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A TWO STEP NEWTON TYPE ITERATION FOR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS IN HILBERT SCALES

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ABSTRACT. In this paper regularized solutions of ill-posed Hammerstein type operator equation $KF(x) = y$, where $K : X \to Y$ is a bounded linear operator with non-closed range and $F : X \to X$ is non-linear, are obtained by a two step Newton type iterative method in Hilbert scales, where the available data is y^δ in place of actual data y with $\|y - y^\delta\| \leq \delta$. We require only a weaker assumption $\|F'(x_0)x\| \sim \|x\|_{-b}$ compared to the usual assumption $\|F'(\widehat{x})x\| \sim \|x\|_{-b}$, where $\widehat{x}$ is the actual solution of the problem, which is assumed to exist, and x_0 is the initial approximation. Two cases, viz-a-viz, (i) when $F'(x_0)$ is boundedly invertible and (ii) $F'(x_0)$ is non-invertible but F is monotone operator, are considered. We derive error bounds under certain general source conditions by choosing the regularization parameter by an a priori manner as well as by using a modified form of the adaptive scheme proposed by Perverzev and Schock [**14**].

1. Introduction. In this paper we present an iterative method which combines Tikhonov regularization with the modified Newton's method in Hilbert scales, for approximately solving the operator equation

$$KF(x) = y. \tag{1.1}$$

Here $K : X \to Y$ is a bounded linear operator with its range $R(K)$ not closed in X, $F : D(F) \subseteq X \to X$ is a nonlinear operator, where $D(F)$

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is the domain of F, and X and Y are Hilbert spaces. We shall use the notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the inner product and the corresponding norm in the Hilbert spaces. The equation (1.1) is, in general, ill-posed, in the sense that a unique solution that depends continuously on the data does not exist and hence need to be regularized to get stable approximate solution. Further, we assume that only approximate data y^δ satisfying

$$\|y - y^\delta\| \le \delta \tag{1.2}$$

is available.

A typical example of equation (1.1) is the classical Hammerstein operator equation (see [**9**, page 430])

$$\int_0^1 k(s,t)h(s,x(s))\, ds = y(t), \quad t \in [0,1], \tag{1.3}$$

where $k(\cdot,\cdot)$ is a non-degenerate kernel which is square integrable, that is,

$$\int_0^1 \int_0^1 |k(s,t)|^2 ds\, dt < \infty$$

and $h : [0,1] \times [0,1] \to \mathbf{R}$ is a suitable function. Then equation (1.3) takes the form (1.1) with $K : L^2[0,1] \to L^2[0,1]$ defined by

$$Ku(t) = \int_0^1 k(s,t)u(s)\, ds, \quad t \in [0,1]$$

and $F : L^2[0,1] \to L^2[0,1]$ is the nonlinear 'superposition operator' defined by

$$F(x)(s) = h(s, x(s)), \quad s \in [0,1].$$

Note that, due to the nonlinearity of F, the solution of (1.1), even when it exists, need not be unique. Therefore, one applies some selection criteria. Following the definition in [**5, 6**], a solution $\widehat{x} \in D(F)$ of (1.1) is called an x_0-minimum norm solution (x_0-MNS) of (1.1), if

$$\|F(\widehat{x}) - F(x_0)\| = \min\{\|F(x) - F(x_0)\| : KF(x) = y,\ x \in D(F)\}. \tag{1.4}$$

In the following, we always assume the existence of an x_0-MNS for exact data y, i.e.,

$$KF(\widehat{x}) = y.$$

One may observe that equation (1.1) is equivalent to

$$K[F(x) - F(x_0)] = y - KF(x_0)$$

for a given x_0, so that the solution $\widehat{x}$ of (1.1) is obtained by solving

$$Kz = y - KF(x_0) \tag{1.5}$$

for z and then solving the non-linear equation

$$F(x) = z + F(x_0). \tag{1.6}$$

An advantage of approximately solving (1.5) and (1.6) to obtain an approximate solution for (1.1) is that one can use any regularization method for linear ill-posed equations for solving (1.5) and any method for solving non-linear equation (1.6). This advantage was exploited by the second author and his collaborators (see [**3–5, 8**]). In [**21**], Shobha et al. considered a two step iterative method for solving (1.1) in a Hilbert space.

In order to improve the rate of convergence many authors have considered the Hilbert scale variant of the regularization methods for solving ill-posed operator equations, for example, [**1, 10–13, 15–17**]. In this paper, we consider the Hilbert scale variant of the method considered in [**21**]. For analyzing (1.1) in the setting of Hilbert scales, we consider a Hilbert scale $\{X_t\}_{t\in\mathbf{R}}$ generated by a strictly positive operator $L : D(L) \to X$ with $D(L)$ dense in X satisfying

$$\|Lx\| \geq \|x\|, \quad x \in D(L).$$

Recall ([**15, 22**]) that the space X_t is the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle x_1, x_2\rangle_t = \langle L^{t/2}x_1, L^{t/2}x_2\rangle$$

i.e.,

$$\|x\|_t = \|L^{t/2}x\|, \quad t \in \mathbf{R}.$$

As in [**21**], we consider two cases of the operator F in (1.1).

Case I. $F'(x_0)^{-1}$ exists and is bounded. Thus, the ill-posedness of (1.1) is essentially due to the non-closedness of the range of the linear operator K, as $F'(x_0)^{-1}$ is a bounded linear operator implies equation (1.6) is well-posed (see [**18**, page 26]). In this case, we consider the sequence $(x^\delta_{n,\alpha,s})$ defined iteratively by

$$y^\delta_{n,\alpha,s} = x^\delta_{n,\alpha,s} - F'(x_0)^{-1}[F(x^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s}] \tag{1.7}$$

and

$$x^\delta_{n+1,\alpha,s} = y^\delta_{n,\alpha,s} - F'(x_0)^{-1}[F(y^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s}] \tag{1.8}$$

where $x^\delta_{0,\alpha,s} := x_0$ is the initial approximation for the solution $\widehat{x}$ of (1.1),

$$z^\delta_{\alpha,s} := F(x_0) + (L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*(y^\delta - KF(x_0)) \tag{1.9}$$

and α is the regularization parameter to be chosen appropriately from the finite set $D_N := \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}$ depending on the inexact data y^δ and the error level δ satisfying (1.2). We use the adaptive parameter selection procedure suggested by Pereverzev and Schock [**14**] for the selection of regularization parameters.

Example 1.1. Consider the non-linear Hammerstein operator equation

$$\int_\Omega k(s,t) f_\lambda(t, x(t))\, dt = y(s) \tag{1.10}$$

with $\Omega \subset \mathbf{R}$ a bounded domain, $k(s,t) : \Sigma_{n=0}^\infty (n+1)^{-2} u_n(s) u_n(t)$, where $u_n(s) = \sqrt{2}\cos(2n\pi s)$ and $f_\lambda : \Omega \times \mathbf{R} \to \mathbf{R}$ is defined as $f_\lambda(s,x) = b(s)g(x(s)) + \lambda c(s)$, where $0 \neq c \in L^p$, $0 < b \in L^{p/(p-q)}$ for some $q \in (2,p)$ (cf. [**2**]) and g is a differentiable function such that $g'(x_0(t)) > \kappa > 0$, for all $t \in \Omega$.

Note that (1.10) is of the form $KF(x) = y$, where $K : L^2(\Omega) \to L^2(\Omega)$ is given by

$$K(x(s)) = \int_\Omega k(s,t) x(t)\, dt$$

and $F : L^2(\Omega) \to L^2(\Omega)$ is given by

$$F(x(s)) = f_\lambda(s, x(s)).$$

Observe that

$$F'(x)h(s) = f_{\lambda x}(s, x(s))h(s) = b(s)g'(x(s))h(s),$$

for $x, y \in B_R(x_0)$ and $h \in L^2(\Omega)$. Further note that, since $g'(x_0(t)) > \kappa > 0$, for all $t \in \Omega$, $F'(x_0)^{-1} = 1/(b(s)g'(x_0))$ exists and is a bounded operator.

