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LOCAL REGULARIZATION OF NONLINEAR VOLTERRA EQUATIONS OF HAMMERSTEIN TYPE

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ABSTRACT. The method of local regularization has been shown to be an effective tool for the reconstruction of solutions of linear and nonlinear inverse problems, especially those problems with special structure or for which non-smooth solutions are expected. In the case of Volterra problems, the method retains the causal structure of the original problem, in contrast to classical regularization methods, and leads to very fast sequential numerical algorithms to solve the inverse problem. Local regularization can be viewed as a generalization of *simplified* (or Lavrentiev) regularization studied by Groetsch and others, and as such can be applied to a wider variety of inverse problems; however, local regularization does not require an a priori estimate of the solution's initial value and, even if this value is known, in numerical tests local regularization frequently outperforms simplified regularization in the quality of reconstructed solution.

In this paper, we study the application of local regularization to the nonlinear Volterra problem of Hammerstein type. We improve upon the results of Lamm and Dai [**25**], where the localized approach led to a two-step solution method, i.e., one regularized linear step followed by one fully nonlinear step. Here we instead take advantage of the local nature of the method in order to simultaneously implement regularization while providing for an effective linearization strategy. The resulting method requires solving a nonlinear equation at one point only, for the initial value of the unknown solution. Thereafter the solution is reconstructed in a fast, sequential, and fully linear manner.

We present convergence results for this new method, discuss its numerical implementation and illustrate its use with numerical examples in which we compare the results of local regularization with another method well-suited for Volterra problems, the method of simplified (or Lavrentiev) regularization. In addition, we show how a modified discrepancy

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principle, similar to that studied by Groetsch and others for the method of simplified regularization, may be used to make an effective a posteriori parameter selection.

1. Introduction. In this paper, we consider the problem of solving a nonlinear Volterra equation of Hammerstein type,

$$\int_0^t k(t,s)g(s,u(s))\,ds = f(t), \quad t \in [0,1], \tag{1.1}$$

for $\overline{u} \in C[0,1]$, given suitable continuous data f. For the moment, we let the kernel k satisfy $k \in C([0,1]^2)$ and assume that the nonlinear function $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ is continuous; further assumptions on k and g are made precise below.

Hammerstein equations of Volterra type appear in many applications, e.g., in chemical absorption kinetics and in models of epidemics ([**1, 15**]). In most practical situations, the equation is ill-posed and in need of stabilization when inexact data f^δ is used in place of "ideal" data f. Unfortunately, working with perturbed or "noisy" data is unavoidable due to modeling or measurement error, round-off error, etc. We henceforth let $f^\delta \in X[0,1]$, where $X[0,1]$ is either $C[0,1]$ or $L^p(0,1)$ for $1 < p < \infty$, and we make the standing assumption that

$$\left\| f - f^\delta \right\|_{X[0,1]} \leq \delta, \tag{1.2}$$

for some fixed $\delta > 0$.

Before considering how to handle noisy data f^δ, we first give conditions on k and g which guarantee that equation (1.1) has a unique solution $\bar{u}$ given "true" data f.

1.1. Solvability of equation (1.1). Inherent in the structure of the Hammerstein integral equation (1.1) is the property that the governing nonlinear operator may be viewed as the composition of a linear integral operator K with a nonlinear operator G. Here $K \in \mathcal{L}(C[0,1])$ is the Volterra integral operator given by

$$Kv(t) = \int_0^t k(t,s)v(s)\,ds, \quad t \in [0,1],\ v \in C[0,1],$$

while $G : C[0,1] \to C[0,1]$ is the Niemytski operator induced by g,

$$Gu(t) = g(t, u(t)), \quad t \in [0,1],\ u \in C[0,1]. \tag{1.3}$$

In terms of these operators, equation (1.1) may then be written as

$$KGu = f. \tag{1.4}$$

For the (inner) *nonlinear problem* in (1.4), namely,

$$\begin{aligned} &\text{Solve} \\ &\qquad Gu = v \\ &\text{for } u \in C[0,1],\ \text{given } v \in C[0,1], \end{aligned} \tag{1.5}$$

well-posedness of this problem is found under classical assumptions on g (see, e.g., [**5, 9**]). These conditions are given in **(g1)** and **(g2)** below, while condition **(g2′)** will be of use in later sections.

Proposition 1.1. *Let $g \in C([0,1] \times \mathbf{R})$ satisfy*

(g1) $\lim_{x\to\infty} g(t,x) = \infty$ *and* $\lim_{x\to-\infty} g(t,x) = -\infty$ *for all* $t \in [0,1]$,

(g2) $(g(t,x) - g(t,y))(x-y) > 0$, *for all* $t \in [0,1], x, y \in \mathbf{R}$, $x \neq y$.

Then there exists a unique operator P which is continuous from $C[0,1]$ onto $C[0,1]$ for which

$$PGu = GPu = u \tag{1.6}$$

for all $u \in C[0,1]$, where G is defined in (1.3). *Further, if g satisfies both* **(g1)** *and the strong monotonicity condition* **(g2′)**,

(g2′) $(g(t,x) - g(t,y))(x-y) \geq \overline{c}|x-y|^2$, *for all* $t \in [0,1]$, $x, y \in \mathbf{R}$, *for some* $\overline{c} > 0$,

then the operator $P : C[0,1] \to C[0,1]$ defined in (1.6) *is uniformly continuous.*

In contrast to the well-posedness of the inner nonlinear problem (1.5) is the ill-posedness of the (outer) *linear problem* associated with the

composition in (1.4). Indeed, for kernels k of practical interest, this linear problem,

$$\begin{aligned} &\text{Solve} \\ &\qquad Kv = f \\ &\text{for } v \in C[0,1], \text{ given } f \in \text{Range}\,(K) \subset C[0,1], \end{aligned} \tag{1.7}$$

is *ill-posed* due to lack of continuous dependence of solutions v on data f. Among the kernels associated with ill-posedness are those most commonly found in the mathematical literature for first-kind linear Volterra equations, namely kernels $k \in C^1([0,1]^2)$ for which $k(t,t) \neq 0$, $t \in [0,1]$. A kernel of this kind is sometimes called *one-smoothing* because for $f \in \text{Range}\,(K) \subset C^1(0,1]$, a single differentiation of equation (1.7) with respect to t leads to the well-posed second-kind equation

$$k(t,t)v(t) + \int_0^t \frac{\partial k}{\partial t}(t,s)v(s)\,ds = f'(t), \quad t \in [0,1],$$

for which there is a unique solution v which depends continuously on $f' \in C([0,1])$ (cf., [**5, 16, 22**]). Because v is the unique solution of the first-kind equation (1.7), it is clear that one-smoothing kernels yield injective operators K; however the solution of (1.7) is not stable under perturbations in f in the $C[0,1]$ norm, rather, it is only stable in the (unnatural) C^1 norm.

Our theory below applies not only to these one-smoothing kernels, but also to so-called ν-smoothing kernels (for $\nu \geq 2$) in the case where the kernel is of convolution type. By increasing ν, the degree of instability of the original linear problem (1.7) is increased, thus allowing for a more severely ill-posed Hammerstein problem (1.1) than one would have under the standard one-smoothing kernel assumption. We note that while ν-smoothing kernels may also be defined for $\nu \geq 2$ in the nonconvolution case, it is not known whether the ν-smoothing nonconvolution operator $(\alpha I + K)^{-1}$ satisfies a needed growth condition as $\alpha \to 0$; this condition *does* hold for both one-smoothing nonconvolution kernels and general ν-smoothing convolution kernels (see Lemma 3.1 below).

The following definition describes the wide class of kernels to which the theory developed in this paper applies.

Definition 1.1. For the purposes of this paper, we will say that the kernel k is *ν-smoothing* and the associated operator K is a *ν-smoothing operator*, $\nu = 1, 2, \ldots$, if k satisfies one of the following conditions:

• If $\nu = 1$, then $k = k(t,s)$ satisfies $k \in C^1([0,1]^2)$ with $k(t,t) \neq 0$, $t \in [0,1]$.

• If $\nu = 2, 3, \ldots$, then there exists a $\kappa \in C^\nu([0,1])$ for which $k(t,s) = \kappa(t-s)$ for all $0 \leq s \leq t \leq 1$, where κ satisfies

$$\frac{d^\ell \kappa}{dt^\ell}(0) = 0, \quad \ell = 0, 1, \ldots, \nu - 2, \quad \text{and} \quad \frac{d^{(\nu-1)}\kappa}{dt^{(\nu-1)}}(0) \neq 0.$$

Without loss of generality we may assume that equation (1.1) has been rescaled so that $k(t,t) = 1$, $t \in [0,1]$, in the case of $\nu = 1$, and $d^{(\nu-1)}\kappa/dt^{(\nu-1)}(0) = 1$ for $\nu \geq 2$.

Henceforth we assume that the kernel k in (1.1) is ν-smoothing for some integer $\nu \geq 1$. Combining the last proposition with well-known results on ill-posedness of linear Volterra equations (see, e.g., [**16**, Chapter 5], [**22**]), we have the following.

Proposition 1.2. *If k is a ν-smoothing kernel for $\nu \geq 1$ and $g \in C([0,1] \times \mathbf{R})$ satisfies* **(g1)** *and* **(g2)**, *then for any $f \in \text{Range}\,(K)$ there exists a unique solution u of* (1.1); *however the equation is ill-posed due to a lack of continuous dependence of solutions u on data $f \in C[0,1]$.*

2. Background. It is worth making brief note of some features of the ill-posed *linear* Volterra problem (1.7) in order to better understand the regularization method proposed in this paper for the stable solution of (1.1); these same features carry over to the nonlinear problem.

