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A SINC QUADRATURE METHOD FOR THE URYSOHN INTEGRAL EQUATION

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ABSTRACT. In this paper, we study the numerical approximation of the Urysohn integral equation with two methods. The methods are developed by means of the sinc approximation with the Single Exponential (SE) and Double Exponential (DE) transformations. These numerical methods combine a sinc Nyström method with the Newton iterative process that involves solving a nonlinear system of equations. We provide an error analysis for the methods. These methods improve conventional results and achieve exponential convergence. Some numerical examples are given to confirm the accuracy and ease of implementation of the methods.

1. Introduction. In this paper, we consider the sinc Nyström method for the numerical solution of the Urysohn integral equations of Fredholm type

$$u(t) - \int_a^b k(t,s,u(s))\,\mathrm{d}s = g(t), \quad t \in [a,b], \tag{1.1}$$

where $u(t)$ is an unknown function to be determined and $k(t,s,u)$ and $g(t)$ are given functions. Equation (1.1) was introduced for the first time by Pavel Urysohn in [**41**]. The Urysohn integral equation includes the Hammerstein equation and many other equations. Equations of this type appear in many applications. For example, they arise as a reformulation of two-point boundary value problems with certain nonlinear boundary conditions [**4, 27**]. Several authors have written a number of papers which establish numerical techniques for finding an approximation of the nonlinear Fredholm integral equations. These

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methods can be categorized into two major types. The first type are those which lead to solving a system of nonlinear equations, and the other uses iterative methods to solve the nonlinear equation directly. We will briefly review these techniques.

In [**22, 23**], Krasnosel'skii et al. have dealt with the theoretical aspect of projection methods, especially the Galerkin method for nonlinear Fredholm integral equations which are involved in the first category. Based on these works, Atkinson and his co-authors extended the projection, iterated projection and discrete projection methods in a series of papers [**3, 4, 6–8**] for this kind of integral equation, and the accelerated projection and iterated projection methods have been proposed in [**12**]. Kumar in [**24–27**] investigated the superconvergence property of the iterated collocation method for Hammerstein integral equations, and these works have been extended and completed for the Galerkin method by Kaneko et al. [**24–26**]. Furthermore, an asymptotic error analysis for the Nyström method has been considered in [**15**]. Hernández et al. have investigated the numerical solution of nonlinear equations, especially nonlinear Fredholm integral equations, through the modified secant and Newton methods [**2, 14, 16**]. Recently, the Multilevel Augmentation Method (MAM) was introduced and improved for Hammerstein integral equation in [**11, 13**] and extended to the Urysohn integral equation case in [**10**].

The aim of this work is to present two numerical schemes for a Nyström method based on sinc quadrature formulas. The first method is given by extending Stenger's idea [**37**] to nonlinear Fredholm integral equations. It is shown that this method has the convergence rate $O(\exp(-C\sqrt{N}))$. The second method is derived by replacing the smoothing transformation employed in the first method, the standard *tanh* transformation, with the so-called double exponential transformation. Such a replacement improves the order of convergence to $O(\exp(-C(N/\log N)))$. For a comprehensive study of single exponential sinc methods, we refer to [**30, 36, 37, 38**], and for double exponential sinc approximation to [**32–35, 39, 40, 43**].

Equation (1.1) can be expressed in operator form as

$$(I-\mathcal{K})u = g, \tag{1.2}$$

where $(\mathcal{K}u)(t) = \int_a^b k(t,s,u(s))\,\mathrm{d}s$. The operator is defined on the Banach space $X = \mathbf{Hol}\,(D) \cap C(\overline{D})$. In this notation, $D \subset \mathbf{C}$ is a simply connected domain which satisfies $(a, b) \subset D$ and $\mathbf{Hol}(D)$ denotes the family of all functions f that are analytic in the domain D. Furthermore, assume (1.2) has at least one solution, and note that the right side of (1.1) is a completely continuous operator [**23**]. In Section 4, the sufficient conditions in which (1.1) has such a solution will be introduced. Let $\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$. Additionally, suppose that the solution $u^*(t)$ to be determined is geometrically isolated [**21**], in other words, there is some ball

$$\mathfrak{B}(u^*, r) = \{u \in X : \|u - u^*\| \leq r\},$$

with $r > 0$, that contains no solution of (1.1) other than u^*. It is assumed that the linear operator $\mathcal{K}'(u^*)$ does not have 1 as an eigenvalue. Then there is a geometrically isolated solution for (1.1) [**27**].

This paper is organized in five sections. In Section 2 we will review the basic properties of the sinc quadrature rule which has been used in our approximation and analysis. Two numerical methods based on sinc approximation are considered in Section 3. We provide in Section 4 a complete convergence analysis for the proposed methods. Finally, in Section 5, we present several numerical experiments. The numerical results are consistent with the theoretical estimates on orders of convergence. The numerical performance of the proposed method is favorable in comparison to that of the multigrid method and the MAM.

