}i_{22}} \\ & \wedge \cdots \wedge \Omega_{i_{2,q-1}i_{2q}}, \end{aligned} \tag{3.13}$$

where $c(I_1)$, $c(I_2)$ denote the numbers of the curvature 2-forms Ω_{jk} with $j > r$ for the combinations $(i_{11}i_{12}, \cdots, i_{1,p-1}i_{1p})$ and $(i_{21}i_{22}, \cdots, i_{2,q-1}i_{2q})$, respectively, and $\sum_{(I_1,I_2)}$ denotes the summation over all such partitions of $I(p+q)$ into $(I_1(p), I_2(q))$. For a fixed $I(p+q)$, let $J(p+q)$ be an even permutation of $I(p+q)$ such that $j_1, \cdots, j_p \in I_1(p)$, and $j_{p+1}, \cdots, j_{p+q} \in I_2(q)$. By denoting $J'(q) = (j_{p+1}, \cdots, j_{p+q})$, using (1.5) and noticing that altogether there are $\binom{p+q}{p}$ such partitions of $I(p+q)$ into $(I_1(p), I_2(q))$, from (3.13) we then obtain

$$\begin{aligned}\Theta_{I(p+q)} &= \frac{1}{(p+q)!} \sum_{(I_1,I_2)} (-1)^{c(I_1)}\delta^{I_1(p)}_{J(p)}\Omega_{i_{11}i_{12}} \wedge \cdots \wedge \Omega_{i_{1,p-1}i_{1p}} \\ &\qquad\qquad \wedge(-1)^{c(I_2)}\delta^{I_2(q)}_{J'(q)}\Omega_{i_{21}i_{22}} \wedge \cdots \wedge \Omega_{i_{2,q-1}i_{2q}} \qquad (3.14)\\ &= \Theta_{J(p)} \wedge \Theta_{J'(q)} .\end{aligned}$$

On the other hand, by the Laplace theorem we can expand the determinant $A_{J(p+q),H(p+q)}$ according to the first p rows. By using this expansion it is easily seen that all $\binom{p+q}{p}$ terms of $A_{J(p+q),H(p+q)}\omega^{H(p+q)}$ are equal so that we have

$$\begin{aligned}&A_{J(p+q),H(p+q)}\omega^{H(p+q)} \\ &\qquad = \frac{(p+q)!}{p!q!} A_{J(p),H(p)}\omega^{H(p)} \wedge A_{J'(q),H'(q)}\omega^{H'(q)} ,\end{aligned} \qquad (3.15)$$

where $H'(q) = (h_{p+1}, \cdots, h_{p+q})$. Substituting (3.3) in (3.14) and using (3.15) we arrive at (3.12), and an application of Lemma 3.1 hence completes the proof of Lemma 3.5.

By repeatedly applying Lemma 3.5 we can easily obtain

COROLLARY 3.5.1. *Let $p_1, \cdots, p_k$ be even positive integers, and $(m_1, \cdots, m_k)$ a k-tuple of nonnegative integers such that $q = \sum_{i=1}^k m_i p_i \leqq n$. On a pseudo-Riemannian manifold M^n of dimension n, if condition* (3.2) *holds for $p_1, \cdots, p_k$, then it also holds for q with*

$$\kappa_q = (\kappa_{p_1})^{m_1} \cdots (\kappa_{p_k})^{m_k} .$$

COROLLARY 3.5.2. *On a pseudo-Riemannian manifold M^n of even dimension n, if condition* (3.2) *holds for some positive even integer p dividing n, then*

$$\Theta_{1\cdots n} = (\kappa_p)^{n/p} \det(a_{ij})\omega^1 \wedge \cdots \wedge \omega^n , \qquad (3.16)$$

where $\omega^1 \wedge \cdots \wedge \omega^n$ is the volume element of M^n.

Combination of Theorem 2.1 with Corollary 3.5.2 gives immediately

THEOREM 3.1. *On a compact orientable pseudo-Riemannian manifold M^n of even dimension n with a pseudo-Riemannian metric of signature r, if condition* (3.2) *holds at every point x for a positive even integer p dividing n, and $(-1)^{[r/2]}(\kappa_p)^{n/p}\det(a_{ij})$ keeps a constant sign, then this sign is the sign of the Euler-Poincaré characteristic $\chi(M^n)$ of M^n. Moreover, under this hypothesis, $\chi(M^n) = 0$ only when $(\kappa_p)^{n/p}\det(a_{ij})$ vanishes identically.*

For the Riemannian case, this theorem was obtained by Chern [8] for $p = 2$, by Thorpe [11] for $a_{ij} = \delta_{ij}$, and jointly by Cheung and Hsiung [5] for a general p.