• The linear Volterra problem (1.7) is causal or non-anticipatory in the sense that, given some $t \in (0,1)$, the value of f on the interval $[0,t]$ depends only on the value of v on $[0,t]$. Thus Volterra problems evolve in a time-like manner and it is natural that solution methods for such problems be sequential in nature [**22**].

• Again for some $t \in (0,1)$, the value of v at t impacts $Kv(\tau) = f(\tau)$ for $\tau \in (t,1]$ (provided the kernel k remains nonzero), so that useful

information about $v(t)$ can be expected to be found in "future" values of the data f. However, it is also the case that $f(\tau)$, $\tau \in (t, 1]$, contains information about $v(s)$ for $s \in (t, \tau]$, so that the usefulness of $f(\tau)$ in the reconstruction of $v(t)$ decreases as τ increases. So although a Volterra problem is naturally solved in a sequential manner (determining $v(t)$ solely from $f(t)$ and previously determined values of v on $[0, t)$), when the data is imperfect–or when computations are subject to error–the reconstruction $v(t)$ can be improved by using a small amount of future data, say, $f(\tau)$ for $\tau \in (t, t + \rho]$, where $\rho > 0$ is small.

• For $t \in (0, 1)$ but t near 1, the interval $(t, t+\rho]$ of available "future" data f is shrinking, with $\rho \to 0$ as $t \to 1$. As a result, reconstructions of solutions with even classical regularization methods tend to deteriorate near the end of the interval $[0, 1]$ due to the loss of useful future data. This is an inherent feature of the Volterra problem itself and is not related to any particular regularization or solution method.

From the above discussion, it is clear that one can only realistically expect to accurately reconstruct the solution v of (1.7) on $[0, 1 - \rho)$ for $\rho > 0$ small, or else require data on a slightly extended interval in order to estimate v on all of $[0, 1]$. We will take the latter approach in this paper and assume that a small amount of data is available on the interval $[0, 1 + \overline{\alpha}]$, for some fixed $\overline{\alpha} > 0$ small. So we must assume that the original equation (1.1) (equivalently (1.4)) holds on this extended interval as well, with definitions of K, G, f^δ, and $\overline{u}$ carried over to this interval, and that the standing assumption (1.2) on data error now becomes

$$\left\| f - f^\delta \right\|_{X[0,1+\overline{\alpha}]} \leq \delta, \tag{2.1}$$

for $\delta > 0$.

2.1. Existing regularization methods for the solution of (1.1). Before discussing regularization methods which work directly on the full nonlinear Hammerstein problem (1.1), we first examine two-step approaches which exploit the linear-nonlinear decomposition in equation (1.4). Any viable regularization method for the stable solution of the (outer) linear problem (1.7) may be used as the first step in a two-step process of solving (1.1); such methods include classical regularization methods due to Tikhonov-Phillips, Landweber, etc. [**10,**

17]. In the second step of this process, the nonlinear problem (1.5) is solved, a step which could require considerable computational resources despite its well-posedness.

In regularizing the linear Volterra problem (1.7), it is well-known that methods such as Tikhonov regularization and other classical methods based on defined functions of $K^\star K$, for $K^\star$ the Hilbert-adjoint of K, are inefficient because the causal nature of the Volterra problem is not preserved [**22**] (i.e., the adjoint $K^\star$ is an anticipatory operator, while the Volterra operator K is non-anticipatory). So we limit our discussion here to two regularization methods, namely, the methods of *simplified* (or Lavrentiev) *regularization* and *local regularization*, for which it is well-known that with either, the causal nature of the original Volterra problem is preserved [**22**].

Simplified regularization has been studied by Groetsch and numerous other authors, although not necessarily in the context of the Volterra problem (see, e.g., [**13, 18, 19, 30, 34**]). For (1.7) in the case of noisy data f^δ, the method takes the form of

$$\alpha v + Kv = f^\delta, \tag{2.2}$$

where $\alpha > 0$ is a regularization parameter. Note however that equation (2.2) imposes the unlikely condition $v(0) = f^\delta(0)/\alpha$ (for example, consider the case of $f^\delta \equiv f$), so the following variation of (2.2) is often used instead, namely,

$$\alpha(v - \overline{v}(0)) + Kv = f^\delta. \tag{2.3}$$

Here the value of $\overline{v}(0)$ must be known a priori, where $\overline{v}$ is used to denote the "true" solution of (1.7). Despite the necessity of this additional information, it is clear from (2.3) that simplified regularization retains the causal structure of the original Volterra problem. In general, the theoretical analysis of the method is limited to only specialized operators K, for example, to self-adjoint (not applicable in the case of Volterra operators) or monotone K. Janno has shown that the linear Volterra operator K is monotone provided K is both one-smoothing and of convolution type, so the theory of simplified regularization does apply in this case (see [**19**, Lemma 3]). It is not known however whether the method is valid for Volterra problems with the more general ν-smoothing operators K.

An alternative to simplified regularization is the method of *local regularization* by which a small amount of future data is used to reconstruct the solution v of (1.7) at any given value of $t \in [0,1]$; the future data is then consolidated or averaged in a way which allows the equation for local regularization to retain a Volterra structure.

We briefly describe here the method of local regularization in the context of the linear problem (1.7), as the method we develop in Section 3 for the full Hammerstein problem is built upon this structure. The idea is that for each $\alpha \in (0, \overline{\alpha}]$, we obtain a small amount of future information from equation (1.7) by advancing the equation from t to $t+\rho$, for $\rho \in [0,\alpha]$, which gives (after splitting the integral),

$$\int_t^{t+\rho} k(t+\rho,s)v(s)\,ds + \int_0^t k(t+\rho,s)v(s)\,ds = f(t+\rho), \tag{2.4}$$

for $t \in [0,1]$. In order to consolidate information on the future interval $[t, t+\rho]$, we integrate both sides of (2.4) with respect to a signed Borel measure $\eta_\alpha = \eta_\alpha(\rho)$, where η_α satisfies conditions to be detailed shortly. (The Lebesgue measure could be used for now, as it satisfies these conditions for small ν.) After simplification, the equation takes the form of

$$\begin{aligned}\int_0^\alpha \int_0^\rho k(t+\rho,t+s)v(t+s)\,ds\,d\eta_\alpha(\rho) &\\ + \int_0^t \int_0^\alpha k(t+\rho,s)\,d\eta_\alpha(\rho)\,v(s)\,ds &\\ = \int_0^\alpha f(t+\rho)\,d\eta_\alpha(\rho), \quad t \in [0,1], &\end{aligned} \tag{2.5}$$

an equation which is still ill-posed and is satisfied by the "true" solution of (1.7), provided the "true" data f is used in (2.5).

To handle the situation where f in (2.5) is replaced by noisy data f^δ, stability is added to the equation by (temporarily) imposing on the first term of (2.5) the rigidity condition $v(t+s) = v(t)$, for s in the small interval $[0,\alpha]$. The condition is only heuristically imposed for all values of $t \in [0,1]$; by this we mean that the rigidity condition actually need never hold precisely but, rather, gives rise to a new equation,

$$b_\alpha v + K_\alpha v = f_\alpha^\delta, \tag{2.6}$$

where for $t \in [0,1]$,

$$b_\alpha(t) := \int_0^\alpha \int_0^\rho k(t+\rho, t+s)\, ds\, d\eta_\alpha(\rho), \tag{2.7}$$

$$f_\alpha^\delta(t) := \int_0^\alpha f^\delta(t+\rho)\, d\eta_\alpha(\rho), \tag{2.8}$$

and K_α is a Volterra operator given by

$$K_\alpha v(t) := \int_0^t k_\alpha(t,s)\, v(s)\, ds, \quad v \in C[0,1], \tag{2.9}$$

with kernel

$$k_\alpha(t,s) := \int_0^\alpha k(t+\rho, s)\, d\eta_\alpha(\rho), \ 0 \le s \le t \le 1. \tag{2.10}$$

In fact we can simplify (2.6) further by noting that if the kernel k is of convolution type, then K_α is also of convolution type and the quantity b_α is independent of t. This latter fact is approximately true in the one-smoothing nonconvolution case because $k(t+\rho, t+s) \approx k(t,t) = 1$ for all $t \in [0,1]$ and ρ, s small. That is, in this case,

$$b_\alpha \approx \int_0^\alpha \rho\, d\eta_\alpha(\rho).$$

Using these observations, we modify (2.6) to obtain the classical *local regularization equation*,

$$a_\alpha v + K_\alpha v = f_\alpha^\delta, \tag{2.11}$$

where the scalar a_α depends on α and ν via

$$a_\alpha = \begin{cases} \displaystyle\int_0^\alpha \rho\, d\eta_\alpha(\rho) & \text{if } \nu = 1, \\ \displaystyle\int_0^\alpha \int_0^\rho \kappa(\rho - s)\, ds\, d\eta_\alpha(\rho), & \text{if } \nu > 1, \end{cases} \tag{2.12}$$

where κ is given in Definition 1.1.