Case II. $F'(x_0)$ is non-invertible, but F is a monotone operator. Recall that ([**20, 23**]) the operator F is said to be a monotone operator if $\langle F(x)-F(y), x-y\rangle \geq 0$, for all $x, y \in D(F)$. In this case, we consider the sequence $(\widetilde{x}^\delta_{n,\alpha,s})$ defined iteratively by

$$\begin{aligned}\widetilde{y}^\delta_{n,\alpha,s} = \widetilde{x}^\delta_{n,\alpha,s} - \left(F'(x_0) + \frac{\alpha}{c}L^{s/2}\right)^{-1} \\ \times \left[F(\widetilde{x}^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s} + \frac{\alpha}{c}L^{s/2}(\widetilde{x}^\delta_{n,\alpha,s} - x_0)\right]\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}\widetilde{x}^\delta_{n+1,\alpha} = \widetilde{y}^\delta_{n,\alpha,s} - \left(F'(x_0) + \frac{\alpha}{c}L^{s/2}\right)^{-1} \\ \times \left[F(\widetilde{y}^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s} + \frac{\alpha}{c}L^{s/2}(\widetilde{y}^\delta_{n,\alpha,s} - x_0)\right]\end{aligned} \tag{1.12}$$

where $\widetilde{x}^\delta_{0,\alpha,s} := x_0$, with x_0, $z^\delta_{\alpha,s}$ and α as in Case I and $0 < c \leq \alpha$.

Example 1.2. In this example, we consider the operator $KF : L^2(0,1) \to L^2(0,1)$ where $K : L^2(0,1) \to L^2(0,1)$ is defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)\,ds$$

and $F : D(F) \subseteq L^2(0,1) \to L^2(0,1)$ is defined by

$$F(u) := \int_0^1 k(t,s)u^3(s)\,ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s & 0 \le s \le t \le 1 \\ (1-s)t & 0 \le t \le s \le 1. \end{cases}$$

Then, for all $x(t), y(t) : x(t) > y(t)$, (see [**20**, subsection 4.3])

$$\langle F(x) - F(y), x - y\rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s)\, ds \right](x-y)(t)\, dt \ge 0.$$

Thus, the operator F is monotone.

The paper is organized as follows. In Section 2, we give preliminaries and the adaptive scheme for choosing the regularization parameter α for Tikhonov regularization of (1.5) in the setting of Hilbert scales. The proposed method and the error estimates for the case when $F'(x_0)$ is invertible are given in subsection 3.1, and the case when $F'(x_0)$ is non-invertible but F is monotone is given in subsection 3.2. We conclude the paper in Section 4.

2. Preliminaries. We assume that the ill-posed nature of the operator K is related to the Hilbert scale $\{X_t\}_{t\in\mathbf{R}}$ according to the relation

$$c_1\|x\|_{-a} \le \|Kx\| \le c_2\|x\|_{-a}, \quad x \in X,$$

for some reals a, c_1 and c_2.

Observe that from the relation $\langle Kx, y\rangle = \langle x, K^*y\rangle = \langle x, L^{-s}K^*y\rangle_s$ for all $x \in X$ and $y \in Y$, we conclude that $L^{-s}K^* : Y \to X$ is the adjoint of the operator K in X. Consequently, $L^{-s}K^*K : X \to X$ is self-adjoint. Further we note that

$$(A_s^*A_s + \alpha I)^{-1}L^{s/2} = L^{s/2}(L^{-s}K^*K + \alpha I)^{-1}$$

where $A_s = KL^{-s/2}$.

One of the crucial results for proving the results in this paper is the following proposition, where f and g are defined by

$$f(t) = \min\{c_1^t, c_2^t\}, \qquad g(t) = \max\{c_1^t, c_2^t\}, \quad t \in \mathbf{R},\ |t| \le 1.$$

Proposition 2.1 (see [**24**, Proposition 2.1]). *For $s \ge 0$ and $|\nu| \le 1$,*

$$f(\nu)\|x\|_{-\nu(s+a)} \le \|(A_s^*A_s)^{\nu/2}x\| \le g(\nu)\|x\|_{-\nu(s+a)}, \quad x \in H.$$

We make use of the relation

$$\|(A_s+\alpha I)^{-1}A_s^p\| \leq \alpha^{p-1}, \quad p>0,\ 0<p\leq 1, \tag{2.13}$$

which follows from the spectral properties of the positive self-adjoint operator A_s, $s>0$.

The following assumption on source condition is based on a source function φ and a property of the source function φ. We will be using this assumption to obtain an error estimate for $\|z_{\alpha,s}^{\delta} - F(\widehat{x})\|$.

Assumption 2.2. *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, \|A_s^*A_s\|] \to (0,\infty)$ *such that the following conditions hold*:

- $\lim_{\lambda\to 0}\varphi(\lambda)=0$,
- $$\sup_{\lambda>0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq \varphi(\alpha) \quad \text{for all } \lambda \in (0, \|A_s^*A_s\|]$$

and

- there exists $v\in X$ with $\|v\| \leq \overline{E}$, $\overline{E}>0$ such that
$$(A_s^*A_s)^{s/2(s+a)}L^{s/2}(F(\hat{x})-F(x_0)) = \varphi(A_s^*A_s)v.$$

Remark 2.3. Note that if $F(\widehat{x})-F(x_0)\in X_t$, i.e., $\|F(\widehat{x})-F(x_0)\|_t \leq E$, for some $0<t\leq 2s+a$, then the above assumption is satisfied. This can be seen as follows.

$$\begin{aligned}&(A_s^*A_s)^{s/2(s+a)}L^{s/2}(F(\widehat{x})-F(x_0))\\ &= (A_s^*A_s)^{t/2(s+a)}(A_s^*A_s)^{(s-t)/(2s+2a)}L^{s/2}(F(\widehat{x})-F(x_0)) = \varphi(A_s^*A_s)v,\end{aligned}$$

where $\varphi(\lambda)=\lambda^{t/2(s+a)}$ and $v=(A_s^*A_s)^{(s-t)/(2s+2a)}L^{s/2}(F(\widehat{x})-F(x_0))$.

Further note that

$$\begin{aligned}\|v\| &\leq g\left(\frac{s-t}{s+a}\right)\|L^{s/2}(F(\widehat{x})-F(x_0))\|_{t-s}\\ &\leq g\left(\frac{s-t}{s+a}\right)\|(F(\widehat{x})-F(x_0))\|_t \leq \overline{E},\end{aligned}$$

where $\overline{E}=g(s-t/s+a)E$.

Theorem 2.4. *Suppose that Assumption* 2.2 *holds, and let* $z_{\alpha,s} := z^0_{\alpha,s}$. *Then*

$$\text{(2.14)} \qquad 1.\ \|z^\delta_{\alpha,s} - z_{\alpha,s}\| \leq \psi(s)\alpha^{-a/[2(s+a)]}\delta,$$

$$\text{(2.15)} \qquad 2.\ \|F(\widehat{x}) - z_{\alpha,s}\| \leq \phi(s)\varphi(\alpha),$$

$$\text{(2.16)} \qquad 3.\ \|F(x_0) - z_{\alpha,s}\| \leq \psi_1(s)\|F(\widehat{x}) - F(x_0)\|,$$

where $\psi(s) = 1/f(s/(s+a))$, $\phi(s) = \overline{E}/f(s/(s+a))$ and $\psi_1(s) = g(s/(s+a))/f(s/(s+a))$.