For studying Pontrjagin classes we need

LEMMA 3.6. *Equation* (3.3) *can be written in the following form*:

$$\Theta_{I(p)} = \kappa_p \tilde{\omega}^{I(p)}, \tag{3.17}$$

where $\tilde{\omega}^{i_\alpha}$ are linear forms defined by

$$\tilde{\omega}^{i_\alpha} = a_{i_\alpha h}\omega^h \qquad (\alpha = 1, \cdots, p). \tag{3.18}$$

PROOF. Let p_1, p_2 be any two positive integers such that $p_1 + p_2 = p$. By using p_1, p_2 for p, q, from (3.15) we then have

$$\begin{aligned} &A_{I(p),H(p)}\omega^{H(p)} \\ &\quad = \frac{p!}{p_1!p_2!} A_{I(p_1),H(p_1)}\omega^{H(p_1)} \wedge A_{I'(p_2),H'(p_2)}\omega^{H'(p_2)} \end{aligned} \tag{3.19}$$

where

$$I'(p_2) = (i_{p_1+1}, \cdots, i_p), \qquad H'(p_2) = (h_{p_1+1}, \cdots, h_p). \tag{3.20}$$

Repeatedly applying the same argument as above to both factors on the right-hand side of (3.19) yields immediately (3.17).

Now we are in a position to prove the following theorem concerning the general curvature conditions for the vanishing of the Pontrjagin classes.

THEOREM 3.2. *On a compact orientable pseudo-Riemannian manifold M^n of dimension n, if condition* (3.2) *holds at every point x for a positive even integer $p \leqq n$, then the kth Pontrjagin class $P_k(M^n)$ of M^n is zero for all $k \geqq p/2$.*

For the Riemannian case, this theorem is due to Chern [8] for $p = 2$, to Thorpe [11] for $a_{ij} = \delta_{ij}$, and jointly due to Cheung and Hsiung [5] for a general p.

PROOF. First, we consider the case $p \leqq 2k < 2p$. Let $(I_1(p), I_2(2k-p))$ be a partition of a fixed $I(2k)$, and $J(2k)$ an even permutation of $I(2k)$ such that $j_1, \cdots, j_p \in I_1(p)$ and $j_{p+1}, \cdots, j_{2k} \in I_2(2k-p)$. By denoting $J'(2k-p) = (j_{p+1}, \cdots, j_{2k})$, from (3.14) we have

$$\Theta_{I(2k)} = \sum_{(J,J')} \Theta_{J(p)} \wedge \Theta_{J'(2k-p)}, \tag{3.21}$$

where $\sum_{(J,J')}$ denotes the summation over all such partitions of $I(2k)$ into $(J(p), J'(2k-p))$. By using condition (3.2) for p and Lemmas 3.1 and 3.6, equation (3.21) is reduced to

$$\Theta_{I(2k)} = \kappa_p \sum_{(J,J')} \tilde{\omega}^{J(p)} \wedge \Theta_{J'(2k-p)}, \tag{3.22}$$

where $\tilde{\omega}^{j_\alpha}$ are linear forms defined by (3.18), so that $\Theta_{I(2k)} \wedge \Theta_{I(2k)}$ is a sum, each term of which contains an exterior factor

$$\tilde{\omega}^{J(p)} \wedge \tilde{\omega}^{\bar{J}(p)}, \tag{3.23}$$

where all the superscripts $j, \bar{j} \in I(2k)$. Since $2k < 2p$, at least two of the j's and $\bar{j}$'s in (3.23) must be equal, so that each of such factors (3.23) is zero. Thus $\Theta_{I(2k)} \wedge \Theta_{I(2k)} = 0$ for all $I(2k)$. By Theorem 2.2 we hence obtain $P_k(M^n) = 0$ for all k with $p/2 \leqq k < p$.

Finally, since condition (3.2) is assumed to hold for p, by Corollary 3.5.1 it also holds for $2^i p$ $(i = 1, 2, \cdots)$. Applying the same arguments as above we therefore have

$$P_k(M^n) = 0 \qquad (2^{i-1}p \leqq k < 2^i p; \quad i = 1, 2, \cdots).$$

Hence $P_k(M^n) = 0$ for all $k \geqq p/2$, and the theorem is proved.

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