2. The quadrature formulae. The sinc function is defined on the whole real line by

$$\operatorname{sinc}(t) = \begin{cases} \sin(\pi t)/(\pi t) & t \neq 0, \\ 1 & t = 0. \end{cases}$$

The sinc numerical methods are based on approximation over the infinite interval $(-\infty, \infty)$, written as

$$f(t) \approx \sum_{j=-N}^{N} f(jh) S(j, h)(t), \quad t \in \mathbf{R},$$

where the basis function $S(j, h)(t)$ is defined by

$$S(j, h)(t) = \operatorname{sinc}\left(\frac{t}{h} - j\right),$$

h is a step size appropriately chosen depending on a given positive integer N, and j is an integer. The sinc approximation and numerical integration are closely related through the following identity

$$\int_{-\infty}^{\infty}\left(\sum_{j=-N}^{N} f(jh)S(j,h)(t) - f(t)\right)\mathrm{d}t = h\sum_{j=-N}^{N} f(jh) - \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t. \tag{2.1}$$

On the other hand, this is a relation between the approximation error of the sinc approximation and the one of integration by the trapezoidal rule [**32**]. The equation (2.1) can be adapted to approximate on general intervals with the aid of appropriate variable transformations $t = \varphi(x)$ as the transformation function $\varphi(x)$ appropriate single exponential (SE) and double exponential (DE) transformations are applied. The single exponential transformation and its inverse can be introduced, respectively, as below [**36**]:

$$\varphi_{SE}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2},$$
$$\phi_{SE}(t) = \log\left(\frac{t-a}{b-t}\right).$$

In order to define a convenient function space, the strip domain

$$D_d = \{z \in \mathcal{C} : |\mathrm{Im}\, z| < d\},$$

for some $d > 0$ is introduced. When incorporated with the SE-transformation, the conditions should be considered on the translated domain

$$\varphi_{SE}(D_d) = \left\{z \in \mathcal{C} : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d\right\}.$$

The following definitions and theorems are considered for further details of the procedure.

Definition 2.1. Let D be a simply connected domain which satisfies $(a,b) \subset D$, and let α and C be positive constants. Then $\mathcal{L}_\alpha(D)$ denotes the family of all functions $f \in \mathbf{Hol}\,(D)$ which satisfy

$$|f(z)| \le C|Q(z)|^\alpha,$$

for all z in D where $Q(z) = (z-a)(b-z)$.

The following theorem involves bounding the error of $(2N+3)$-point sinc quadrature for f on (a,b). When incorporated with the SE-transformation, the quadrature rule is designated as the SE-sinc quadrature.

Theorem 2.2 [**36**]. *Let $(fQ) \in \mathcal{L}_\alpha(\varphi_{SE}(D_d))$ for d with $0 < d < \pi$. Let N be a positive integer and h be selected by the formula*

$$h = \sqrt{\frac{\pi d}{\alpha N}}.$$

Then there exists a constant C, which is independent of N, such that

$$\left| \int_a^b f(t)\,\mathrm{d}t - h \sum_{j=-N}^{N} f(\varphi_{SE}(jh))\varphi'_{SE}(jh) \right| \leq C \exp(-\sqrt{\pi d \alpha N}). \tag{2.2}$$

The double exponential transformation can be used instead of the single exponential transformation. The DE-transformation and its inverse are

$$\varphi_{DE}(x) = \frac{b-a}{2} \tanh\left(\frac{\pi}{2}\sinh(x)\right) + \frac{b+a}{2},$$
$$\phi_{DE}(t) = \log\left[\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right) + \sqrt{1 + \left\{\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right)\right\}^2}\right].$$

This transformation maps D_d onto the domain

$$\varphi_{DE}(D_d)$$
$$= \left\{ z \in \mathcal{C} : \left| \arg\left[\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right) + \sqrt{1 + \left\{\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right)\right\}^2}\right] \right| < d \right\}.$$

If we use the DE-transformation instead of the SE-transformation, the DE-sinc quadrature is achieved. The rate of convergence is accelerated as the next theorem states.

Theorem 2.3 [**33**]. *Let* $(fQ) \in \mathcal{L}_\alpha(\varphi_{DE}(D_d))$ *for* d *with* $0 < d < \pi/2$. *Assume that* N *is a positive integer and* h *is selected by the formula*

$$h = \frac{\log(2dN/\alpha)}{N}.$$

Then there exists a constant C, *which is independent of* N, *such that*

(2.3)

$$\left| \int_a^b f(t)\,\mathrm{d}t - h \sum_{j=-N}^{N} f(\varphi_{DE}(jh))\varphi'_{DE}(jh) \right| \le C \exp\left(\frac{-2\pi dN}{\log(2dN/\alpha)} \right).$$

3. Sinc Nyström method.

3.1. SE-sinc scheme. In the SE-sinc Nyström method we approximate the integral operator in (1.1) by the quadrature formula (2.2). Let $u \in \mathbf{Hol}\,(\varphi_{SE}(D_d))$ and $k(t,\cdot,u(\cdot))Q(\cdot) \in \mathcal{L}_\alpha(\varphi_{SE}(D_d))$ for all $t \in [a,b]$ and $u \in \mathfrak{B}$. Then the integral in (1.1) can be approximated by Theorem 2.2 and the following discrete SE-operator can be defined:

$$\mathcal{K}_N^{SE}(u)(t) = h \sum_{j=-N}^{N} k(t, t_j^{SE}, u(t_j^{SE}))\varphi'_{SE}(jh).$$