The linear theory of local regularization is well-developed [**2, 3, 6, 20–28, 31**]. For suitably defined measures η_α and all $\alpha > 0$ sufficiently small, it is known that the coefficient a_α is positive for $\nu = 1, 2, \ldots$, and thus the local regularization equation (2.11) is a well-posed second-kind Volterra equation with unique solution $v_\alpha^\delta \in C[0,1]$, where v_α^δ depends continuously on the data $f^\delta \in X[0,1]$. Further, if we make an appropriate selection of $\alpha = \alpha(\delta)$ as $\delta \to 0$, we have the usual regularized convergence result that $v_\alpha^\delta \to \overline{v}$ in $C[0,1]$ as $\delta \to 0$.

2.2. Implementation of existing methods for the solution of (1.1). In practice either simplified regularization or local regularization may be used to first solve the (outer) ill-posed linear Volterra problem (1.7) to obtain a stable reconstruction of v; in the second step we must apply nonlinear solution techniques in order to determine an estimate u for the solution of the nonlinear problem (1.5). Theoretical results for a two-step process based on simplified regularization are, as expected, limited to specialized operators K, while the analysis of the approach based on local regularization may be found in [**25**] for general ν-smoothing convolution operators K and continuous data f^δ.

Alternatively, simplified regularization can be very easily applied *directly* to the full nonlinear Hammerstein problem (1.4) (equivalently, (1.1)) in the case of noisy data f^δ. That is, if $\overline{u}(0)$ is known, the regularization equation becomes

$$\alpha(u - \overline{u}(0)) + KGu = f^\delta, \tag{2.13}$$

which may be used to solve the Hammerstein problem in a single step instead of two. The method still preserves the causal nature of the original problem and leads naturally to a sequential solution method since the solution u_α^δ of (2.13) may be found via

$$\begin{aligned} &u_\alpha^\delta(0) = \overline{u}(0), \\ &u_\alpha^\delta(t) = \overline{u}(0) + \frac{1}{\alpha} f^\delta(t) - \frac{1}{\alpha} \int_0^t k(t,s) g(s, u_\alpha^\delta(s))\, ds, \quad t \in (0,1], \end{aligned} \tag{2.14}$$

where the integral in the final term of (2.14) can be evaluated without using the value of u_α^δ at t. So in (2.14), the value of the function u_α^δ at t is computed directly from its values on $[0,t)$ and the method

of simplified regularization for the nonlinear Hammerstein problem becomes a sequential, fully-*linear* solution method.

Similarly, a direct application of local regularization directly to the full nonlinear Hammerstein problem can be written, in the case of noisy data,

$$a_\alpha Gu + K_\alpha Gu = f_\alpha^\delta, \tag{2.15}$$

which also leads to a sequential method because the solution u_α^δ of (2.15) may be found from

$$g(t, u_\alpha^\delta(t)) = \frac{1}{a_\alpha} f_\alpha^\delta(t) - \frac{1}{a_\alpha} \int_0^t k_\alpha(t,s) g(s, u_\alpha^\delta(s))\, ds, \quad t \in (0,1], \tag{2.16}$$

where again the last integral in (2.16) is evaluated without using the value of u_α^δ at t. However, in contrast to simplified regularization, one must still solve a nonlinear problem at every step to recover the value of $u_\alpha^\delta(t)$ from $g(t, u_\alpha^\delta(t))$.

The purpose of this paper is to remedy this situation for the method of local regularization, making modifications which lead to a fully linear solution method for $u_\alpha^\delta(t)$, $t \in (0,1]$, such as occurs for the method of simplified regularization. In the next section, we derive the modified method and develop an associated convergence theory. Finally, in Section 4, we show by example how our new method is capable of outperforming the method of simplified regularization.

3. Local regularization for the full Hammerstein problem. The needed modification of (2.15) is based on the simple observation that the (small) regularization interval on which the method is based is also a good place to facilitate a linearization of any nonlinear terms, in this case, the nonlinear Gu in the first term of (2.15).

To this end, let $\tau_\alpha \in C[0,1]$ be given satisfying

$$0 \le \tau_\alpha(t) \le \min\{t, \alpha\}, \quad t \in [0,1]. \tag{3.1}$$

Then we seek u_α^δ satisfying the new local regularization equation,

$$a_\alpha \mathcal{G}_{\tau_\alpha} u + K_\alpha Gu = f_\alpha^\delta \tag{3.2}$$

where for $w \in C[0,1]$ and $t \in [0,1]$,

(3.3)
$$\mathcal{G}_{\tau_\alpha} w(t) := g(t, w(t-\tau_\alpha(t))) + g_x(t, w(t-\tau_\alpha(t)))\,[w(t) - w(t-\tau_\alpha(t))]\,.$$

If $\tau_\alpha \equiv 0$, then (3.2) reduces to (2.15). Otherwise, if $\tau_\alpha(t) > 0$ for all $t \in (0,1]$, then equation (3.2) is the same as (2.15) at the point $t = 0$ only and $u_\alpha^\delta(0)$ is the solution of the nonlinear equation

$$g(0, u(0)) = \frac{1}{a_\alpha} \int_0^\alpha f^\delta(\rho)\, d\eta_\alpha(\rho). \tag{3.4}$$

Once $u_\alpha^\delta(0)$ has been determined, $u_\alpha^\delta(t)$ can be found for $t \in (0,1]$ directly from $u_\alpha^\delta(s)$, $s \in [0,t)$, via

$$\begin{aligned} u_\alpha^\delta(t) = u_\alpha^\delta(t-\tau_\alpha(t)) &- \frac{g(t, u_\alpha^\delta(t-\tau_\alpha(t)))}{g_x(t, u_\alpha^\delta(t-\tau_\alpha(t)))} \\ &+ \frac{1}{a_\alpha\, g_x(t, u_\alpha^\delta(t-\tau_\alpha(t)))} \left[\int_0^\alpha f^\delta(t+\rho) d\eta_\alpha(\rho) \right] \\ &- \frac{1}{a_\alpha g_x(t, u_\alpha^\delta(t-\tau_\alpha(t)))} \left[\int_0^t k_\alpha(t,s)\, g(s, u_\alpha^\delta(s))\, ds \right]. \end{aligned} \tag{3.5}$$

(Based on (g2′) and (3.14) below, equations (3.4)–(3.5) are well-defined.) As before, the integral in the final term of (3.5) can be evaluated without using the value of u_α^δ at t, so the local regularization method associated with equation (3.2) is a linear, sequential regularization method for all $t > 0$. See Section 4 for more discussion regarding implementation of this method.

Note that we deliberately did not require the size of the linearization interval to be the same as the size of the regularization interval; indeed, although one may need to increase the size of the regularization parameter α when the noise level δ is large, this should generally not affect (in a significant way) the interval associated with linearization.

Remark 3.1. It is worth noting that while equation (2.15) is a second-kind nonlinear equation of Volterra type, the new equation (3.2) with $\tau_\alpha(t) > 0$, $t \in (0,1]$, is a *functional integral equation*, or a *delay integral equation* with discrete finite delays.

In the remainder of this section we investigate the well-posedness of equation (3.2) and perform a convergence analysis appropriate for the regularization method.

3.1. Theoretical analysis. The following definitions will be needed in this section.

For any fixed $\epsilon > 0$, we define a modulus of continuity for $\overline{u} \in C[0, 1+\overline{\alpha}]$,

$$\mu(\epsilon, \overline{u}) := \max\{|\overline{u}(t) - \overline{u}(s)|\,;\ t, s \in [0, 1+\overline{\alpha}],\ |t-s| \le \epsilon\}\,, \tag{3.6}$$

where $\mu(\epsilon, \overline{u}) \to 0$ as $\epsilon \to 0$. We can obtain more specific results when $\overline{u}$ satisfies additional smoothness assumptions, so we also consider the possibility of $\bar{u}$ satisfying the Hölder condition

$$|\overline{u}(t) - \overline{u}(s)| \le L_\xi |t-s|^\xi, \tag{3.7}$$

for $0 < \xi \le 1$, $L_\xi > 0$, and all $t, s \in [0, 1+\overline{\alpha}]$.

We define the interval J as follows:

$$J \supset \{\overline{u}(t) : t \in [0, 1+\overline{\alpha}]\},\ \ J \text{ open}, \tag{3.8}$$

and use the notation $C([0,1] \times J)$ to denote the space of bounded continuous functions on $[0,1] \times J$ with norm denoted by $\|\cdot\|_J$,

$$\|h\|_J := \sup_{(t,x)\in[0,1]\times J} |h(t,x)| < \infty, \quad h \in C([0,1] \times J).$$

As usual, $h \in C^1([0,1] \times J)$ if h is differentiable and $h, h_t, h_x \in C([0,1] \times J)$.

Definition 3.1. We call $\{\eta_\alpha\}_{\alpha\in(0,\overline{\alpha}]}$ a *local-regularizing family of measures* for problem (1.4) provided that, for $\alpha \in (0, \overline{\alpha}]$, each η_α is a signed Borel measure on $[0, \alpha]$, normalized such that $\eta_\alpha([0,\alpha]) = 1$, and the family $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ satisfies the following conditions:

(A1) For each $j = 0, 1, \ldots, \nu$,

$$\int_0^\alpha \rho^j d\eta_\alpha(\rho) = \alpha^j\,(c_j + \mathcal{O}(\alpha)) \quad \text{for all} \ \ \alpha \in (0, \overline{\alpha}], \tag{3.9}$$

where the constants $c_0, c_1, \ldots, c_\nu \in \mathbf{R}$ and $c_\nu \neq 0$ are such that the roots of the polynomial $p_\nu(x)$, defined by

$$p_\nu(x) = \frac{c_\nu}{\nu!}x^\nu + \frac{c_{\nu-1}}{(\nu-1)!}x^{\nu-1} + \cdots + \frac{c_1}{1!}x + \frac{c_0}{0!}, \tag{3.10}$$

have negative real part.