Proof. Note that

$$\begin{aligned} \|z^\delta_{\alpha,s} - z_{\alpha,s}\| &= \|(L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*(y^\delta - y)\| \\ &= \|L^{-s/2}(A_s^*A_s + \alpha I)^{-1}A_s^*(y^\delta - y)\| \end{aligned}$$

now by taking $\nu = s/(s+a)$ and $x = (A_s^*A_s + \alpha I)^{-1}A_s^*(y^\delta - y)$ in Proposition 2.1, we have

$$\begin{aligned} \text{(2.17)} \quad & \|z^\delta_{\alpha,s} - z_{\alpha,s}\| \\ &\leq \frac{1}{f(s/(s+a))}\|(A_s^*A_s)^{s/(2(s+a))}(A_s^*A_s + \alpha I)^{-1}A_s^*(y^\delta - y)\| \\ &= \frac{1}{f(s/(s+a))}\|(A_s^*A_s + \alpha I)^{-1}(A_s^*A_s)^{(2s+a)/(2(s+a))}(y^\delta - y)\| \\ &\leq \frac{1}{f(s/(s+a))}\|(A_s^*A_s + \alpha I)^{-1}(A_s^*A_s)^{(2s+a)/(2(s+a))}\|\delta. \end{aligned}$$

We note that relation (2.13) with $p = (2s+a)/(2(s+a))$ gives

$$\text{(2.18)} \qquad \|(A_s^*A_s + \alpha I)^{-1}(A_s^*A_s)^{(2s+a)/(2(s+a))}\| \leq \alpha^{-a/(2(s+a))}.$$

Now (2.14) follows from (2.17) and (2.18). Further, we observe that

$$\begin{aligned} \|z_{\alpha,s} - F(\widehat{x})\| &= \|[(L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*K - I](F(\widehat{x}) - F(x_0))\| \\ &= \|\alpha L^{-s/2}(A_s^*A_s + \alpha I)^{-1}L^{s/2}(F(\widehat{x}) - F(x_0))\| \\ &\leq \frac{1}{f(s/2(s+a))}\|\alpha(A_s^*A_s)^{s/(2(s+a))}(A_s^*A_s + \alpha I)^{-1} \\ &\qquad L^{s/2}(F(\widehat{x}) - F(x_0))\| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{f(s/2(s+a))}\|\alpha(A_s^*A_s+\alpha I)^{-1}(A_s^*A_s)^{s/(2(s+a))}\\
&\qquad\qquad\qquad\qquad L^{s/2}(F(\widehat{x})-F(x_0))\|\\
&= \frac{1}{f(s/2(s+a))}\|(A_s^*A_s+\alpha I)^{-1}\varphi(A_s^*A_s)v\|.
\end{aligned}$$

So, by Assumption 2.2, we have that

$$\|z_{\alpha,s}-F(\widehat{x})\| \leq \frac{1}{f(s/(s+a))}\varphi(\alpha)\overline{E}.$$

Again,

$$\begin{aligned}
\|z_{\alpha,s}-F(x_0)\| &= \|(L^{-s}K^*K+\alpha I)^{-1}L^{-s}K^*K(F(\widehat{x})-F(x_0))\|\\
&= \|L^{-s/2}(A_s^*A_s+\alpha I)^{-1}A_s^*A_sL^{s/2}(F(\widehat{x})-F(x_0))\|\\
&\leq \frac{1}{f(s/(s+a))}\|(A_s^*A_s)^{s/(2(s+a))}(A_s^*A_s+\alpha I)^{-1}\\
&\quad (A_s^*A_s)L^{s/2}(F(\widehat{x})-F(x_0))\|\\
&= \frac{1}{f(s/(s+a))}\|(A_s^*A_s+\alpha I)^{-1}(A_s^*A_s)\|\\
&\quad \|(A_s^*A_s)^{s/(2(s+a))}L^{s/2}(F(\widehat{x})-F(x_0))\|\\
&\leq \frac{g(s/(s+a))}{f(s/(s+a))}\|L^{s/2}(F(\widehat{x})-F(x_0))\|_{-s}\\
&\leq \psi_1(s)\|F(\widehat{x})-F(x_0)\|.
\end{aligned}$$

This completes the proof of Theorem 2.4.

2.1. Error bounds and parameter choice in Hilbert scales. Let $C_s = \max\{\phi(s),\psi(s)\}$. Then, by (2.14), (2.15) and triangle inequality, we have

$$\|F(\widehat{x})-z_{\alpha,s}^{\delta}\| \leq C_s(\varphi(\alpha)+\alpha^{-a/(2(s+a))}\delta). \tag{2.19}$$

The error estimate $\varphi(\alpha)+\alpha^{-a/(2(s+a))}\delta$ in (2.19) attains minimum for the choice $\alpha := \alpha(\delta,s,a)$ which satisfies $\varphi(\alpha)=\alpha^{-a/(2(s+a))}\delta$. Clearly $\alpha(\delta,s,a)=\varphi^{-1}(\psi_{s,a}^{-1}(\delta))$, where

$$\psi_{s,a}(\lambda)=\lambda[\varphi^{-1}(\lambda)]^{a/(2(s+a))}, \quad 0<\lambda\leq\|A_s\|^2, \tag{2.20}$$

and in this case

$$\|F(\widehat{x}) - z^{\delta}_{\alpha,s}\| \le 2C_s\psi^{-1}_{s,a}(\delta),$$

which has at least optimal order with respect to δ, s and a (cf. [**14**]).

2.2. Adaptive scheme and stopping rule. In this paper we consider the adaptive scheme suggested by Pereverzev and Schock in [**14**], modified suitably, for choosing the parameter α which does not involve even the regularization method in an explicit manner.

Let $i \in \{0, 1, 2, \dots, N\}$ and $\alpha_i = \mu^i\alpha_0$ where $\mu = \eta^{2(1+s/a)}$, $\eta > 1$, and $\alpha_0 = \delta^{2(1+s/a)}$. Let

$$l := \max\{i : \varphi(\alpha_i) \le \alpha_i^{-a/(2(s+a))}\delta\} < N \tag{2.21}$$

and

$$k := \max\{i : \|z^{\delta}_{\alpha_{i,s}} - z^{\delta}_{\alpha_{j,s}}\| \le 4\alpha_j^{-a/(2(s+a))}\delta,\ j = 0, 1, 2, \dots, i-1\}. \tag{2.22}$$

Analogous to the proof of Theorem 4.3 in [**5**], we have the following theorem.

Theorem 2.5 (cf. [**5**, Theorem 4.3]). *Let l be as in* (2.21), *k as in* (2.22), *$\psi_{s,a}$ as in* (2.20) *and $z^{\delta}_{\alpha_k,s}$ as in* (1.9) *with $\alpha = \alpha_k$. Then $l \le k$ and*

$$\|F(\widehat{x}) - z^{\delta}_{\alpha_k,s}\| \le C_s\left(2 + \frac{4\eta}{\eta - 1}\right)\eta\psi^{-1}_{s,a}(\delta),$$

where C_s is as in (2.19).

3. The method and convergence analysis. We will be using the following assumption to prove Theorems 3.3 and 3.11.

Assumption 3.1 (cf. [**20**, Assumption 3 (A3)]). *There exists a constant $k_0 \ge 0$ such that, for every $x, u \in D(F)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v)$, $\|\Phi(x, u, v)\| \le k_0\|v\|\|x - u\|$.*

3.1.

Case I: $F(x_0)^{-1}$ is a bounded linear operator. Consider the two step iterative method defined as in (1.7) and (1.8) with α_k in place of α.

We assume that F possesses a uniformly bounded Fréchet derivative for all $x \in D(F)$, i.e., $\|F'(x)\| \leq M$, for all x in a neighborhood of x_0 and for some $M > 0$, and $\|F'(x_0)^{-1}\| := \beta$, $\beta > 0$. Let

$$e^{\delta}_{n,\alpha_k,s} := \|y^{\delta}_{n,\alpha_k,s} - x^{\delta}_{n,\alpha_k,s}\|, \quad \text{for all } n \geq 0, \tag{3.23}$$

and let $\delta_0 < 1/(4k_0\beta\psi(s))\alpha_0^{a/(2(s+a))}$ and $\|\widehat{x} - x_0\| \leq \rho$, with

$$\rho < \frac{1}{\psi_1(s)M}\left[\frac{1}{4k_0\beta} - \psi(s)\alpha_0^{-a/(2(s+a))}\delta_0\right]$$

and

$$\gamma_\rho := \beta[\psi_1(s)M\rho + \psi(s)\alpha^{-a/(2(s+a))}\delta].$$

Further, let

$$r_1 = \frac{1 - \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}$$

and

$$r_2 = \min\left\{\frac{1}{k_0}, \frac{1 + \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}\right\}.$$

For $r \in (r_1, r_2)$, let

$$q = k_0 r. \tag{3.24}$$

Then $q < 1$.