The Nyström method applied to (1.1) is to find u_N^{SE} such that

$$u_N^{SE}(t) - h \sum_{j=-N}^{N} k(t, t_j^{SE}, u(t_j^{SE}))\varphi'_{SE}(jh) = g(t), \tag{3.1}$$

where the points t_j^{SE} are defined by the formula

$$t_j^{SE} = \varphi_{SE}(jh), \quad j = -N, \dots, N.$$

Solving (3.1) reduces to solving a finite dimensional nonlinear system. For any solution of (3.1) the values $u_N^{SE}(t_j^{SE})$ at the quadrature points satisfy the nonlinear system

$$u_N^{SE}(t_i^{SE}) - h \sum_{j=-N}^{N} k(t_i^{SE}, t_j^{SE}, u(t_j^{SE}))\varphi'_{SE}(jh) = g(t_i^{SE}), \quad i = -N, \dots, N. \tag{3.2}$$

Conversely, given a solution $u_N^{SE}(t_i^{SE})$, $i=-N,\dots,N$, of the system (3.2), then the function u_N^{SE} defined by

$$u_N^{SE}(t) = h\sum_{j=-N}^{N} k(t,t_j^{SE},u(t_j^{SE}))\varphi_{SE}'(jh) + g(t),$$

is readily seen to satisfy (3.1).

We rewrite equation (3.1) in operator notation as

$$(I-\mathcal{K}_N^{SE})u_N^{SE} = g. \tag{3.3}$$

Atkinson in [**3**] by using the Leray-Schauder theorem proved that, under certain differentiability assumptions on $\mathcal{K}$ and $\mathcal{K}_N^{SE}$, (3.3) has a unique solution in a neighborhood of an isolated solution of (1.1), and these approximation solutions converge to an isolated solution for sufficiently large N. We assume that $k_u(t,s,u)\equiv(\partial k(t,s,u))/(\partial u)$ is continuous for all $t,s\in[a,b]$ and $u\in\mathfrak{B}$. This assumption implies that $\mathcal{K}$ is Fréchet differentiable [**3**] with

$$\mathcal{K}'(u)x(t) = \int_a^b k_u(t,s,u(s))x(s)\,\mathrm{d}s, \quad t\in[a,b],\ x\in X.$$

Furthermore, the continuity assumption is considered for second partial derivative of the kernel, $k_{uu}(t,s,u)$, leading to the existence and the boundedness of the second Fréchet derivative with

$$\mathcal{K}''(u)(x,y)(t) = \int_a^b k_{uu}(t,s,u(s))x(s)y(s)\,\mathrm{d}s,$$
$$t\in[a,b], \quad x,y\in X.$$

Similar to $\mathcal{K}_N^{SE}$, $(\mathcal{K}_N^{SE})'$ and $(\mathcal{K}_N^{SE})''$ can be defined by the SE-sinc quadrature formula as follows:

$$(\mathcal{K}_N^{SE})'(u)x(t) = h\sum_{j=-N}^{N} k_u(t,t_j^{SE},u(t_j^{SE}))\varphi_{SE}'(jh)x(t_j^{SE}),$$

and

$$(\mathcal{K}_N^{SE})''(u)(x,y)(t) \\ = h\sum_{j=-N}^{N} k_{uu}(t, t_j^{SE}, u(t_j^{SE}))\varphi'_{SE}(jh)x(t_j^{SE})y(t_j^{SE}). \tag{3.4}$$

3.2. DE-sinc scheme. The DE-sinc Nyström case is focused on in this part. In the DE-case we assume that the solution of (1.1) belongs to $\mathbf{Hol}\,(\varphi_{DE}(D_d))$ and $k(t,\cdot,u(\cdot))Q(\cdot) \in \mathcal{L}_\alpha(\varphi_{DE}(D_d))$ for all $t \in [a,b]$ and $u \in \mathfrak{B}$. So with the aid of the DE-sinc quadrature formula, the discrete DE-sinc operator is defined as:

$$\mathcal{K}_N^{DE}(u)(t) = h\sum_{j=-N}^{N} k(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh).$$

In this case, $u_N^{DE}(t)$ is found from collocating

$$u_N^{DE}(t) - h\sum_{j=-N}^{N} k(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh) = g(t), \tag{3.5}$$

at the DE-sinc quadrature points

$$t_j^{DE} = \varphi_{DE}(jh), \quad j = -N, \dots, N.$$

Finally, the DE-sinc Nyström solution

$$u_N^{DE}(t) = h\sum_{j=-N}^{N} k(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh) + g(t),$$

can be determined by a nonlinear system with the unknown coefficient $u_N^{DE}(t_i^{DE})$ for $i = -N, \dots, N$. Equation (3.5) can be rewritten in operator form as follows:

$$(I - \mathcal{K}_N^{DE})u_N^{DE} = g. \tag{3.6}$$

Similar to the SE-case, $(\mathcal{K}_N^{DE})'(u)$ and $(\mathcal{K}_N^{DE})''(u)$ are defined as follows:

$$(\mathcal{K}_N^{DE})'(u)x(t) = h \sum_{j=-N}^{N} k_u(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh)x(t_j^{DE}),$$

and

$$(\mathcal{K}_N^{DE})''(u)(x,y)(t) = h \sum_{j=-N}^{N} k_{uu}(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh)x(t_j^{DE})y(t_j^{DE}).$$