(A2) There exists $\widetilde{C} > 0$ such that

$$\int_0^\alpha |h(\rho)|\, d\,|\eta_\alpha|\,(\rho) \leq \widetilde{C}\,\|h\|_{X[0,\alpha]}, \quad \text{for all } \alpha \in (0, \overline{\alpha}],$$

for all $h \in X[0,\alpha]$. Here $|\eta_\alpha|$ denotes the total variation of the measure η_α.

Remark 3.2. Standard classes of measures used in local regularization satisfy both assumptions (A1) and (A2) [**24**]. While condition (A1) is not used explicitly in this paper, it is needed for essential results (e.g., Lemma 3.1 below) upon which this paper relies. Condition (A2) implies that $f_\alpha^\delta \in C[0,1]$ when $f^\delta \in X[0, 1+\overline{\alpha}]$, for f_α^δ defined in (2.8). Indeed, this follows from (A2) and from continuity of translations on $X[0, 1+\overline{\alpha}]$. Further, if f_α is defined as expected from f, i.e.,

$$f_\alpha(t) := \int_0^\alpha f(t+\rho)\, d\eta_\alpha(\rho), \quad t \in [0,1], \tag{3.11}$$

then

$$\left\| f_\alpha - f_\alpha^\delta \right\| \leq \widetilde{C} \left\| f - f^\delta \right\|_{X[0,1+\alpha]} \leq \widetilde{C}\delta.$$

Theorem 3.1 is the main well-posedness/approximation result associated with equation (3.2).

Theorem 3.1. *Let $\overline{u}$ denote the solution of* (1.1) *with "true" data $f \in C[0, 1+\overline{\alpha}]$ and where $g \in C^1([0,1+\overline{\alpha}] \times J)$ satisfies* (g1) *and* (g2′). *For equation* (3.2), *assume that $f^\delta \in X[0, 1+\overline{\alpha}]$ satisfies* (2.1) *and that $\{\eta_\alpha\}$ is a local-regularizing family of measures. Then for $\overline{\alpha}$ sufficiently small the following are true.*

(1) *If* $\delta \sim \alpha^{\nu}\lambda(\alpha, \overline{u})$, *where*

$$\lambda(\alpha, \overline{u}) := \max\{\alpha,\, \mu(\alpha, \overline{u})\}, \tag{3.12}$$

then for every $\alpha \in (0, \overline{\alpha}]$ *there exists a unique solution* $u^{\delta}_{\alpha(\delta)} \in C([0,1]; J)$ *of* (3.2) *for which*

$$\|u^{\delta}_{\alpha(\delta)} - \overline{u}\| = \mathcal{O}\big(\lambda(\alpha(\delta), \overline{u})\big) \longrightarrow 0 \quad \textit{as} \quad \delta \to 0.$$

(1′) *If, in addition,* $\overline{u}$ *satisfies the Hölder condition* (3.7) *with Hölder exponent* ξ, *and if* $\alpha = \alpha(\delta)$ *is selected satisfying* $\alpha(\delta) \sim \delta^{1/(\nu+\xi)}$ *as* $\delta \to 0$, *then*

$$\|u^{\delta}_{\alpha(\delta)} - \overline{u}\| = \mathcal{O}\big(\delta^{\xi/(\nu+\xi)}\big) \longrightarrow 0 \quad \textit{as} \quad \delta \to 0.$$

(2) *The mapping* $f^{\delta} \in \{h \in X[0, 1+\overline{\alpha}] \mid \|h - f\| \le \delta\} \to u^{\delta}_{\alpha} \in C([0,1]; J)$ *is continuous for each* $\alpha \in (0, \overline{\alpha}]$.

Before proving Theorem 3.1, we state two lemmas which facilitate the arguments in the proof. The first lemma contains known results regarding the quantities K_{α} and a_{α} defined in (2.9) and (2.12), respectively.

Lemma 3.1 [**23, 24, 31**]. *Let* $\{\eta_{\alpha}\}_{\alpha \in (0,\overline{\alpha}]}$, *be a local-regularizing family of measures. Then there exists a constant* $\overline{C}$ *for which*

$$\|k\|_{C[0,\alpha]} \le \overline{C}\alpha^{\nu-1} \tag{3.13}$$

for every $\alpha \in (0, \overline{\alpha}]$. *Further, if* $\overline{\alpha} > 0$ *is sufficiently small, there are constants* $0 < C_1 < C_2$ *such that*

$$C_1\alpha^{\nu} \le a_{\alpha} \le C_2\alpha^{\nu}, \quad \alpha \in (0, \overline{\alpha}], \tag{3.14}$$

and constants $C > 0$ *and* $\mathcal{M} > 0$ *such that if* k *satisfies* $\|k^{(\nu)}\|_{C[0,\overline{\alpha}]} \le C$ *then* $(a_{\alpha}I + K_{\alpha})^{-1} : C[0,1] \to C[0,1]$ *is a bounded linear operator, with operator norm satisfying*

$$\left\| (a_\alpha I + K_\alpha)^{-1} \right\|_{\mathcal{L}(C[0,1])} \leq \frac{\mathcal{M}}{a_\alpha}, \tag{3.15}$$

for all $\alpha \in (0, \overline{\alpha}]$.

In this second lemma, we establish some useful technical estimates.

Lemma 3.2. *Let* g *satisfy the assumptions of Theorem* 3.1*, and let* $\Delta \in [0, \overline{\alpha}]$ *be fixed. Let* $v, w \in \widetilde{M}_z$*, where for some* $z \in C[0, 1+\Delta]$ *and* $c_z > 0$*,*

$$\widetilde{M}_z := \{u \in C([0,1]; J) \mid \|Gu - Gz\| \leq c_z\}.$$

If $s_\alpha(\cdot)$ *and* $r_\alpha(\cdot)$ *satisfy*

$$0 \leq s_\alpha(t),\ r_\alpha(t) \leq \min\{t, \alpha\}, \quad t \in [0, 1+\Delta],$$

then

$$\|v - z\| \leq \frac{c_z}{\overline{c}}, \tag{3.16}$$

(3.17)
$$\begin{aligned} &\big\| g(\cdot, v(\cdot - s_\alpha)) - g(\cdot, w(\cdot - s_\alpha)) - g_x(\cdot, w(\cdot - r_\alpha))\big(v(\cdot - s_\alpha) - w(\cdot - s_\alpha)\big) \big\| \\ &\qquad \leq \overline{m}\left(\frac{4c_z}{\bar{c}} + 2\mu(\alpha, z) \right) \|v(\cdot - s_\alpha) - w(\cdot - s_\alpha)\|, \end{aligned}$$

and thus

(3.18)
$$\begin{aligned} &\|g(\cdot, v(\cdot - s_\alpha)) - g(\cdot, w(\cdot - s_\alpha)) - g_x(\cdot, w(\cdot - r_\alpha))(v(\cdot - s_\alpha) - w(\cdot - s_\alpha))\| \\ &\qquad \leq \frac{2\overline{m}c_z}{\overline{c}} \left(\frac{4c_z}{\bar{c}} + 2\mu(\alpha, z) \right). \end{aligned}$$

Proof. Using (g2′) we have $\|v - z\| \leq 1/\overline{c}\|Gv - Gz\|$, from which (3.16) follows.

Let $t \in [0, 1+\Delta]$, and let $s = s_\alpha(t)$ and $r = r_\alpha(t)$. Then using the assumed regularity of g,

$$\begin{aligned}|g(t, v(t-s)) - g(t, w(t-s)) - g_x(t, w(t-r))(v(t-s) - w(t-s))| \\ = |g_x(t, \psi(t, s, v, w)) - g_x(t, w(t-r))|\,|v(t-s) - w(t-s)|,\end{aligned}$$

for $\min\{v(t-s), w(t-s)\} \le \psi(t, s, v, w) \le \max\{v(t-s), w(t-s)\}$, so

$$\begin{aligned}&|\psi(t, s, v, w) - w(t-r)| \\ &\quad \le |\psi(t, s, v, w) - w(t-s)| + |w(t-s) - z(t-s)| \\ &\qquad + |z(t-s) - z(t)| + |z(t) - z(t-r)| + |z(t-r) - w(t-r)| \\ &\quad \le |v(t-s) - w(t-s)| + \frac{2c_z}{\overline{c}} + 2\mu(\alpha, z) \\ &\quad \le \frac{4c_z}{\overline{c}} + 2\mu(\alpha, z).\end{aligned}$$