Lemma 3.2. *Let $e^{\delta}_{0,\alpha_k,s}$ be as in* (3.23). *Then $e^{\delta}_{0,\alpha_k,s} \leq \gamma_\rho$.*

Proof. Observe that

$$\begin{aligned} e^{\delta}_{0,\alpha_k,s} &= \|y^{\delta}_{0,\alpha_k,s} - x^{\delta}_{0,\alpha_k,s}\| \\ &= \|F'(x_0)^{-1}(F(x_0) - z^{\delta}_{\alpha_k,s})\| \\ &\leq \beta\|F(x_0) - z^{\delta}_{\alpha_k,s}\| \\ &\leq \beta[\|F(x_0) - z_{\alpha_k,s}\| \\ &\quad + \|z_{\alpha_k,s} - z^{\delta}_{\alpha_k,s}\|]. \end{aligned} \tag{3.25}$$

Now, using (2.14) and (2.16) in (3.25), one can see that

$$\begin{aligned} e^{\delta}_{0,\alpha_k,s} &\leq \beta[\psi_1(s)\|F(\widehat{x})-F(x_0)\|+\psi(s)\alpha^{-a/(2(s+a))}\delta] \\ &\leq \beta[\psi_1(s)M\rho+\psi(s)\alpha^{-a/(2(s+a))}\delta]=\gamma_\rho. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.3. *Let $e^{\delta}_{n,\alpha_k,s}$ and q be as in equation* (3.23) *and* (3.24)*, respectively, $y^{\delta}_{n,\alpha_k,s}$ and $x^{\delta}_{n,\alpha_k,s}$ defined as in* (1.7) *and* (1.8)*, respectively, with $\alpha=\alpha_k$ and $\delta\in(0,\delta_0]$. Then, under Assumption* 3.1 *and Lemma* 3.2*, $x^{\delta}_{n,\alpha_k,s}$, $y^{\delta}_{n,\alpha_k,s}\in B_r(x_0)$ and the following estimates hold for all $n\geq 0$:*

(a) $\|x^{\delta}_{n,\alpha_k,s}-y^{\delta}_{n-1,\alpha_k,s}\|\leq q\|y^{\delta}_{n-1,\alpha_k,s}-x^{\delta}_{n-1,\alpha_k,s}\|$;

(b) $\|x^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n-1,\alpha_k,s}\|\leq (1+q)\|y^{\delta}_{n-1,\alpha_k,s}-x^{\delta}_{n-1,\alpha_k,s}\|$;

(c) $\|y^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n,\alpha_k,s}\|\leq q^2\|y^{\delta}_{n-1,\alpha_k,s}-x^{\delta}_{n-1,\alpha_k,s}\|$;

(d) $e^{\delta}_{n,\alpha_k,s}\leq q^{2n}\gamma_\rho$, *for all $n\geq 0$.*

Proof. Suppose $x^{\delta}_{n,\alpha_k,s}$, $y^{\delta}_{n,\alpha_k,s}\in B_r(x_0)$. Then

$$\begin{aligned} &x^{\delta}_{n+1,\alpha_k,s}-y^{\delta}_{n,\alpha_k,s} \\ &\qquad= y^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n,\alpha_k,s}+F'(x_0)^{-1}[F(x^{\delta}_{n,\alpha_k,s})-(F(y^{\delta}_{n,\alpha_k,s})] \\ &\qquad= F'(x_0)^{-1}[F'(x_0)(y^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n,\alpha_k,s}) \\ &\qquad\quad -(F(y^{\delta}_{n,\alpha_k,s})-F(x^{\delta}_{n,\alpha_k,s}))], \end{aligned}$$

and hence, by Assumption 3.1, we have

$$\begin{aligned} \|x^{\delta}_{n+1,\alpha_k,s}-y^{\delta}_{n,\alpha_k,s}\| &= \Big\|F'(x_0)^{-1}\int_0^1 F'(x_0)\Phi(x_0,x^{\delta}_{n,\alpha_k,s} \\ &\qquad +t(y^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n,\alpha_k,s}),y^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n,\alpha_k,s})\,dt\Big\| \\ &\leq k_0 r\|y^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n,\alpha_k,s}\|. \end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality:

$$\|x^{\delta}_{n,\alpha_k,s}-x^{\delta}_{n-1,\alpha_k,s}\|\leq\|x^{\delta}_{n,\alpha_k,s}-y^{\delta}_{n-1,\alpha_k,s}\|+\|y^{\delta}_{n-1,\alpha_k,s}-x^{\delta}_{n-1,\alpha_k,s}\|.$$

Again, (c) follows from (a), Assumption 3.1 and the following expression.

$$e^{\delta}_{n,\alpha_k,s} = \left\| F'(x_0)^{-1} \int_0^1 [F'(x_0) - F'(x^{\delta}_{n,\alpha_k,s} + t(x^{\delta}_{n,\alpha_k,s} - y^{\delta}_{n-1,\alpha_k,s}))] \times (x^{\delta}_{n,\alpha_k,s} - y^{\delta}_{n-1,\alpha_k,s})\, dt \right\|$$

and (d) follows from (c). Now we show that $x^{\delta}_{n,\alpha_k,s}$, $y^{\delta}_{n,\alpha_k,s} \in B_r(x_0)$ by induction.

Note that, by (b) and Lemma 3.2,

$$\begin{aligned} \|x^{\delta}_{1,\alpha_k,s} - x_0\| &\leq (1+q)e^{\delta}_{0,\alpha_k,s} \leq \frac{e^{\delta}_{0,\alpha_k,s}}{1-q} \\ &\leq \frac{\gamma_\rho}{1-q} < r \end{aligned} \tag{3.26}$$

i.e., $x^{\delta}_{1,\alpha_k,s} \in B_r(x_0)$. Again, note that, by (3.26) and (c), we have

$$\begin{aligned} \|y^{\delta}_{1,\alpha_k,s} - x_0\| &\leq \|y^{\delta}_{1,\alpha_k,s} - x^{\delta}_{1,\alpha_k,s}\| + \|x^{\delta}_{1,\alpha_k,s} - x_0\| \\ &\leq (1+q+q^2)e^{\delta}_{0,\alpha_k,s} \\ &\leq \frac{\gamma_\rho}{1-q} < r, \end{aligned}$$

i.e., $y^{\delta}_{1,\alpha_k,s} \in B_r(x_0)$. Further, let us assume that $x^{\delta}_{m,\alpha_k,s}$, $y^{\delta}_{m,\alpha_k,s} \in B_r(x_0)$, for some $m \geq 0$. Then, using (b), (3.26) and Lemma 3.2, we have

$$\begin{aligned} \|x^{\delta}_{m+1,\alpha_k,s} - x_0\| &\leq \|x^{\delta}_{m+1,\alpha_k} - x^{\delta}_{m,\alpha_k,s}\| + \cdots + \|x^{h,\delta}_{1,\alpha_k,s} - x_0\| \\ &\leq (q+1)(q^{2m} + q^{2(m-1)} + \cdots + 1)e^{\delta}_{0,\alpha_k,s} \\ &\leq (q+1)\frac{1-(q^{2m+1})}{1-q^2}e^{\delta}_{0,\alpha_k,s} \\ &\leq \frac{\gamma_\rho}{1-q} < r, \end{aligned}$$

i.e., $x^{\delta}_{m+1,\alpha_k,s} \in B_r(x_0)$ and

$$\begin{aligned} \|y^{\delta}_{m+1,\alpha_k,s} - x_0\| &\leq \|y^{\delta}_{m+1,\alpha_k,s} - x^{\delta}_{m+1,\alpha_k,s}\| \\ &\quad + \|x^{\delta}_{m+1,\alpha_k,s} - x_0\| \\ &\leq (q^{2(m+1)} + \cdots + q^3 + q^2 + q + 1)e^{\delta}_{0,\alpha_k,s} \\ &\leq \frac{\gamma_\rho}{1-q} < r, \end{aligned}$$

i.e., $y^\delta_{m+1,\alpha_k,s} \in B_r(x_0)$. Thus, by induction, $x^\delta_{n,\alpha_k,s}$, $y^\delta_{n,\alpha_k,s} \in B_r(x_0)$, for all $n \geq 0$. This completes the proof of Theorem 3.3. $\square$

Theorem 3.4. *Let $x^\delta_{n,\alpha_k,s}$ and $y^\delta_{n,\alpha_k,s}$ be as in* (1.7) *and* (1.8), *respectively, with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$, and let the assumptions of Theorem* 3.3 *hold. Then $(x^\delta_{n,\alpha_k,s})$ is a Cauchy sequence in $B_r(x_0)$ and converges, say, to $x^\delta_{\alpha_k,s} \in \overline{B_r(x_0)}$. Further, $F(x^\delta_{\alpha_k,s}) = z^\delta_{\alpha_k,s}$ and*

$$\|x^\delta_{n,\alpha_k,s} - x^\delta_{\alpha_k,s}\| \leq Cq^{2n},$$

where $C = (\gamma_\rho)/(1-q)$.