4. Convergence analysis.

4.1. Sinc Nyström method. The convergence of the two sinc Nyström methods which was introduced in the previous sections is discussed in the present section. We first consider the SE-case. For the following lemma, D represents either $\varphi_{SE}(D_d)$ or $\varphi_{DE}(D_d)$. In this lemma, the sufficient conditions to have a completely continuous operator have been investigated.

Lemma 4.1 [**23**]. *Let the kernel $k(t,s,u)$ be continuous and have a continuous partial derivative $(\partial k(t,s,u))/(\partial u)$ for all $t,s \in D$ and $u \in \mathfrak{B}$. Then $\mathcal{K} : X \to X$ is a completely continuous operator and is differentiable at each point of $\mathfrak{B}$.*

Our basic assumption is that (1.1) has an analytic solution. The sufficient conditions to have such a solution have been mentioned in [**23**, page 83]. We presume that those conditions are satisfied here. Our idea for deriving the order of convergence is based on collectively compact operator theory [**1**]. For ease of reference, the following required conditions are mentioned from [**3, 42**].

$\mathcal{C}_1$. $\{\mathcal{K}_N^{SE} : N \geq 1\}$ is a collectively compact family on X.

$\mathcal{C}_2$. $\mathcal{K}_N^{SE}$ is pointwise convergent to $\mathcal{K}$ on X.

$\mathcal{C}_3$. For $N \geq 1$, $\mathcal{K}_N^{SE}$ possesses continuous first and bounded second Fréchet derivatives on $\mathfrak{B}$. Moreover,

$$\|(K_N^{SE})''\| \leq \alpha < \infty,$$

where α is a constant.

It is more convenient to rewrite the quadrature rule defined in Theorem 2.2 in the following notation. Let $Q_N^{SE} : X \to \mathbf{R}$ be a discrete operator defined by

$$Q_N^{SE} f = h \sum_{j=-N}^{N} f(t_j^{SE}) \varphi'_{SE}(jh), \tag{4.1}$$

and let $Q : X \to \mathbf{R}$ be an integral operator defined by $Qf = \int_a^b f(t)\,\mathrm{d}t$. Kress et al. in [**24**] have concluded from Steklov's theorem that $Q_N^{SE} f \to Qf$ for all $f \in C[a,b]$. Additionally, it is easily proved by the Banach-Steinhaus theorem that Q_N^{SE} is uniformly bounded [**34**]. Now, the following theorem is stated to prove that $\mathcal{K}_N^{SE}$ satisfies the conditions $\mathcal{C}_1$-$\mathcal{C}_3$.

Theorem 4.2. *Assume that $k(t,\cdot,u(\cdot))Q(\cdot) \in \mathcal{L}_\alpha(\varphi_{SE}(D_d))$ for $0 < d < \pi$ and $k_{uu}(t,s,u)$ is continuous for all $t,s \in [a,b]$ and $u \in \mathfrak{B}$. Then the conditions $\mathcal{C}_1$-$\mathcal{C}_3$ are fulfilled.*

Proof. From the continuity of the kernel and the above discussion, the family

$$S = \{\mathcal{K}_N^{SE} u \mid N \geq 1,\ u \in \mathfrak{B}\},$$

is uniformly bounded. Furthermore, note that the function $k(t,s,u)$ is uniformly continuous on $[a,b] \times [a,b] \times \mathfrak{B}$, and therefore we can conclude from the uniform boundedness of Q_N^{SE} that S is a family of equicontinuous functions. So $\mathcal{C}_1$ follows from the Arzelà-Ascoli theorem.

Due to Theorem 2.2 and the relevant discussion to (4.1), the condition $\mathcal{C}_2$ holds. By considering (3.4) on $\mathfrak{B}$ and the continuity of $k_{uu}(t,s,u)$, $\mathcal{C}_3$ is easily concluded. $\square$

Lemma 4.3. *Let $I - \mathcal{K}'(u^*)$ be nonsingular and the assumptions of Theorem 4.2 fulfilled. Then for sufficiently large N, the linear operators*

$I - (\mathcal{K}_N^{SE})'(u^*)$ *are nonsingular*; *furthermore,*

$$\|(I - (\mathcal{K}_N^{SE})'(u^*))^{-1}\| \leq M,$$

where M is a constant independent of N.