Thus

$$\begin{aligned}\|g(\cdot, v(\cdot - s)) - g(\cdot, w(\cdot - s)) - g_x(\cdot, w(\cdot - r))(v(\cdot - s) - w(\cdot - s))\| \\ \le \overline{m}\|\psi(\cdot, s, v, w) - w(\cdot - r)\|\,\|v(\cdot - s) - w(\cdot - s)\| \\ \le \overline{m}\left(\frac{4c_z}{\overline{c}} + 2\mu(\alpha, z)\right)\|v(\cdot - s) - w(\cdot - s)\|,\end{aligned}$$

proving (3.17). The inequality in (3.18) then follows from (3.16). □

We now return to the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $\alpha \in (0, \overline{\alpha}]$ be fixed. Applying the same arguments that we used to derive equation (2.5) to the full Hammerstein equation (1.1), we obtain the nonlinear analog of (2.5), an equation still satisfied by $\overline{u}$,

$$\begin{aligned}(3.19)\quad &\int_0^\alpha \int_0^\rho k(t+\rho, t+s) g(t+s, u(t+s))\, ds\, d\eta_\alpha(\rho) \\ &\qquad + \int_0^t \int_0^\alpha k(t+\rho, s)\, d\eta_\alpha(\rho)\, g(s, u(s))\, ds \\ &\qquad = \int_0^\alpha f(t+\rho)\, d\eta_\alpha(\rho),\end{aligned}$$

for $t \in [0,1]$. That is,

$$a_\alpha G\overline{u} + K_\alpha G\overline{u} = f_\alpha + \left(a_\alpha G\overline{u} - D_\alpha G\overline{u}\right), \tag{3.20}$$

where f_α is given by (3.11) and $D_\alpha : C[0, 1+\alpha] \to C[0,1]$ is defined, for $v \in C[0, 1+\overline{\alpha}]$, by

$$D_\alpha v(t) := \int_0^\alpha \int_0^\rho k(t+\rho, t+s) v(t+s)\, ds\, d\eta_\alpha(\rho), \quad t \in [0,1].$$

But u_α^δ satisfies (3.2), rewritten here as

$$a_\alpha Gu + K_\alpha Gu = f_\alpha^\delta + a_\alpha \left[Gu - \mathcal{G}_{\tau_\alpha} u\right], \tag{3.21}$$

so subtracting equation (3.20) from equation (3.21), u_α^δ also solves

$$\left(a_\alpha I + K_\alpha\right)\left[Gu - G\overline{u}\right] = F_\alpha^\delta + a_\alpha \left[Gu - \mathcal{G}_{\tau_\alpha} u\right], \tag{3.22}$$

where F_α^δ is given by

$$F_\alpha^\delta := f_\alpha^\delta - f_\alpha + \left[D_\alpha G\overline{u} - a_\alpha G\overline{u}\right]. \tag{3.23}$$

Therefore u_α^δ satisfies

$$u = H_\alpha u, \tag{3.24}$$

for $H_\alpha : C([0,1]; J) \to C[0,1]$ defined for $w \in C([0,1]; J)$ via

$$H_\alpha w := P\left[\left(a_\alpha I + K_\alpha\right)^{-1}\left(F_\alpha^\delta + a_\alpha Gw - a_\alpha \mathcal{G}_{\tau_\alpha} w\right) + G\overline{u}\right], \tag{3.25}$$

where P was defined in Proposition 1.1.

Proof of (1). For $\overline{\alpha} > 0$ sufficiently small, there is a $\gamma > 0$ satisfying

$$\delta \leq \gamma \alpha^\nu \lambda(\alpha, \overline{u}) \tag{3.26}$$

for all $\alpha \in (0, \overline{\alpha}]$. Let

$$\theta = \mathcal{M}\left[1 + 2\left\|g_x\right\|_J + \frac{\widetilde{C}\gamma}{C_1} + \frac{2\widetilde{C}\overline{C}}{C_1}\left(\left\|g_x\right\|_J + \left\|g_t\right\|_J\right) + \frac{c_2}{2C_1}\left\|k\right\|_{C^1(J)}\left\|g\right\|_J\right].$$

Then, making use of (3.8) and the fact that $\lambda(\alpha, \overline{u}) \to 0$ as $\alpha \to 0$, it follows that for $\overline{\alpha} > 0$ sufficiently small and all $\alpha \in (0, \overline{\alpha}]$ we have

$$\left\{ y \in \mathbf{R} \;\middle|\; \max_{t\in[0,1]} |\overline{u}(t) - y| \le \frac{\theta}{\overline{c}}\, \lambda(\alpha, \overline{u}) \right\} \subset J, \tag{3.27}$$

$$\frac{4\mathcal{M}\overline{m}\,\theta\lambda(\alpha,\overline{u})}{\overline{c}} \left(\frac{4\,\theta}{\overline{c}} + 2 \right) \le 1, \tag{3.28}$$

and, for some fixed $d \in (0,1)$,

$$\frac{\mathcal{M}}{\overline{c}} \left[2\bar{m} \left(\frac{4\theta\lambda(\alpha,\overline{u})}{\bar{c}} + 2\lambda(\alpha,\overline{u}) \right) + \|g_x\|_J \, \frac{2\theta + \overline{c}}{\overline{c}} \lambda(\alpha,\overline{u}) \right] \le d. \tag{3.29}$$

Henceforth, we fix this value of $\overline{\alpha}$. In what follows we also make use of the bounds (3.16)–(3.18) from Lemma 3.2, using the parameter values $c_z = \theta\lambda(\alpha,\overline{u})$, $\Delta = 0$, $z = \overline{u}$, and $\widetilde{M}_z = M_\alpha^\theta$ (defined below) in that lemma.

The verification of (1) comes from showing that for all $\alpha \in (0, \overline{\alpha}]$, the mapping H_α is a contraction on the set

$$M_\alpha^\theta := \left\{ u \in C([0,1]; J) \;\middle|\; \left\| Gu - G\overline{u} \right\| \le \theta\lambda(\alpha,\overline{u}) \right\}.$$

Let $\alpha \in (0, \overline{\alpha}]$ be arbitrary. The proof of the desired result is in three parts.

Part 1. $M_\alpha^\theta \subset C[0,1]$ is closed, nonempty.

Let $u_i \in M_\alpha^\theta$ satisfy $u_i \to u \in C[0,1]$. Then $Gu_i \to Gu$ and thus $\|Gu - G\overline{u}\| \le \theta\lambda(\alpha,\overline{u})$. It follows from (3.16) that $\|u(t) - \overline{u}(t)\| \le \theta\lambda(\alpha,\overline{u})/\overline{c}$ for all $t \in [0,1]$. Thus $u(t) \in J$, for all $t \in [0,1]$, and $u \in M_\alpha^\theta$. In addition, $\overline{u} \in M_\alpha^\theta$ so M_α^θ is not empty.

Part 2. $H_\alpha u \in M_\alpha^\theta$, for all $u \in M_\alpha^\theta$.

From the definitions of F_α^δ and $H_\alpha u$ in (3.23) and (3.25), respectively, we have

$$\begin{aligned} \|GH_\alpha u - G\overline{u}\| &\le \frac{\mathcal{M}}{a_\alpha} \left\| a_\alpha Gu - a_\alpha \mathcal{G}_{\tau_\alpha} u + F_\alpha^\delta \right\| \\ &\le \mathcal{M} \sum_{i=1}^{6} T_i, \end{aligned} \tag{3.30}$$

where

$$
\begin{aligned}
T_1 &:= \|Gu(\cdot) - G\overline{u}(\cdot) - g_x(\cdot, u(\cdot - \tau_\alpha(\cdot)))\,[u(\cdot) - \overline{u}(\cdot)]\| \\
T_2 &:= \|G\overline{u}(\cdot) - g(\cdot, \overline{u}(\cdot - \tau_\alpha(\cdot)))\| \\
T_3 &:= \|g(\cdot, \overline{u}(\cdot - \tau_\alpha(\cdot))) - g(\cdot, u(\cdot - \tau_\alpha(\cdot))) \\
&\qquad -g_x(\cdot, u(\cdot - \tau_\alpha(\cdot)))\,[\overline{u}(\cdot - \tau_\alpha(\cdot)) - u(\cdot - \tau_\alpha(\cdot))]\| \\
T_4 &:= \|g_x(\cdot, u(\cdot - \tau_\alpha(\cdot)))\,[\overline{u}(\cdot - \tau_\alpha(\cdot)) - \overline{u}(\cdot)]\| \\
T_5 &:= \frac{\left\|f_\alpha^\delta - f_\alpha\right\|}{a_\alpha} \\
T_6 &:= \frac{1}{a_\alpha}\left\|D_\alpha G\overline{u}(\cdot) - a_\alpha G\overline{u}(\cdot)\right\|.
\end{aligned}
$$

Estimates on T_1 and T_3 follow from (3.18) and (3.28),

$$
T_i \le \frac{2\overline{m}\,\theta\lambda(\alpha,\overline{u})}{\overline{c}}\left(\frac{4\,\theta\lambda(\alpha,\overline{u})}{\overline{c}} + 2\lambda(\alpha,\overline{u})\right) \le \frac{\lambda(\alpha,\overline{u})}{2}, \quad \text{for } i = 1, 3,
$$

while estimates of T_2 and T_4 are straightforward,

$$
T_i \le \|g_x\|_J\,\lambda(\alpha,\overline{u}), \quad \text{for } i = 2, 4.
$$

Using Lemma 3.1 and (3.26),

$$
T_5 = \frac{1}{a_\alpha}\left\|f_\alpha - f_\alpha^\delta\right\| \le \frac{\widetilde{C}\delta}{C_1\alpha^\nu} \le \frac{\widetilde{C}\gamma}{C_1}\lambda(\alpha,\overline{u}).
$$