Proof. Using relations (b) and (c) of Theorem 3.3, we obtain

$$\begin{aligned}\|x^\delta_{n+m,\alpha_k,s} - x^\delta_{n,\alpha_k,s}\| &\leq \sum_{i=0}^{i=m-1} \|x^\delta_{n+i+1,\alpha_k,s} - x^\delta_{n+i,\alpha_k,s}\| \\ &\leq \sum_{i=0}^{i=m-1} (1+q)e^\delta_{n+i,\alpha_k,s} \\ &\leq \sum_{i=0}^{i=m-1} (1+q)q^{2(n+i)}e^\delta_{0,\alpha_k,s} \\ &= (1+q)q^{2n}e^\delta_{0,\alpha_k,s} + (1+q)q^{2(n+1)}e^\delta_{0,\alpha_k,s} \\ &\quad + \cdots + (1+q)q^{2(n+m)}e^\delta_{0,\alpha_k,s} \\ &\leq (1+q)q^{2n}(1+q^2+q^{2(2)}+\cdots+q^{2m})e^\delta_{0,\alpha_k,s} \\ &\leq q^{2n}\left[\frac{1-(q^2)^{m+1}}{1-q}\right]\gamma_\rho \leq Cq^{2n}.\end{aligned}$$

Thus, $x^\delta_{n,\alpha_k,s}$ is a Cauchy sequence in $B_r(x_0)$, and hence it converges, say, to $x^\delta_{\alpha_k,s} \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}\|F(x^\delta_{n,\alpha_k,s}) - z^\delta_{\alpha_k,s}\| &= \|F'(x_0)(x^\delta_{n,\alpha_k,s} - y^\delta_{n,\alpha_k,s})\| \\ &\leq \|F'(x_0)\|\|x^\delta_{n,\alpha_k,s} - y^\delta_{n,\alpha_k,s}\| \\ &\leq Me^\delta_{n,\alpha_k,s} \leq Mq^{2n}\gamma_\rho.\end{aligned} \tag{3.27}$$

Now, by letting $n \to \infty$ in (3.27) we obtain $F(x^\delta_{\alpha_k,s}) = z^\delta_{\alpha_k,s}$. This completes the proof. $\square$

Hereafter, we assume that $\|\widehat{x} - x_0\| < \rho \leq r$.

Theorem 3.5. *Suppose that Assumption* 3.1 *holds. Then*

$$\|\widehat{x} - x^{\delta}_{\alpha_k,s}\| \leq \frac{\beta}{1-q}\|F(\widehat{x}) - z^{\delta}_{\alpha_k,s}\|.$$

Proof. Note that $q < 1$ and, by Assumption 3.1, we have

$$\begin{aligned}
\|\widehat{x} - x^{\delta}_{\alpha_k,s}\| &\leq \|\widehat{x} - x^{\delta}_{\alpha_k,s} + F'(x_0)^{-1} \\
&\quad \times [F(x^{\delta}_{\alpha_k,s}) - F(\widehat{x}) + F(\widehat{x}) - z^{\delta}_{\alpha_k,s}]\| \\
&\leq \|F'(x_0)^{-1}[F'(x_0)(\widehat{x} - x^{\delta}_{\alpha_k,s}) + F(x^{\delta}_{\alpha_k,s}) - F(\widehat{x})]\| \\
&\quad + \|F'(x_0)^{-1}(F(\widehat{x}) - z^{\delta}_{\alpha_k,s})\| \\
&\leq k_0\|x_0 - \widehat{x} - t(x^{\delta}_{\alpha_k,s} - \hat{x})\|\|\widehat{x} - x^{\delta}_{\alpha_k,s}\| + \beta\|F(\widehat{x}) - z^{\delta}_{\alpha_k,s}\| \\
&\leq k_0 r\|\widehat{x} - x^{\delta}_{\alpha_k,s}\| + \beta\|F(\widehat{x}) - z^{\delta}_{\alpha_k,s}\|.
\end{aligned}$$

Hence, the theorem is proved. □

Theorem 3.4 and Theorem 3.5 together imply the following theorem.

Theorem 3.6. *Let $x^{\delta}_{n,\alpha_k,s}$ be as in* (1.7) *with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$, and let the assumptions in Theorems* 3.4 *and* 3.5 *hold. Then*

$$\|\widehat{x} - x^{\delta}_{n,\alpha_k,s}\| \leq Cq^{2n} + \frac{\beta}{1-q}\|F(\hat{x}) - z^{\delta}_{\alpha_k,s}\|,$$

where C is as in Theorem 3.4.

Theorem 3.7. *Let $x^{\delta}_{n,\alpha_k,s}$ be as in* (1.7) *with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$, and let the assumptions in Theorems* 2.5 *and* 3.6 *hold. Let*

$$n_k := \min\{n : q^{2n} \leq \alpha_k^{-a/2(s+a)}\delta\}.$$

Then

$$\|\widehat{x} - x^{\delta}_{n_k,\alpha_k,s}\| = O(\psi^{-1}_{s,a}(\delta)).$$

3.2.

Case II: $F'(x_0)$ is non-invertible and F is a monotone operator. In this section, let X be a real Hilbert space. We consider the two step iterative method defined as in (1.11) and (1.12) with α_k in place of α for approximating the zero $\widetilde{x}^\delta_{\alpha_k,s}$ of the equation,

$$F(x) + \frac{\alpha_k}{c} L^{s/2}(x - x_0) = z^\delta_{\alpha_k,s}, \tag{3.28}$$

and then we show that $\widetilde{x}^\delta_{\alpha_k,s}$ is an approximation to the solution $\widehat{x}$ of (1.1).

Let $F'(x_0) \in L(X)$ be a bounded positive self-adjoint operator on X and $B_s := L^{-s/4}F'(x_0)L^{-s/4}$. Usually, for the analysis of regularization methods in Hilbert scales, an assumption of the form (cf. [**1, 13**])

$$\|F'(\widehat{x})x\| \sim \|x\|_{-b}, \quad x \in X, \tag{3.29}$$

on the degree of ill-posedness is used. In this paper, instead of (3.29), we require only a weaker assumption:

$$d_1\|x\|_{-b} \le \|F'(x_0)x\| \le d_2\|x\|_{-b}, \quad x \in D(F), \tag{3.30}$$

for some reals b, d_1 and d_2.

Note that (3.30) is simpler than (3.29). Now we define f_1 and g_1 by

$$f_1(t) = \min\{d_1^t, d_2^t\}, \qquad g_1(t) = \max\{d_1^t, d_2^t\}, \quad t \in \mathbf{R}, \ |t| \le 1.$$

One of the crucial steps for proving the results in this paper is the following proposition.

Proposition 3.8 (see [**7**], Proposition 3.1). *For $s > 0$ and $|\nu| \le 1$,*

$$f_1(\nu/2)\|x\|_{(-\nu(s+b))/2} \le \|B_s^{\nu/2}x\| \le g_1(\nu/2)\|x\|_{(-\nu(s+b))/2}, \quad x \in H.$$

Let

$$\psi_2(s) := \frac{g_1(-s/(2(s+b)))}{f_1(s/(2(s+b)))}, \qquad \overline{\psi_2(s)} := \frac{g_1(s/(2(s+b)))}{f_1(s/(2(s+b)))},$$

and let

$$\widetilde{e}^{\delta}_{n,\alpha_k} := \|\widetilde{y}^{\delta}_{n,\alpha_k} - \widetilde{x}^{\delta}_{n,\alpha_k}\|, \quad \text{for all } n \geq 0. \tag{3.31}$$

Let $\delta_0 < 1/[4k_0\psi(s)\psi_2(s)\overline{\psi_2(s)}]\alpha_0^{a/[2(s+a)]}$ and $\|\widehat{x} - x_0\| \leq \rho$, with

$$\rho < \frac{1}{M\psi_1(s)}\left[\frac{1}{4k_0\psi_2(s)\overline{\psi_2(s)}} - \psi(s)\alpha_0^{-a/[2(s+a)]}\delta_0\right]$$

and

$$\widetilde{\gamma}_\rho := \psi_2(s)\Big[\psi_1(s)M\rho + \psi(s)\alpha_0^{-a/[2(s+a)]}\delta_0\Big].$$

Further, let

$$\widetilde{r}_1 = \frac{1 - \sqrt{1 - 4k_0\overline{\psi_2(s)}\widetilde{\gamma}_\rho}}{2\overline{\psi_2(s)}k_0}$$

and

$$\widetilde{r}_2 = \min\left\{\frac{1}{k_0}, \frac{1 + \sqrt{1 - 4k_0\overline{\psi_2(s)}\widetilde{\gamma}_\rho}}{2\overline{\psi_2(s)}k_0}\right\}.$$

For $\widetilde{r} \in (\widetilde{r}_1, \widetilde{r}_2)$, let

$$\widetilde{q} = \overline{\psi_2(s)}k_0\widetilde{r}. \tag{3.32}$$

Then $\widetilde{q} < 1$.