Proof. Condition $\mathcal{C}_3$ is satisfied and $\{\mathcal{K}_N^{SE}(u^*) \mid N \geq N_1\}$ is equidifferentiable. Therefore, according to [**1**, Theorem 6.10], $\{(\mathcal{K}_N^{SE})'(u) \mid N \geq N_1\}$ is a collectively compact family of operators. Moreover, from conditions $\mathcal{C}_1$, $\mathcal{C}_2$ and Theorem 6.11 of [**1**], we can conclude that $(\mathcal{K}_N^{SE})'(u)$ is pointwise convergent to $\mathcal{K}'(u)$ for all $u \in \mathfrak{B}$. So, the final result has been obtained from the existence of $(I - \mathcal{K}'(u^*))^{-1}$ and the theory of collectively compact operators. $\square$

Now we are ready to formulate the main result.

Theorem 4.4. *Suppose that the assumptions of Lemma* 4.3 *hold. Then there exists a positive integer N_1 such that, for all $N \geq N_1$,* (3.3) *has a unique solution $u_N^{SE} \in X$. Furthermore, there exists a constant C independent of N such that*

$$\|u^* - u_N^{SE}\| \leq C \exp(-\sqrt{\pi d \alpha N}).$$

Proof. By subtracting (1.1) from (3.3) and adding the term $\mathcal{K}'(u^*)(u^* - u_N^{SE})$ on both sides, the following term has been obtained

$$\begin{aligned} &(I - (\mathcal{K}_N^{SE})'(u^*))(u^* - u_N^{SE}) \\ &= \mathcal{K}(u^*) - \mathcal{K}_N^{SE}(u^*) - [\mathcal{K}_N^{SE}(u_N^{SE}) - \mathcal{K}_N^{SE}(u^*) - (\mathcal{K}_N^{SE})'(u^*)(u_N^{SE} - u^*)]. \end{aligned} \tag{4.2}$$

By applying $\|\cdot\|$ on both sides of (4.2) and Lemma 4.3, we achieve the following relation

$$\begin{aligned} \|u^* - u_N^{SE}\| \leq M\{&\|\mathcal{K}(u^*) - \mathcal{K}_N^{SE}(u^*)\| \\ &+ \|\mathcal{K}_N^{SE}(u_N^{SE}) - \mathcal{K}_N^{SE}(u^*) - (\mathcal{K}_N^{SE})'(u^*)(u_N^{SE} - u^*)\|\}. \end{aligned}$$

The second term on the right-hand side has been bounded by the term $(1/2)\alpha\|u^* - u_N^{SE}\|^2$ by condition $\mathcal{C}_3$, and the finite result has been obtained from Theorem 2.2. $\square$

The conditions $\mathcal{C}_1$–$\mathcal{C}_3$ can be defined for the DE-case by replacing the SE-transformation φ_{SE} with DE-transformation φ_{DE}. Also, the DE-discrete operator Q_N^{DE} can be defined by

$$Q_N^{DE} f = h \sum_{j=-N}^{N} f(t_j^{DE}) \varphi'_{DE}(jh).$$

We assume $0 < d < (\pi/2)$ in Theorem 4.2; then similar conclusions are achieved for the DE-case. The proof of the similar theorems goes almost in the same way as in the SE-case. Consequently, we refrain from going into details and only state the final theorem.

Theorem 4.5. *Suppose that the same hypotheses of Lemma* 4.3 *are satisfied for the* DE *case. Then there exists a positive integer* N_1 *such that, for all* $N \geq N_1$, (3.6) *has a unique solution* $u_N^{DE} \in X$. *Furthermore, there exists a constant* C *independent of* N *such that*

$$\|u^* - u_N^{DE}\| \leq C \exp\left(\frac{-2\pi d N}{\log(2dN/\alpha)}\right).$$

4.2. Sinc collocation method. The application of the sinc collocation method for nonlinear Fredholm integral equations has been discussed in [**31**]. That application is actually the discrete sinc collocation method. In this section we are trying to give an error bound to the mentioned method based on the sinc Nyström method. In [**5**, subsection 4.3], the iterated discrete collocation has been discussed where the integration nodes belong to the set of the collocation points, as it happens in the discrete sinc collocation method. As the following theorem shows, in this case, the iterated discrete collocation method is the Nyström method.

Theorem 4.6 [**6**]. *Suppose that the hypotheses of Lemma* 4.3 *are fulfilled. Furthermore, let* u^* *be an isolated solution of* (1.1). *Then the iterated discrete sinc collocation method coincides with the sinc Nyström method* $u - \mathcal{K}_N^{SE} u = g$.

It is more convenient to introduce the notations z_N^{SE} for the discrete sinc collocation solution and $\widetilde{z}_N^{SE}$ for the iterated discrete sinc collocation solution. In the following, we mention some theorems related to the SE-sinc and DE-sinc collocation methods. Firstly, we state the following lemma which is used subsequently.

Lemma 4.7 [**36**]. *Let $h > 0$. Then it holds that*:

$$\sup_{x\in\mathbf{R}} \sum_{-N-1}^{N+1} |S(j,h)(x)| \le \frac{2}{\pi}(3+\log(N+1)).$$

Based on Lemma 4.7 it may be seen that $\|\mathcal{P}_N^{SE}\| \le C_{SE}\log(N+1)$ and $\|\mathcal{P}_N^{DE}\| \le C_{DE}\log(N+1)$ where C_{SE} and C_{DE} are constants independent of N.