Finally,

$$
T_6 \le T_{6,1} + T_{6,2},
$$

where

$$
\begin{aligned}
T_{6,1} &:= \frac{1}{a_\alpha}\left\|\int_0^\alpha \int_0^\rho k(\cdot+\rho, \cdot+s)\right. \\
&\qquad\qquad \left.\times\,[g(\cdot+s, \overline{u}(\cdot+s)) - g(\cdot, \overline{u}(\cdot))]\;ds\,d\eta_\alpha(\rho)\right\|, \\
&\le \frac{1}{C_1\alpha^\nu}\int_0^\alpha \left\|\int_0^\rho k(\cdot+\rho,\cdot+s)[g(\cdot+s,\overline{u}(\cdot+s)) - g(\cdot,\overline{u}(\cdot))]\,ds\right\| d|\eta_\alpha|(\rho) \\
&\le \frac{\widetilde{C}\cdot\overline{C}\alpha^{\nu-1}}{C_1\alpha^\nu}\sup_{\rho\in[0,\alpha]}\left|\int_0^\rho \|g(\cdot+s,\overline{u}(\cdot+s)) - g(\cdot,\overline{u}(\cdot))\|\;ds\right| \\
&\le \frac{\widetilde{C}\overline{C}}{C_1\alpha}\int_0^\alpha \left(\|g_x\|_J\,\mu(\alpha,\overline{u}) + \|g_t\|\,\alpha\right)\,ds \\
&\le \frac{2\widetilde{C}\overline{C}}{C_1}\left(\|g_x\|_J + \|g_t\|_J\right)\lambda(\alpha,\overline{u})
\end{aligned}
$$

and

$$T_{6,2} := \frac{1}{a_\alpha} \left\| \left(\int_0^\alpha \int_0^\rho k(\cdot + \rho, \cdot + s)\, ds\, d\eta_\alpha(\rho) - a_\alpha \right) g(\cdot, \overline{u}(\cdot)) \right\|.$$

But $T_{6,2} = 0$ in the case of $\nu = 2, 3, \ldots$, while for $\nu = 1$ we have

$$T_{6,2} \leq \frac{c_2 \alpha^2}{2C_1 \alpha} \|k\|_{C^1(J)} \|g\|_J \leq \frac{c_2 \lambda(\alpha, \overline{u})}{2C_1} \|k\|_{C^1(J)} \|g\|_J.$$

Combining these estimates and using the definition of θ, it follows that

$$\|GH_\alpha u - G\overline{u}\| \leq \theta \lambda(\alpha, \overline{u}),$$

so that $H_\alpha u$ satisfies one of the conditions to belong to M_α^θ. This last inequality also gives

$$\|H_\alpha u - \overline{u}\| \leq \frac{\theta}{\overline{c}}\, \lambda(\alpha, \overline{u}),$$

so that from (3.27), $H_\alpha u(t) \in J$ for all $t \in [0,1]$ and all $\alpha \in (0, \overline{\alpha}]$. Further, from the definition of H_α and the fact that $F_\alpha^\delta \in C[0,1]$, we have $H_\alpha u$ continuous, and thus $H_\alpha u \in C([0,1]; J)$. It follows that $H_\alpha u \in M_\alpha^\theta$.

Part 3. H_α is a contraction on M_α^θ.

Let $u_1, u_2 \in M_\alpha^\theta$. Then $H_\alpha u_i \in M_\alpha^\theta$, $i = 1, 2$, and

$$\begin{aligned} \|H_\alpha u_1 - H_\alpha u_2\| &\leq \frac{1}{\overline{c}} \|GH_\alpha u_1 - GH_\alpha u_2\| \\ &\leq \frac{\mathcal{M}}{\overline{c}} \|(Gu_1 - \mathcal{G}_{\tau_\alpha} u_1) - (Gu_2 - \mathcal{G}_{\tau_\alpha} u_2)\| \\ &\leq \frac{\mathcal{M}}{\overline{c}} \sum_{i=1}^{3} S_i, \end{aligned}$$

where

$$\begin{aligned} S_1 &:= \|Gu_1 - Gu_2 - g_x(\cdot, u_1(\cdot - \tau_\alpha(\cdot)))[u_1 - u_2]\| \\ S_2 &:= \|g(\cdot, u_2(\cdot - \tau_\alpha(\cdot))) - g(\cdot, u_1(\cdot - \tau_\alpha(\cdot))) \\ &\qquad - g_x(\cdot, u_2(\cdot - \tau_\alpha(\cdot)))[u_2(\cdot - \tau_\alpha(\cdot)) - u_1(\cdot - \tau_\alpha(\cdot))]\| \\ S_3 &:= \|[g_x(\cdot, u_1(\cdot - \tau_\alpha(\cdot))) \\ &\qquad - g_x(\cdot, u_2(\cdot - \tau_\alpha(\cdot)))][u_1(\cdot - \tau_\alpha(\cdot)) - u_2(\cdot)]\|. \end{aligned}$$

Then from (3.17) we have

$$S_i \leq \overline{m}\left(\frac{4\theta\lambda(\alpha,\overline{u})}{\overline{c}} + 2\lambda(\alpha,\overline{u})\right)\|u_1 - u_2\|, \quad i = 1,2,$$

while for S_3, we note that

$$\begin{aligned}
&\|u_1(\cdot - \tau_\alpha(\cdot)) - u_2(\cdot)\| \\
&\leq \|u_1(\cdot - \tau_\alpha(\cdot)) - \overline{u}(\cdot - \tau_\alpha(\cdot))\| + \|\overline{u}(\cdot - \tau_\alpha(\cdot)) - \overline{u}(\cdot)\| + \|\overline{u}(\cdot) - u_2(\cdot)\| \\
&\leq \lambda(\alpha,\overline{u})\,\frac{2\theta + \overline{c}}{\overline{c}},
\end{aligned}$$

so

$$\begin{aligned}
S_3 &\leq \|g_x\|_J \,\|u_1(\cdot - \tau_\alpha(\cdot)) - u_2(\cdot - \tau_\alpha(\cdot))\|\,\|u_1(\cdot - \tau_\alpha(\cdot)) - u_2(\cdot)\| \\
&\leq \|g_x\|_J \,\|u_1 - u_2\|\,\lambda(\alpha,\overline{u})\,\frac{2\theta + \overline{c}}{\overline{c}}.
\end{aligned}$$

It follows that

$$\|H_\alpha u_1 - H_\alpha u_2\| \leq d\|u_1 - u_2\|,$$

where $d \in (0,1)$ is defined in (3.29).

Thus for any $\alpha \in (0,\overline{\alpha}]$, there is $u_\alpha^\delta \in C([0,1];J)$ which is the unique fixed point of H_α and the unique solution of (3.2), completing the proof of (1).

Proof of (1′). For $\overline{u}$ satisfying (3.7), it follows that $\mu(\alpha,\overline{u}) = \mathcal{O}(\alpha^\xi) = \mathcal{O}(\delta^{\xi/\nu+\xi})$. So for (1′), the quantity $\lambda(\alpha,\overline{u})$ given by (3.12) satisfies $\lambda(\alpha,\overline{u}) = \mathcal{O}(\delta^{\xi/\nu+\xi})$ as $\delta \to 0$ and the result follows by using this value of $\lambda(\alpha,\overline{u})$ and repeating the arguments for (1).

Proof of (2). For $i = 1,2$, let $f_i^\delta \in \{h \in X[0,1+\overline{\alpha}] \,\big|\, \|h - f\|_{X[0,1+\overline{\alpha}]} \leq \delta\}$ and define $H_{\alpha,i}$ as usual, now with f_i^δ replacing f^δ. Then there exists a unique $u_{\alpha,i}^\delta \in C([0,1];J)$ satisfying $u_{\alpha,i}^\delta = H_{\alpha,i}u_{\alpha,i}^\delta$, for $i = 1,2$. It follows that

$$\begin{aligned}
\left\|u_{\alpha,1}^\delta - u_{\alpha,2}^\delta\right\| &= \left\|H_{\alpha,1}u_{\alpha,1}^\delta - H_{\alpha,2}u_{\alpha,2}^\delta\right\| \\
&\leq \left\|H_{\alpha,1}u_{\alpha,1}^\delta - H_{\alpha,1}u_{\alpha,2}^\delta\right\| + \left\|H_{\alpha,1}u_{\alpha,2}^\delta - H_{\alpha,2}u_{\alpha,2}^\delta\right\| \\
&\leq d\left\|u_{\alpha,1}^\delta - u_{\alpha,2}^\delta\right\| \\
&\quad + \left\|\int_0^\alpha f_1^\delta(\cdot + \rho)\,d\eta_\alpha(\rho) - \int_0^\alpha f_2^\delta(\cdot + \rho)\,d\eta_\alpha(\rho)\right\|
\end{aligned}$$

for $d \in (0,1)$ defined in (3.29) above, so we have

$$(1-d)\left\|u^{\delta}_{\alpha,1} - u^{\delta}_{\alpha,2}\right\| \le \widetilde{C}\left\|f^{\delta}_1 - f^{\delta}_2\right\|_{X[0,1+\overline{\alpha}]}$$

or

$$\left\|u^{\delta}_{\alpha,1} - u^{\delta}_{\alpha,2}\right\| \le \frac{\widetilde{C}}{1-d}\left\|f^{\delta}_1 - f^{\delta}_2\right\|_{X[0,1+\overline{\alpha}]}. \qquad \square$$

4. Discretization and numerical implementation. We briefly describe one numerical implementation of the new local regularization equation (3.2) by defining a simple discrete collocation method which is applied to equation (3.5) (equivalently, to (3.2)).