Lemma 3.9. *Let $\widetilde{e}^{\delta}_{0,\alpha_k,s}$ be as in* (3.31)*, and let Proposition* 3.8 *hold. Then $\widetilde{e}^{\delta}_{0,\alpha_k,s} < \widetilde{\gamma}_\rho$.*

Proof. Observe that

(3.33)
$$\begin{aligned}\widetilde{e}^{\delta}_{0,\alpha_k,s} &= \|\widetilde{y}^{\delta}_{0,\alpha_k,s} - \widetilde{x}^{\delta}_{0,\alpha_k,s}\| \\ &= \left\|\left(F'(x_0) + \frac{\alpha_k}{c}L^{s/2}\right)^{-1}(F(x_0) - z^{\delta}_{\alpha_k,s})\right\|\end{aligned}$$

$$
\begin{aligned}
&\le \left\| L^{-s/4}\left(L^{-s/4}F'(x_0)L^{-s/4}+\frac{\alpha_k}{c}I\right)^{-1} L^{-s/4} \right.\\
&\qquad\qquad\qquad\qquad \left. \times (F(x_0)-z^{\delta}_{\alpha_k,s})\right\|\\
&\le \frac{1}{f_1(s/(2(s+b)))}\left\| B_s^{s/(2(s+b))}\left(B_s+\frac{\alpha_k}{c}I\right)^{-1} L^{-s/4}\right.\\
&\qquad\qquad\qquad\qquad \left. \times (F(x_0)-z^{\delta}_{\alpha_k,s})\right\|\\
&\le \frac{1}{f_1(s/(2(s+b)))}\left\| \left(B_s+\frac{\alpha_k}{c}I\right)^{-1} B_s^{s/(s+b)}B_s^{-s/(2(s+b))}\right.\\
&\qquad\qquad\qquad\qquad \left. \times L^{-s/4}(F(x_0)-z^{\delta}_{\alpha_k,s})\right\|\\
&\le \frac{g_1(-s/(2(s+b)))}{f_1(s/(2(s+b)))}\left(\frac{\alpha_k}{c}\right)^{-b/(s+b)} \|F(x_0)-z^{\delta}_{\alpha_k,s}\|\\
&\le \psi_2(s)[\|F(x_0)-z_{\alpha_k,s}\|+\|z_{\alpha_k,s}-z^{\delta}_{\alpha_k,s}\|].
\end{aligned}
$$

Now, using (2.14) and (2.16) in (3.33), one can see that

$$
\begin{aligned}
\widetilde{e}^{\delta}_{0,\alpha_k,s} &\le \psi_2(s)[\psi_1(s)\|F(\widehat{x})-F(x_0)\|+\psi(s)\alpha^{-a/(2(s+a))}\delta]\\
&\le \psi_2(s)[\psi_1(s)M\rho+\psi(s)\alpha_0^{-a/2(s+a)}\delta_0]=\widetilde{\gamma}_\rho. \qquad \square
\end{aligned}
$$

Lemma 3.10. *Let Proposition* 3.8 *hold. Then, for all* $h \in X$*,*

$$
\left\| \left(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\right)^{-1} F'(x_0)h\right\| \le \overline{\psi_2(s)}\|h\|.
$$

Proof. Observe that, by Proposition 3.8,

$$
\begin{aligned}
&\left\| \left(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\right)^{-1} F'(x_0)h\right\|\\
&\qquad = \left\| L^{-s/4}\left(L^{-s/4}F'(x_0)L^{-s/4}+\frac{\alpha_k}{c}I\right)^{-1} L^{-s/4}\right.\\
&\qquad\qquad\qquad\qquad \left. \times F'(x_0)L^{-s/4}L^{s/4}h\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{f_1(s/(2(s+b)))}\left\|B_s^{s/(2(s+b))}\left(B_s+\frac{\alpha_k}{c}I\right)^{-1}B_sL^{s/4}h\right\|\\
&\leq \frac{1}{f_1(s/(2(s+b)))}\left\|\left(B_s+\frac{\alpha_k}{c}I\right)^{-1}B_s\right\|\|B_s^{s/(2(s+b))}L^{s/4}h\|\\
&\leq \frac{g_1(s/(2(s+b)))}{f_1(s/(2(s+b)))}\|L^{s/4}h\|_{-s/2}\\
&\leq \frac{g_1(s/(2(s+b)))}{f_1(s/(2(s+b)))}\|h\|.
\end{aligned}
$$

This completes the proof of Lemma 3.10. $\square$

Theorem 3.11. *Let* $\widetilde{e}^{\delta}_{n,\alpha_k,s}$ *and* $\widetilde{q}$ *be as in equations* (3.31) *and* (3.32)*, respectively,* $\widetilde{y}^{\delta}_{n,\alpha_k,s}$ *and* $\widetilde{x}^{\delta}_{n,\alpha_k,s}$ *as defined in* (1.11) *and* (1.12)*, respectively, with* $\alpha=\alpha_k$ *and* $\delta\in(0,\delta_0]$*. Then, under Assumption* 3.1 *and Lemma* 3.9*,* $\widetilde{x}^{\delta}_{n,\alpha_k,s}$*,* $\widetilde{y}^{\delta}_{n,\alpha_k,s}\in B_{\tilde{r}}(x_0)$*, and the following estimates hold for all* $n\geq 0$:

(a) $\|\widetilde{x}^{\delta}_{n,\alpha_k,s}-\widetilde{y}^{\delta}_{n-1,\alpha_k,s}\|\leq\widetilde{q}\|\widetilde{y}^{\delta}_{n-1,\alpha_k,s}-\widetilde{x}^{\delta}_{n-1,\alpha_k,s}\|$;

(b) $\|\widetilde{x}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n-1,\alpha_k,s}\|\leq(1+\widetilde{q})\|\widetilde{y}^{\delta}_{n-1,\alpha_k,s}-\widetilde{x}^{\delta}_{n-1,\alpha_k,s}\|$;

(c) $\|\widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s}\|\leq\widetilde{q}^2\|\widetilde{y}^{\delta}_{n-1,\alpha_k,s}-\widetilde{x}^{\delta}_{n-1,\alpha_k,s}\|$;

(d) $\widetilde{e}^{\delta}_{n,\alpha_k,s}\leq\widetilde{q}^{2n}\widetilde{\gamma}_{\rho}$, *for all* $n\geq 0$.