The following theorem gives us an error bound for the sinc interpolation.

Theorem 4.8 [**36**]. *Let $f \in \mathcal{L}_\alpha(\varphi_{SE}(D))$ for d with $0 < d < \pi$. Suppose that N is a positive integer, and h is given by the formula $h = \sqrt{(\pi d)/(\alpha N)}$. Then there exists a constant C independent of N, such that*

$$\left\| f(t) - \sum_{j=-N}^{N} f(\varphi_{SE}(jh))S(j,h)(\phi_{SE}(t))\right\| \le C\sqrt{N}\exp\Big(-\sqrt{\pi d\alpha N}\Big).$$

A similar theorem for the DE-case can be stated as follows.

Theorem 4.9 [**39**]. *Let $f \in \mathcal{L}_\alpha(\varphi_{DE}(D_d))$ for d with $0 < d < (\pi/2)$, let N be a positive integer and let h be selected by the formula $h = [\log(2dN/\alpha)]/N$. Then there exists a constant C which is independent of N, such that*

$$\left\| f(t) - \sum_{j=-N}^{N} f(\varphi_{DE}(jh))S(j,h)(\phi_{DE}(t))\right\| \le C\exp\left(\frac{-\pi dN}{\log(2dN/\alpha)}\right).$$

The SE and DE-sinc interpolation operators $\mathcal{P}_N^{SE}$ and $\mathcal{P}_N^{DE}$ which are utilized in [**31**] are considered. The following relation is easily proved:

$$u^* - z_N^{SE} = [u^* - \mathcal{P}_N^{SE}u^*] + \mathcal{P}_N^{SE}[u^* - \tilde{z}_N^{SE}]. \tag{4.3}$$

Now we are ready to state the main result of this section.

Theorem 4.10. *Let the hypotheses of Theorem* 4.8 *for the* SE-*case and Theorem* 4.9 *for the* DE-*case and Theorem* 4.6 *be fulfilled. Then the following error bound holds for the* SE-sinc *and* DE-sinc *collocation methods, respectively,*

$$\|u^* - z_N^{SE}\| \leq \sqrt{N}\log(N+1)\exp\left(-\sqrt{\pi d\alpha N}\right),$$
$$\|u^* - z_N^{DE}\| \leq \log(N+1)\exp\left(\frac{-\pi dN}{\log(2dN/\alpha)}\right).$$

Proof. By applying $\|\cdot\|$ on both sides of (4.3), the following relation has been obtained

$$\|u^* - z_N^{SE}\| \leq \|u^* - \mathcal{P}_N^{SE}u^*\| + \|\mathcal{P}_N^{SE}\|\|u^* - \tilde{z}_N^{SE}\|;$$

the first term of the right side and the interpolation operator can be bounded by Theorem 4.8 and Lemma 4.7, respectively. So, the final result can be concluded from Theorems 4.4 and 4.6. The second inequality has been investigated in a similar way. ◻

5. Numerical experiments. In this section, the theoretical results of the previous sections are used for some numerical examples. The numerical experiments are implemented in *Mathematica* 7. The programs are executed on a PC with 2.00 GHz Intel Core 2 dual processor with 2 GB RAM. In order to analyze the error of the method the following notations are introduced:

$$e_{\max} = \max\left\{|u(t_i) - u_N(t_i)| : t_i = \frac{i}{1000}, i = 1(1)1000\right\},$$

and $e_{\max}$ approximate $\|u - u_N\|_\infty$. For the solution of the nonlinear system which arises in the formulation of the methods, one may use the steepest descent method, the Newton method or a mathematical software package. In our experiments we have used Mathematica's routine `FindRoot`. This routine needs an initial guess to solve the nonlinear systems. If the initial guess is selected badly, this routine may fail to converge to the desired solution. In these examples, an initial point is selected by the steepest descent method [**9**]. As we saw in Section 4, the convergence of the two methods depends on two parameters: α and d. In fact, the parameter d indicates the size of the holomorphic domain of u, and α is the order of the Hölder constant of kQ [**33**]. So, due to the smoothness of the kernels, it is assumed that $\alpha = 1$ for all examples. The important parameter d values are 3.14 and 1.57 for the SE-sinc and DE-sinc methods, respectively. $e_{\max}$ is reported for $N = 10\,(10)\,100$. In table form, we present the computing time T_N^{SE} and T_N^{DE} measured in seconds when SE-sinc and DE-sinc are used, respectively. Additionally, SESN and DESN are the abbreviations for Single Exponential Sinc Nyström and Double Exponential Sinc Nyström methods, respectively. These tables and figures show that, by increasing N, the error is reduced significantly. As expected, the results show that the convergence rate of the DE-sinc Nyström method is much faster than the SE-sinc scheme.

Example 5.1. The following Urysohn integral equation is considered

$$u(t) - \int_0^1 \frac{\mathrm{d}s}{2 + t + u(s)} = g(t), \quad t \in [0, 1], \tag{5.1}$$

where $g(t)$ is chosen so that $u^*(t) = \cos(0.3\,\pi t)$ is a solution of (5.1). This equation has been solved in [**29**] by three algorithms based on the multigrid method. Table 1 shows the error results achieved for the SE and DE-sinc Nyström methods.