To this end, we let N be a positive integer, sufficiently large so that for $\Delta t = 1/N$ and

$$t_i := i\Delta t, \quad i = 1, 2, \ldots, N, N+1, \ldots,$$

there is an integer $R \ge 1$ satisfying

$$0 < (1+\overline{\alpha}) - t_{N+R} \le \varepsilon,$$

where $\varepsilon > 0$ is a given (small) tolerance. In the discrete setting, the regularization parameter $\alpha \in (0, \overline{\alpha}]$ satisfies

$$\alpha = t_L, \quad \text{for some fixed } L \in \{1, 2, \ldots, R\},$$

while the discrete form τ_i of $\tau_\alpha(t)$ in (3.1) satisfies

$$0 \le \tau_i \le \min\{t_i, \alpha\} = \min\{i, L\}\Delta t, \quad i = 1, \ldots N.$$

To simplify the exposition in what follows, we fix τ_i at its smallest nonzero value, i.e.,

$$\tau_i = \Delta t, \quad i = 1, \ldots N,$$

a choice that appears to be reasonable for many examples.

We assume that a reasonable quadrature scheme is used to approximate integrals appearing in equation (3.5), and use the notation $[\cdot]_Q$ to indicate this scheme. Thus, the computed value of $f^{\delta}_{\alpha}(t)$ evaluated

at $t = t_i$ and $\alpha = t_L$ will be designated, for $i = 1, \ldots, N$, by the scalar f_i^δ, where

$$f_i^\delta := \left[\int_0^\alpha f^\delta(t_i + \rho)\, d\eta_\alpha(\rho) \right]_Q = [f_\alpha^\delta(t_i)]_Q. \tag{4.1}$$

Similarly, the computed value $\widehat{a}_\alpha$ of a_α is given by

$$\widehat{a}_\alpha := [a_{t_L}]_Q.$$

For $j = 1, \ldots, N$, let χ_j denote the characteristic function on the interval $[t_{j-1}, t_j)$. Then we seek the vector $\mathbf{c} = (c_j)$ (where $c_j = c_j(\delta, L, N)$, for $j = 1, \ldots, N$) so that for $t \in [0, 1]$,

$$u(t) := \sum_{j=1}^{N} c_j \chi_j(t) \tag{4.2}$$

satisfies equation (3.5) at the collocation point t_i, $i = 1, \ldots, N$. This leads to the system of equations, for $i = 2, \ldots, N$,

$$c_i = c_{i-1} - \frac{g(t_i, c_{i-1})}{g_x(t_i, c_{i-1})} + \frac{f_i^\delta}{\widehat{a}_\alpha\, g_x(t_i, c_{i-1})} \tag{4.3}$$

$$- \frac{1}{\widehat{a}_\alpha g_x(t_i, c_{i-1})} \left[\int_0^{t_i} k_\alpha(t_i, s)\, g\left(s, \sum_{j=1}^{i-1} c_j \chi_j(s) \right) ds \right]_Q, \tag{4.4}$$

so it is clear that, beginning with c_2, each c_i is determined in a sequential manner without requiring the solution of a nonlinear equation.

There is no defined value for c_0 so we cannot use equation (4.4) to recover c_1. If we have a good estimate of the desired solution $\overline{u}$ at $t = 0$, then that value could be used for c_0 and then equation (4.4) used to find c_1. Otherwise, we may find c_1 satisfying the discretized regularization equation evaluated at the collocation point t, $t \to t_1^-$,

$$\widehat{a}_\alpha g(t_1, c_1) = f_1^\delta - \left[\int_0^{t_1} k_\alpha(t_1, s)\, g(s, c_1)\, ds \right]_Q, \tag{4.5}$$

a nonlinear equation in c_1.

4.1. Numerical examples. In each of the examples below, we specify the ν-smoothing kernel k, the nonlinear function g, and a solution $\overline{u}$ from which the true data f is obtained by computing $KG\overline{u}$ exactly; the data is then discretized with a given value of N and uniformly distributed random error is added to generate the discretization $\mathbf{f}^\delta = (f_i^\delta)$ of f^δ. In order to better make comparisons with simplified (Lavrentiev) regularization in some of the examples below, we report the relative error between discretized f and f^δ on the original interval $[0,1]$; the error on the extended interval $[0, 1+\overline{\alpha}]$ is slightly larger. Further, to avoid confusion we denote the regularization parameters for local regularization and simplified (Lavrentiev) regularization α_{loc} and α_{lav}, respectively.

In each example of local regularization we let η_α be given by the normalized Lebesgue measure on $[0,\alpha]$, and we use (4.5) to determine c_1. For simplified (Lavrentiev) regularization, we use the exact value of $\overline{u}(0)$ in the discrete form of equation (2.3), even though this value is rarely available.

Finally, except for Example 4.4 below in which a modified discrepancy principle for local regularization is utilized, results presented in the tables and figures that follow are those associated with the regularization parameter giving the smallest relative error in reconstructed solutions. In figures, the dashed curve represents the true solution $\overline{u}$ while the solid curve is that of the reconstructed solution.

Example 4.1. In this first example we let $k(t) = 1$, a one-smoothing convolution kernel, $\overline{u}(t) = 8(t-0.4)^2 + 1$ and $g(t,u) = u^3$. The discretization level is given by $N = 200$, and we give results for both local regularization and simplified regularization in the case of several different levels of relative data error. The reconstructed solutions for local regularization and simplified regularization are graphed in Figure 1 in the case of 2% relative data error.

In Table 1, the following quantities are given for each regularization method and for levels 0.5%, 1%, 2% and 4% of relative data error: (1) the relative solution error; (2) the base-2 logarithm of the ratio of current to previous relative solution error, a quantity corresponding to s in a hypothesized convergence rate of $C\delta^s$; and (3) the value of the regularization parameter used in each case.

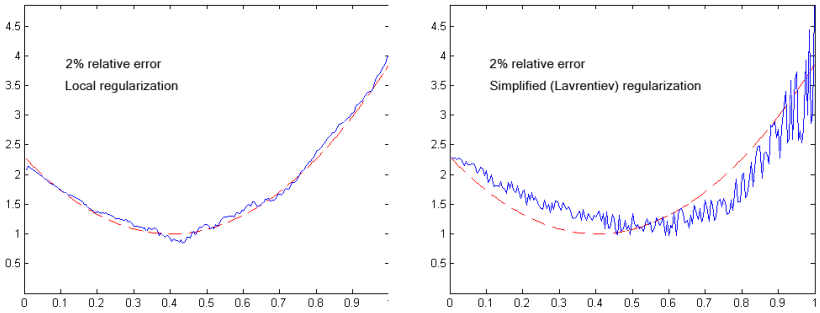

FIGURE 1. Regularization methods applied to the one-smoothing problem in Example 4.1, with a 2% relative data error. *Left*: Local regularization, *Right*: Simplified (Lavrentiev) regularization.

TABLE 1. Example 4.1 (one-smoothing convolution kernel).

Relative	**Local regularization**			**Simplified regularization**		
data error	Rel. error	$\log_2$(ratio)	α_{loc}	Rel. error	$\log_2$(ratio)	α_{lav}
0.005	0.0206	–	0.060	0.1026	–	0.40
0.01	0.0287	0.48	0.080	0.1423	0.47	0.50
0.02	0.0409	0.51	0.095	0.1942	0.45	0.80
0.04	0.0604	0.56	0.120	0.2619	0.43	1.25

Example 4.2. We now let $k(t) = 0.5t^2$, a three-smoothing convolution kernel, but use the same g and $\overline{u}$ as in Example 4.1. Thus this problem is more severely ill-posed than that seen in Example 4.1 and, in fact, the theory of simplified (Lavrentiev) regularization has not been developed for ν-smoothing problems when $\nu > 1$. Indeed, this method fails to find a reasonable reconstruction of $\overline{u}$ even when the "true" data f is used (i.e., for 0% relative error in data). In Figure 2 the results for simplified regularization are shown for four different values of α_{lav} and for 0% relative data error. It was observed that the relative error in reconstructed solutions apparently reaches its infimum as $\alpha_{\text{lav}} \to \infty$, with the constant-valued solution in this case equal to the user-supplied scalar, $\overline{u}(0)$.

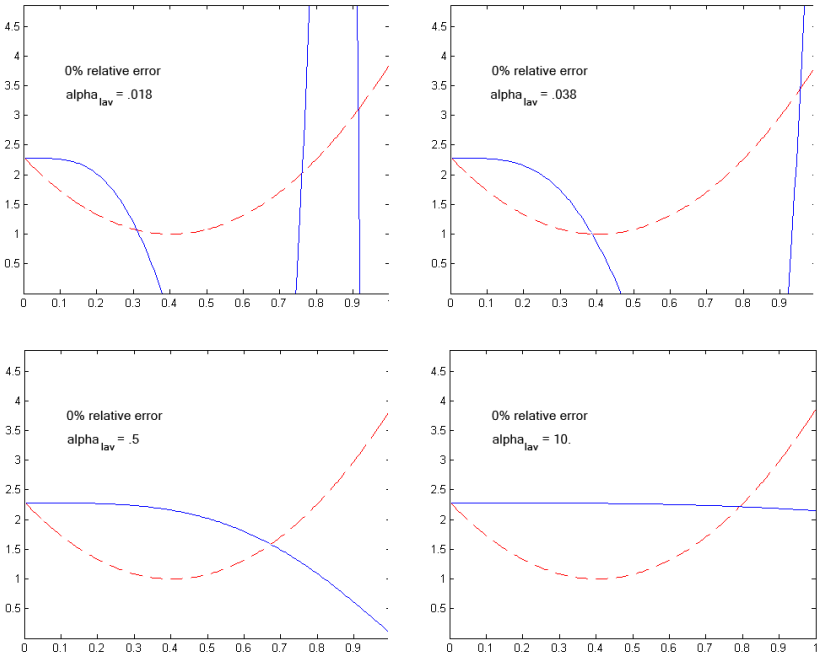

FIGURE 2. Simplified (Lavrentiev) regularization applied to the three-smoothing problem in Example 4.2 in the case of 0% relative error in data. *Top row*: $\alpha_{\text{lav}} = .018, .038$. *Bottom row*: $\alpha_{\text{lav}} = 5, 10$. The relative solution error was observed to decrease with increasing α_{lav}.