Proof. If $\widetilde{x}^{\delta}_{n,\alpha_k,s}$, $y^{\delta}_{n,\alpha_k,s}\in B_{\tilde{r}}(x_0)$, then by Assumption 3.1,

$$
\begin{aligned}
\widetilde{x}^{\delta}_{n+1,\alpha_k,s}-\widetilde{y}^{\delta}_{n,\alpha_k,s}&=\left(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\right)^{-1}\\
&\quad\times[F'(x_0)(\widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s})\\
&\qquad\qquad-(F(\widetilde{y}^{\delta}_{n,\alpha_k,s})-F(\widetilde{x}^{\delta}_{n,\alpha_k,s}))]\\
&=\left(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\right)^{-1}\int_0^1[F'(x_0)-F'(\widetilde{x}^{\delta}_{n,\alpha_k,s}\\
&\quad+t(\widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s}))](\widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s})\,dt\\
&=\left(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\right)^{-1}F'(x_0)\int_0^1\Phi(x_0,\widetilde{x}^{\delta}_{n,\alpha_k,s}\\
&\quad+t(\widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s}),\widetilde{y}^{\delta}_{n,\alpha_k}-\widetilde{x}^{\delta}_{n,\alpha_k,s})\,dt,
\end{aligned}
$$

and hence, by Lemma 3.10 and Assumption 3.1, we have

$$\begin{aligned}\|\widetilde{x}^{\delta}_{n+1,\alpha_k,s}-\widetilde{y}^{\delta}_{n,\alpha_k,s}\| \le \overline{\psi_2(s)}\bigg\| \int_0^1 \Phi(x_0, \widetilde{x}^{\delta}_{n,\alpha_k,s} \\ + t(\widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s}), \widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s})\, dt\bigg\| \\ \le \overline{\psi_2(s)} k_0 r \|\widetilde{y}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s}\|.\end{aligned}$$

This proves (a).

Now (b) follows from (a) and the triangle inequality:

$$\begin{aligned}\|\widetilde{x}^{\delta}_{n,\alpha_k,s}-\widetilde{x}^{\delta}_{n-1,\alpha_k,s}\| \le \|\widetilde{x}^{\delta}_{n,\alpha_k,s}-\widetilde{y}^{\delta}_{n-1,\alpha_k,s}\| \\ + \|\widetilde{y}^{\delta}_{n-1,\alpha_k,s}-\widetilde{x}^{\delta}_{n-1,\alpha_k,s}\|.\end{aligned}$$

Again, (c) follows from (a), Assumption 3.1, Lemma 3.10 and the following expression

$$\begin{aligned}\widetilde{e}^{\delta}_{n,\alpha_k,s} &= \bigg\| \left(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\right)^{-1} [F'(x_0)(\widetilde{x}^{\delta}_{n,\alpha_k,s}-\widetilde{y}^{\delta}_{n-1,\alpha_k,s}) \\ &\qquad - (F(\widetilde{x}^{\delta}_{n,\alpha_k,s}) - F(\widetilde{y}^{\delta}_{n-1,\alpha_k,s}))]\bigg\| \\ &= \bigg\| \left(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\right)^{-1} \\ &\qquad \times \int_0^1 [F'(x_0) - (F'(\widetilde{y}^{\delta}_{n-1,\alpha_k,s}) \\ &\qquad + t(\widetilde{y}^{\delta}_{n-1,\alpha_k,s}-\widetilde{x}^{\delta}_{n,\alpha_k,s}))](\widetilde{x}^{\delta}_{n,\alpha_k,s}-\widetilde{y}^{\delta}_{n-1,\alpha_k,s})\, dt\bigg\|.\end{aligned}$$

Further, (d) follows from (c).

The remaining part of the proof is analogous to the proof of Theorem 3.3. □

Next, we go to the main result of this section.

Theorem 3.12. *Let* $\widetilde{y}^{\delta}_{n,\alpha_k,s}$ *and* $\widetilde{x}^{\delta}_{n,\alpha_k,s}$ *be as in* (1.11) *and* (1.12), *respectively, with* $\alpha = \alpha_k$, $\delta \in (0, \delta_0]$ *and the assumptions of Theorem* 3.11 *holding. Then* $(\widetilde{x}^{\delta}_{n,\alpha_k,s})$ *is a Cauchy sequence in* $B_{\tilde{r}}(x_0)$

and converges, say, to $\widetilde{x}^{\delta}_{\alpha_k,s} \in \overline{B_{\tilde{r}}(x_0)}$. *Further,* $F(\widetilde{x}^{\delta}_{\alpha_k,s}) - z^{\delta}_{\alpha_k,s} + (\alpha_k/c)L^{s/2}(\widetilde{x}^{\delta}_{\alpha_k,s} - x_0) = 0$ *and* $\|\widetilde{x}^{\delta}_{n,\alpha_k,s} - \widetilde{x}^{\delta}_{\alpha_k,s}\| \leq \widetilde{C}\widetilde{q}^{2n}$, *where* $\widetilde{C} = (\widetilde{\gamma}_\rho)/(1-\widetilde{q})$.

Proof. Analogous to the proof of Theorem 3.4, one can see that $(\widetilde{x}^{\delta}_{n,\alpha_k,s})$ is a Cauchy sequence in $B_{\widetilde{r}}(x_0)$, and hence it converges, say, to $\widetilde{x}^{\delta}_{\alpha_k,s} \in \overline{B_{\widetilde{r}}(x_0)}$. Observe that, from (3.11),

$$
\begin{aligned}
(3.34)\quad \|F(\widetilde{x}^{\delta}_{n,\alpha_k,s}) - z^{\delta}_{\alpha_k,s} &+ \frac{\alpha_k}{c}L^{s/2}(\widetilde{x}^{\delta}_{n,\alpha_k,s} - x_0)\| \\
&= \left\|\left(F'(x_0) + \frac{\alpha_k}{c}L^{s/2}\right)(\widetilde{y}^{\delta}_{n,\alpha_k,s} - \widetilde{x}^{\delta}_{n,\alpha_k,s})\right\| \\
&\leq \left\|F'(x_0) + \frac{\alpha_k}{c}L^{s/2}\right\|_{X_s\to X}\|\widetilde{y}^{\delta}_{n,\alpha_k,s} - \widetilde{x}^{\delta}_{n,\alpha_k,s}\| \\
&\leq \left\|F'(x_0) + \frac{\alpha_k}{c}L^{s/2}\right\|_{X_s\to X}\widetilde{e}^{\delta}_{n,\alpha_k,s} \\
&\leq \left\|F'(x_0) + \frac{\alpha_k}{c}L^{s/2}\right\|_{X_s\to X}\widetilde{q}^{2n}\widetilde{\gamma}_\rho.
\end{aligned}
$$

Now, by letting $n \to \infty$ in (3.34), we obtain $F(\widetilde{x}^{\delta}_{\alpha_k,s}) + (\alpha_k/c)L^{s/2}(\widetilde{x}^{\delta}_{\alpha_k,s} - x_0) = z^{\delta}_{\alpha_k,s}$. This completes the proof. □

In addition to Assumption 2.2, we use the following assumption to obtain the error estimate for $\|\widehat{x} - \widetilde{x}^{\delta}_{\alpha_k,s}\|$.

Assumption 3.13. *There exists a continuous, strictly monotonically increasing function* $\varphi_1 : (0, \|B_s\|] \to (0,\infty)$ *such that the following conditions hold*:

- $\lim_{\lambda\to 0}\varphi_1(\lambda) = 0$,
- $$\sup_{\lambda>0}\frac{\alpha\varphi_1(\lambda)}{\lambda+\alpha} \leq \varphi_1(\alpha) \quad \text{for all } \lambda \in (0, \|B_s\|],$$
- there exists $w \in X$ with $\|w\| \leq E_2$, such that
$$B_s^{s/(2(s+b))}L^{s/4}(x_0 - \hat{x}) = \varphi_1(B_s)w.$$

• For each $x \in B_{\tilde{r}}(x_0)$, there exists a bounded linear operator $G(x, x_0)$ (cf. [**19**]) such that

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\| \leq k_1$.

Remark 3.14. If $x_0 - \widehat{x} \in X_{t_1}$, i.e., $\|x_0 - \widehat{x}\|_{t_1} \leq E_1$ for some positive constant E_1 and $0 \leq t_1 \leq s + b$, then, as in Remark 2.3, we have $B_s^{s/(2(s+b))} L^{s/4}(x_0 - \widehat{x}) = \varphi_1(B_s)w$ where $\varphi_1(\lambda) = \lambda^{t_1/(s+b)}$, $w = B_s^{(s-2t_1)/(2(s+b))} L^{s/4}(\widehat{x} - x_0)$ and $\|w\| \leq g_1((s - 2t_1)/(2(s + b)))E_1 =: E_2$.

Assume that $k_1 < (1/(1 - c))[(1/\overline{\psi_2(s)}) - k_0\tilde{r}]$ and, for the sake of simplicity, assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$ for $\alpha > 0$. Let $\psi_3(s) := E_2/[f_1(s/(2(s + b)))]$.