Example 5.2. In this example, we consider solving the equation

$$u(t) - \int_0^1 \sin(\pi(t+s))u^2(s)\,\mathrm{d}s = \sin(\pi t) - \frac{4}{3\pi}\cos(\pi t),$$
$$t \in [0, 1].$$

The equation has an isolated solution $u^*(t) = \sin(\pi t)$.

TABLE 1. Numerical results for Example 5.1.

N	SESN Method	T_N^{SE}	DESN Method	T_N^{DE}
10	6.83E − 06	23.963	1.91E − 11	23.978
20	1.24E − 08	24.195	2.77E − 16	24.539
30	2.33E − 10	24.757	2.77E − 16	24.897
40	1.21E − 11	25.351	2.22E − 16	25.069
50	6.71E − 13	25.974	2.77E − 16	24.741
60	4.31E − 14	26.692	2.22E − 16	25.958
70	3.33E − 15	27.566	2.77E − 16	27.253
80	4.44E − 16	28.502	3.33E − 16	27.097
90	2.22E − 16	29.312	2.77E − 16	28.173
100	2.77E − 16	30.124	2.77E − 16	29.437

TABLE 2. Numerical results for Example 5.2.

N	SESN Method	T_N^{SE}	DESN Method	T_N^{DE}
10	2.43E − 02	0.951	9.85E − 03	0.920
20	1.05E − 03	1.124	5.56E − 06	0.701
30	7.17E − 05	1.373	2.59E − 09	1.373
40	6.52E − 06	1.700	3.43E − 13	1.700
50	7.23E − 07	1.919	2.22E − 15	1.919
60	9.41E − 08	2.543	1.81E − 15	2.543
70	1.38E − 08	2.480	1.66E − 15	2.480
80	2.26E − 09	3.682	1.81E − 15	3.682
90	4.03E − 10	4.550	2.22E − 15	4.055
100	7.74E − 11	4.524	2.22E − 15	4.524

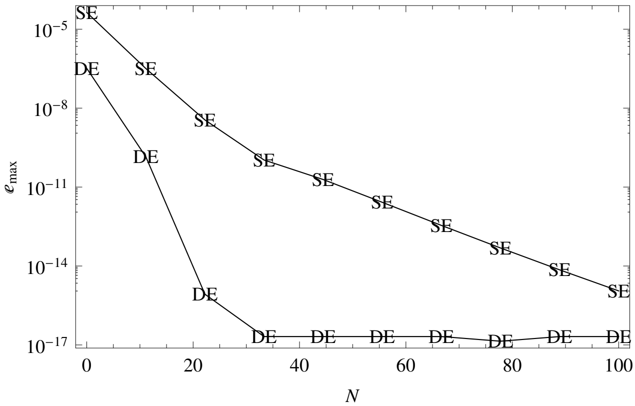

FIGURE 1. The SE and DE-sinc Nyström results for Example 5.3.

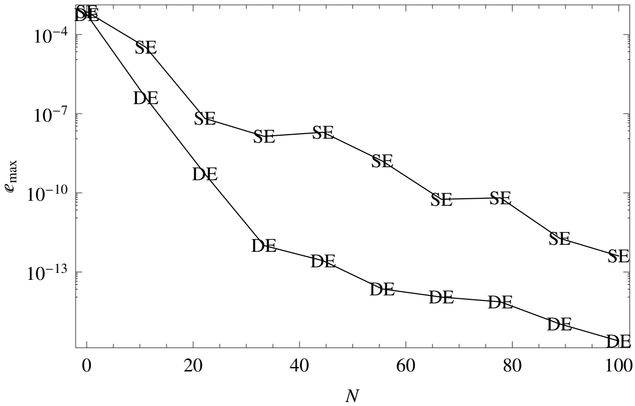

FIGURE 2. The SE and DE-sinc Nyström results for Example 5.5.

This equation has been approximated by the Multilevel Augmentation Method (MAM) based on collocation and Galerkin methods via the piecewise linear polynomial basis in [**11**]. By comparing the results of Table 2 with Tables 6.1 and 6.4 of [**11**], we can conclude that the SE and DE-results are better than the MAM results. However, the MAM is applicable for a larger class of functions.

Example 5.3. In this example, we apply the methods to the following nonlinear Fredholm integral equation of the second kind:

$$u(t) - \frac{1}{16}\int_0^1 [(1+t)\,s\,\sin(u(s)) + u^2(s)]\,\mathrm{d}s = t^2 - \frac{1+t}{32}(1-\cos(1)) - \frac{1}{80}, \quad t \in [0,1].$$

The equation has an isolated solution $u^*(t) = t^2$. This example has been mentioned in [**10**] where the author tried to extend the MAM for the Urysohn integral equation. By comparing the results of Table 1 in [**10**] and Figure 2, we can claim that the SE and DE-sinc methods have better results.