In contrast, the method of local regularization performs well when relative data error levels range from 0% to 4% and higher, as can be seen in Figure 3. The findings for both regularization methods in the case of noisy data are summarized in Table 2.

Example 4.3. In this example, we apply local regularization to a one-smoothing nonconvolution kernel, $k(t,s) = 1 + ts$, $0 \le s \le t \le 1$. Here $\overline{u} = -3t + 5$, $g(u) = \exp(u)$, and discretization is determined by $N = 100$. In Figure 4 we show the results of local regularization when applied to the case of 0.1% relative data error. Additional nonconvolution examples may be found in [**29**].

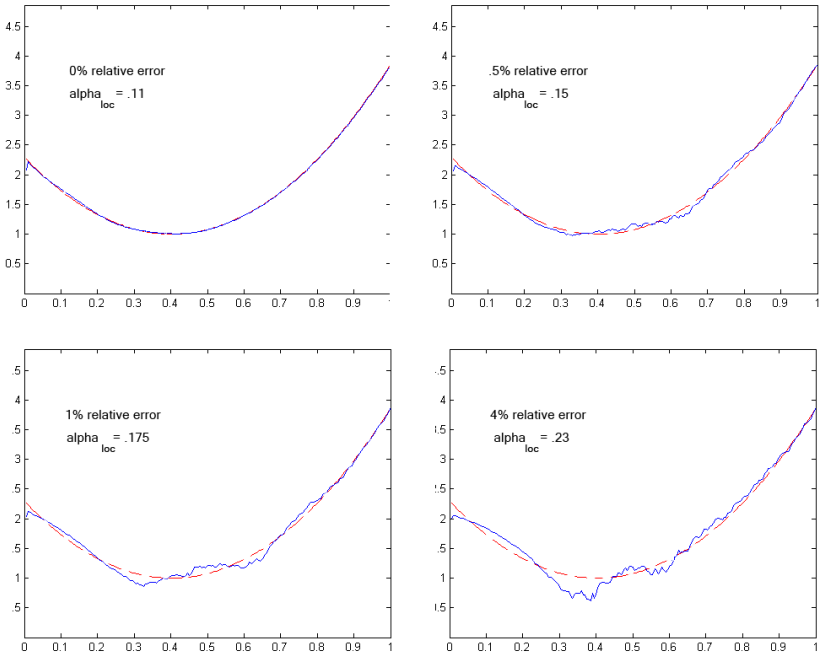

FIGURE 3. Local regularization applied to the three-smoothing problem in Example 4.2. *Top row*: 0% and 0.5% relative data error. *Bottom row*: 1%, and 4% relative data error.

TABLE 2. Example 4.2 (three-smoothing convolution kernel).

Relative	**Local regularization**			**Simplified regularization**		
data error	Rel. error	$\log_2$(ratio)	α_{loc}	Rel. error	$\log_2$(ratio)	α_{lav}
0.005	0.0278	–	0.150	0.4860	–	∞
0.01	0.0425	0.61	0.175	0.4860	0	∞
0.02	0.0545	0.36	0.205	0.4860	0	∞
0.04	0.0686	0.33	0.230	0.4860	0	∞

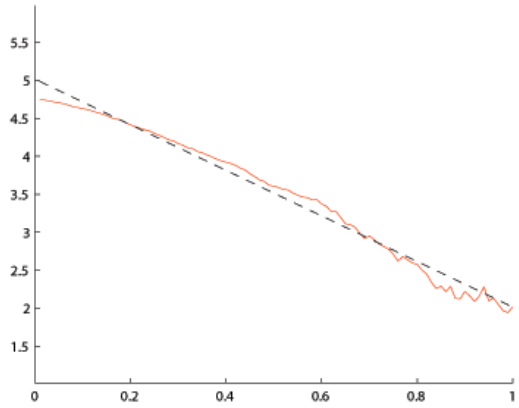

FIGURE 4. Local regularization applied to the one-smoothing nonconvolution problem in Example 4.3, with $\alpha_{\text{loc}} = 0.25$. The relative error in the data is 0.1%.

4.2. A modified discrepancy principle for parameter selection. We briefly describe a discrepancy principle for the *a posteriori* selection of the local regularization parameter $\alpha = \alpha_{\text{loc}}$. The theoretical justification for the use of the principle is well-established in the case of linear Volterra problems of convolution type [**2, 3**]. A development of the convergence theory for the nonconvolution linear problem and for the Hammerstein problem is beyond the scope of this paper and will be presented elsewhere.

For the discrepancy principle, we require the usual assumption (2.1) on data error, as well as a new condition on the signal-to-noise ratio, namely, that there is $\widetilde{\tau} \in (1, 2)$ for which the noisy data f^δ satisfies

$$\left\| f^\delta \right\| > (\widetilde{\tau} + 1)\delta.$$

Define the operator $A_\alpha : C[0,1] \to C[0,1]$ by

$$A_\alpha w := K_\alpha G w - a_\alpha \left(G w - \mathcal{G}_{\tau_\alpha} w \right), \quad w \in C[0,1], \tag{4.6}$$

so that equation (3.21) (equivalently (3.2)) may be written in the form

$$a_\alpha G u + A_\alpha u = f_\alpha^\delta. \tag{4.7}$$

Then the *modified discrepancy principle* for local regularization of the Hammerstein problem is as follows. Choose the regularization parameter $\alpha = \alpha(\delta)$ as the smallest $\alpha \in (0, \overline{\alpha}]$ so that

$$a_\alpha^m \left\| A_\alpha u_\alpha^\delta - f_\alpha^\delta \right\| = \widetilde{\tau}\delta,$$

for some fixed $m \in (0,1)$. (Any $\alpha \in (0, \overline{\alpha})$ satisfying (4.8) would be acceptable.)

Remark 4.1. Note that the principle above resembles the class of modified discrepancy principles originally studied by Engl, Groetsch, Schock and others for use with Tikhonov and simplified (Lavrentiev) regularization in [**10, 18, 32, 34**] and considered more recently in [**11, 13, 30**].

Under suitable conditions on η_α and the linearization parameter τ, it can be shown that, for $\delta > 0$ sufficiently small or for $\|f^\delta\|/\delta$ sufficiently large, there exists a smallest $\alpha = \alpha(\delta) \in (0, \overline{\alpha}]$ satisfying (4.8). Further, if the selection of $\alpha(\delta)$ is made using (4.8), it follows that $\alpha(\delta) \to 0$ and $\|u_\alpha^\delta - \overline{u}\| \to 0$ as $\delta \to 0$.

Example 4.4. To illustrate application of the modified discrepancy principle (4.8) in a numerical example, we repeat Example 4.2; here the choice of f^δ from that example is fixed so that the relative data error is 1%. The value of δ in (2.1) associated with this f^δ is $\delta = 0.0067$. We use $m = .01$ and $\widetilde{\tau} = \sqrt{2}$ in (4.8) and, for $\overline{\alpha} = 0.25$, seek the smallest $\alpha \in (0, \overline{\alpha}]$ satisfying (4.8). Implementation of the modified discrepancy principle in the discretized setting determines $\alpha(\delta) = 0.165$, where the corresponding reconstruction has a relative error of 0.044. These results are illustrated in Figure 5 below.

5. Conclusion. We have developed the theory of local regularization for nonlinear Volterra problems of Hammerstein type in the case of one-smoothing nonconvolution kernels and ν-smoothing convolution kernels for $\nu > 1$. We have discussed practical implementation of the method and demonstrated with numerical examples the way in which local regularization can outperform simplified (or Lavrentiev) regularization. Finally, we have used a modified discrepancy principle in a numerical example to illustrate the *a posteriori* selection of the local regularization parameter α.

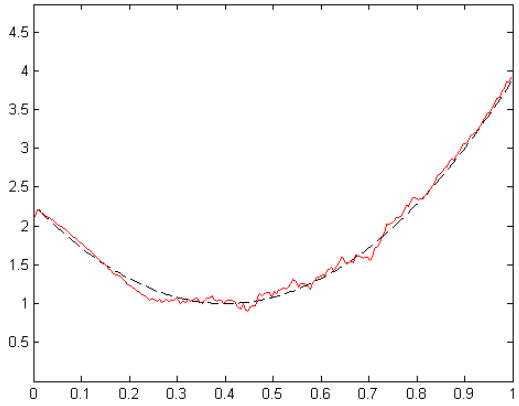

FIGURE 5. Three-smoothing problem with 1% relative error in the data. The value of $\alpha(\delta) = 0.165$ is selected using the modified discrepancy principle (4.8); for u_α^δ determined using this α, there is a 4.4% relative error in the recovered solution.

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