Theorem 3.15. *Suppose* $\widetilde{x}^{\delta}_{\alpha_k,s}$ *is the solution of* (3.28) *and Assumptions* 3.1 *and* 3.13 *hold. Then*

$$\|\widehat{x} - \widetilde{x}^{\delta}_{\alpha_k,s}\| = O(\psi^{-1}_{s,a}(\delta)).$$

Proof. Note that $c(F(\widetilde{x}^{\delta}_{\alpha_k,s}) - z^{\delta}_{\alpha_k,s}) + \alpha_k L^{s/2}(\widetilde{x}^{\delta}_{\alpha_k,s} - x_0) = 0$, so

$$\begin{aligned}
(F'(x_0) + \alpha_k L^{s/2})(\widetilde{x}^{\delta}_{\alpha_k,s} - \widehat{x}) &= (F'(x_0) + \alpha_k L^{s/2})(\widetilde{x}^{\delta}_{\alpha_k,s} - \widehat{x}) \\
&\quad - c(F(\widetilde{x}^{\delta}_{\alpha_k,s}) - z^{\delta}_{\alpha_k,s}) \\
&\quad - \alpha_k L^{s/2}(\widetilde{x}^{\delta}_{\alpha,s} - x_0) \\
&= \alpha_k L^{s/2}(x_0 - \widehat{x}) + F'(x_0)(\widetilde{x}^{\delta}_{\alpha_k,s} - \widehat{x}) \\
&\quad - c[F(\widetilde{x}^{\delta}_{\alpha_k,s}) - z^{\delta}_{\alpha_k,s}] \\
&= \alpha_k L^{s/2}(x_0 - \widehat{x}) + F'(x_0)(\widetilde{x}^{\delta}_{\alpha_k,s} - \widehat{x}) \\
&\quad - c[F(\widetilde{x}^{\delta}_{\alpha_k,s}) - F(\widehat{x}) + F(\widehat{x}) - z^{\delta}_{\alpha_k,s}] \\
&= \alpha_k L^{s/2}(x_0 - \widehat{x}) - c(F(\widehat{x}) - z^{\delta}_{\alpha_k,s}) \\
&\quad + F'(x_0)(x^{\delta}_{\alpha_k,s} - \widehat{x}) - c[F(\widetilde{x}^{\delta}_{\alpha_k,s}) - F(\widehat{x})].
\end{aligned}$$

Thus, since $0 < c < \alpha_k$, we have

(3.35)
$$\begin{aligned}\|\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}\| &\le \|\alpha_k(F'(x_0) + \alpha_k L^{s/2})^{-1}L^{s/2}(x_0 - \widehat{x})\| \\ &\quad + \|(F'(x_0) + \alpha_k L^{s/2})^{-1} \\ &\quad \times c(F(\widehat{x}) - z^\delta_{\alpha_k,s})\| + \|(F'(x_0) + \alpha_k L^{s/2})^{-1} \\ &\quad \times [F'(x_0)(\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}) - c(F(\widetilde{x}^\delta_{\alpha_k,s}) - F(\widehat{x}))]\| \\ &\le \Gamma_1 + \overline{\psi^2(s)}\|F(\widehat{x}) - z^\delta_{\alpha_k,s}\| + \Gamma_2,\end{aligned}$$

where

$$\begin{aligned}\Gamma_1 &:= \|\alpha_k(F'(x_0) + \alpha_k L^{s/2})^{-1}L^{s/2}(x_0 - \widehat{x})\|, \\ \Gamma_2 &:= \|(F'(x_0) + \alpha_k L^{s/2})^{-1}[F'(x_0)(\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}) \\ &\qquad - c(F(\widetilde{x}^\delta_{\alpha_k,s}) - F(\widehat{x}))]\|.\end{aligned}$$

Note that, by Assumption 3.13,
(3.36)
$$\begin{aligned}\Gamma_1 &\le \|\alpha_k L^{-s/4}(B_s + \alpha_k I)^{-1}L^{s/4}(x_0 - \hat{x})\| \\ &\le \frac{1}{f_1((s/(2(s+b))))}\|\alpha_k(B_s + \alpha_k I)^{-1}B_s^{s/((2(s+b)))}L^{s/4}(x_0 - \widehat{x})\| \\ &\le \frac{1}{f_1((s/2(s+b)))}\varphi_1(\alpha_k)E_2\end{aligned}$$

and

(3.37)
$$\begin{aligned}\Gamma_2 &= \Bigg\|(F'(x_0) + \alpha_k L^{s/2})^{-1} \\ &\qquad \times \int_0^1 [F'(x_0) - cF'(\widehat{x} + t(\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}))](\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x})\,dt\Bigg\| \\ &\le \Bigg\|(F'(x_0) + \alpha_k L^{s/2})^{-1} \\ &\qquad \times \int_0^1 [F'(x_0) - F'(\widehat{x} + t(\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}))](\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x})\,dt\Bigg\| \\ &\qquad + (1-c)\Bigg\|(F'(x_0) + \alpha_k L^{s/2})^{-1}F'(x_0)\end{aligned}$$

$$\times \int_0^1 G(\widehat{x} + t(\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}), x_0)(\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x})\, dt\bigg\|$$
$$\leq \overline{\psi_2(s)} k_0 \widetilde{r} \|\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}\| + \overline{\psi_2(s)}(1-c)k_1 \|\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}\|.$$

The last step follows from Lemma 3.10, Proposition 3.8, Assumptions 3.1 and 3.13. Hence, by (3.35)–(3.37), we have

$$\begin{aligned}\|\widetilde{x}^\delta_{\alpha_k,s} - \widehat{x}\| &\leq \frac{\psi_3(s)\varphi_1(\alpha_k) + \overline{\psi^2(s)}\|F(\widehat{x}) - z^\delta_{\alpha_k,s}\|}{1 - [(1-c)k_1 + k_0\widetilde{r}]\overline{\psi_2(s)}} \\ &\leq \frac{\psi_3(s)\varphi_1(\alpha_k) + \overline{\psi^2(s)} C_s (2 + (4\eta)/(\eta - 1))\eta(\psi^{-1}_{s,a}(\delta))}{1 - [(1-c)k_1 - k_0\widetilde{r}]\overline{\psi_2(s)}} \\ &= O(\psi^{-1}_{s,a}(\delta)).\end{aligned}$$

This completes the proof of Theorem 3.15. □

The following Theorem is a consequence of Theorems 3.12 and 3.15.

Theorem 3.16. *Let* $\widetilde{x}^\delta_{n,\alpha_k,s}$ *be as in* (1.12) *with* $\alpha = \alpha_k$ *and* $\delta \in (0, \delta_0]$*, and let the assumptions in Theorems* 3.12 *and* 3.15 *hold. Then*

$$\|\widehat{x} - \widetilde{x}^\delta_{n,\alpha_k,s}\| \leq \widetilde{C} q^{2n} + O(\psi^{-1}_{s,a}(\delta))$$

where $\widetilde{C}$ *is as in Theorem* 3.12.

Theorem 3.17. *Let* $\widetilde{x}^\delta_{n,\alpha_k,s}$ *be as in* (1.12) *with* $\alpha = \alpha_k$ *and* $\delta \in (0, \delta_0]$*, and let the assumptions in Theorem* 3.16 *hold. Let*

$$n_k := \min\{n : \widetilde{q}^{2n} \leq \alpha_k^{-a/(2(s+a))}\delta\}.$$

Then

$$\|\widehat{x} - \widetilde{x}^\delta_{n_k,\alpha_k,s}\| = O(\psi^{-1}_{s,a}(\delta)).$$

4. Conclusion. In this paper we present a two step iterative regularization method for obtaining an approximate solution of an ill-posed Hammerstein type operator equation $KF(x) = y$ in the Hilbert scale setting where K is a bounded linear operator and F is a nonlinear operator. It is assumed that the available data is y^δ in place of the exact data y. Two cases of F are discussed: (a) $F'(x_0)^{-1}$ exists and is bounded and (b) F is monotone and $F'(x_0)$ is non-invertible. We considered the Hilbert space $(X_t)_{t\in\mathbf{R}}$ generated by L for the analysis where $L : D(L) \to X$ is a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. In order to choose the regularization parameter α, we used the adaptive scheme of Pereverzev and Schock [**14**].

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