Example 5.4. Consider

$$u(t) - \int_0^1 \frac{\mathrm{d}s}{t+s+u(s)} = g(t), \quad t \in [0,1],$$

with $g(t)$ chosen so that $u^*(t) = 1/(1+t)$. This Urysohn integral equation was introduced and solved in [**8**] by projection and iterated projection methods. Tables 1 and 2 in [**8**] report the Galerkin and iterated Galerkin solutions based on a piecewise polynomial space. Table 3 shows the SE and DE-sinc Nyström results.

TABLE 3. Numerical results for Example 5.4.

N	SESN Method	T_N^{SE}	DESN Method	T_N^{DE}
10	1.67E − 05	22.791	4.19E − 06	23.556
20	3.01E − 07	24.493	2.01E − 10	24.493
30	8.76E − 09	24.866	2.02E − 14	25.069
40	2.63E − 10	26.301	4.44E − 16	24.227
50	7.59E − 12	25.506	4.44E − 16	25.427
60	7.54E − 13	25.788	4.44E − 16	27.112
70	1.61E − 13	26.052	4.44E − 16	27.238
80	2.46E − 14	27.503	6.66E − 16	31.574
90	3.77E − 15	28.346	4.44E − 16	32.258
100	1.11E − 15	28.782	4.44E − 16	37.643

TABLE 4. Numerical results for Example 5.6.

N	SESN Method	T_N^{SE}	DESN Method	T_N^{DE}
10	3.35E − 05	0.297	4.38E − 07	0.343
20	1.88E − 07	0.389	4.14E − 13	0.437
30	3.34E − 09	0.578	1.11E − 16	0.453
40	1.08E − 10	0.748	1.11E − 16	0.500
50	5.22E − 12	0.891	1.11E − 16	0.733
60	3.31E − 13	1.405	1.11E − 16	0.718
70	2.59E − 14	1.419	0.00E − 00	1.935
80	2.44E − 15	1.825	0.00E − 00	2.901
90	2.22E − 16	2.153	0.00E − 00	4.352
100	1.11E − 16	2.558	0.00E − 00	5.382

Example 5.5. For the following nonlinear Fredholm integral equation

$$u(t) - t\int_0^1 \frac{4s + \pi\sin(\pi s)}{[u(s)]^2 + s^2 + 1}\,\mathrm{d}s = -2t\log(3) + \sin(\frac{\pi}{2}t), \quad t \in [0, 1].$$

The exact solution is $u^*(t) = \sin((\pi/2)t)$. The numerical results are shown in Figure 2.

Example 5.6. We consider the following integral equation

$$u(t) + \int_0^1 \frac{t}{2}\cos(u(s))\,\mathrm{d}s = t, \quad t \in [0, 1],$$

introduced by Döring in [**13**]. Its exact solution is $u^*(t) = \mathrm{q}t$, where q is a solution of the nonlinear equation

$$2t^2 - 2t + \sin(t) = 0.$$

In [**2**] the Chebyshev-Newton Type Method (CNTM) is considered. This method constructs a family of iterative processes free of derivatives, such as the classic secant method. By comparing Table 4 with

Table 2 in [**2**], it is concluded that the presented methods are as efficient as the CNTM.

Example 5.7. The final example is the following integral equation:

$$u(t) - \int_0^1 \sin(s+t)\arctan(u(s))\,\mathrm{d}s = g(t), \quad t \in [0,1],$$

where $g(t)$ is chosen so that the isolated solution is $u(t) = \exp(t)$. Table 5 reports the SE and DE-sinc Nyström solutions. This nonlinear equation has been approximated in [**18**] by the wavelet collocation method. Table 9 in [**18**] represents the full and fast collocation solutions and full and fast multilevel solutions. Comparison of that table and Table 5 shows that the error of the sinc Nyström methods is lower.

6. Conclusion. Finding exact solutions for nonlinear Fredholm integral equations is often not possible. So approximating these solutions is very important. Many authors have proposed different methods. In this research, two numerical methods based on sinc quadrature, the SE-sinc and DE-sinc Nyström methods have been suggested. It has been shown theoretically and numerically that both schemes are extremely accurate and achieve exponential convergence with respect to

TABLE 5. Numerical results for Example 5.7.

N	SESN Method	T_N^{SE}	DESN Method	T_N^{DE}
10	1.64E − 04	17.297	6.74E − 06	17.675
20	1.35E − 06	17.566	1.23E − 10	17.455
30	3.22E − 08	17.847	7.32E − 15	17.894
40	1.36E − 09	17.752	2.55E − 15	17.596
50	8.31E − 11	18.302	2.33E − 15	17.893
60	6.61E − 12	18.142	2.55E − 15	18.471
70	6.44E − 13	18.564	2.44E − 15	20.063
80	7.56E − 14	19.204	2.33E − 15	22.136
90	1.18E − 14	19.484	2.33E − 15	23.259
100	3.77E − 15	20.031	2.33E − 15	25.257

N. These two methods have some strengths and weaknesses. In comparison with each other, as the theorems show, it is understood that the SE-sinc quadrature formulas are applicable to larger classes of functions than the DE-sinc quadrature formulas, whereas the DE-sinc formula are more efficient for well-behaved functions. In comparison with other methods, for example, the MAM and CNTM, their advantage is exponential convergence of approximate solution.

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