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EXPONENTIAL STABILIZATION OF VOLTERRA INTEGRAL EQUATIONS WITH SINGULAR KERNELS

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ABSTRACT. We consider exponential stabilization of mechanical systems consisting of rigid and flexible members by feedback acting on the rigid parts. The flexible material is assumed to be linearly viscoelastic of convolution type with a completely monotone kernel. No better decay rate can be obtained by stabilization than the essential growth rate of the unperturbed system. A formula for the essential growth rate is given, in particular it depends only on the convolution kernel. It is negative (i.e. exponential stabilization is possible) if and only if the kernel decays exponentially. Two mechanical models are discussed.

1. Introduction. In this paper we treat stabilization problems for certain systems of partial-differential integral equations with possible singular kernels arising in the theory of linear viscoelasticity. We have in mind a mechanical system consisting of one or more rigid bodies with flexible members attached to them. To stabilize the motion of the system (including the vibrations of the flexible parts), a control force or torque is applied to the rigid parts. The control is obtained by a linear feedback law from positions and velocities of the rigid parts.

If the flexible parts are made of perfectly elastic material, the motion of the uncontrolled system is energy conserving. The fact that observations and control act on rigid parts of positive mass, makes the feedback a bounded compact perturbation. It is known (e.g. [**12**]) that such perturbations cannot stabilize the system exponentially. Intuitively, one may imagine that due to inertial effects the control is too slow to compensate high frequency vibrations, in fact most of the energy of short waves is reflected at the interfaces between the flexible

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and the rigid members and goes back to the flexible part. (This is quite unlike stabilization by energy dissipating boundary conditions, where the wave reflection behavior is influenced directly and very significant stabilization is possible.)

What makes real mechanical systems still stabilizable is the energy dissipation in the flexible material, such as internal friction and viscoelastic effects. We discuss in this paper how viscoelasticity affects the exponential stabilizability of the system. We shall be concerned here with removing the principal obstable - i.e., that no compact perturbation can stabilize the system. The construction of an efficient feedback is a separate problem (and not a trivial one).

Let the essential growth rate of the system be the least exponential growth rate of solutions that can be obtained by compact perturbation. Thus a negative essential growth rate means precisely exponential stabilizability. We assume that stress σ and strain ε are related by a linear constitutive equation of convolution type with a completely monotone kernel a:

$$\sigma(t) = E\varepsilon(t) + \int_0^t a(t-s)\frac{\partial}{\partial s}\varepsilon(s)\,ds.$$

It turns out that the essential growth rate is only determined by the kernel and that the special structure of the mechanical system has no bearing on it. This is no surprise, since here the essential growth rate measures how much the physical properties of the material contribute to energy dissipation in the system. We derive a formula for the essential growth rate in terms of the kernel. Instead of writing down the formula here, let us give a rough verbal explanation. First, exponential stability can never be obtained at a better rate than that at which the memory of the material fades, i.e., the exponential decay of the kernel a is a lower bound for the essential growth rate. In particular, fractional derivative laws allow no exponential stabilization, as the kernels behave like roots of time. If the kernel decays exponentially, the essential growth rate of the system is always negative. Besides the decay rate of the kernels, the high frequency modes of the elastic system (with memory effects ignored) contribute to the essential growth rate in a twofold manner. If the kernel has a finite derivative at 0, the system behaves essentially like a damped wave equations, i.e., the poles come close to a line parallel to the imaginary axis for large imaginary parts. The real part of this line is a lower bound for the essential growth rate and given by $a'(0)/2(E+a(0))$. This is the case for most of the traditional kernels for Boltzmann viscoelasticity, in particular, finite sums of exponentials, as they arise in more or less complex spring-dashpot models.

If the derivative of the kernel is unbounded near 0, poles with large imaginary parts have real parts tending to $-\infty$, hence there is no limitation to stabilizability from this point of view. However, the high frequency modes of the elastic system may still allow nonoscillating solutions with slow exponential decay, thus causing the essential growth rate to be somewhat larger than the growth rate of the kernel a, while still negative. This is the typical situation for fractional derivative models modified to have exponential decay of the kernel.

Our paper is strongly motivated by and should be seen in the context of a variety of recent investigations of qualitative effects of singular kernels in viscoelasticity, all somehow related to the decay of high frequency modes. Typical problems of this kind concern stability, propagation of singularities, compactness and uniform continuity of solution operators. Instead of attempting a detailed discussion of existing literature, we refer the reader to the monograph [**25**], in particular to its section on hyperbolicity of Volterra equations and to its extensive bibliography. Let us just briefly recall that it has been known for a while that kernels with finite derivatives at 0 exhibit a behavior resembling frictional damping. (For propagation of singularities see, e.g., [**1**], the problem of essential stability has been treated in [**10**].) Though constitutive equations with singular kernels have been around at least since the thirties (see, e.g., the references in [**3, 4, 5**]), the current intense interest in their qualitative properties seems to have started with Renardy's paper [**24**], where it is shown that this type of kernel yields wave propagation behavior which is somehow hybrid between hyperbolic and parabolic (cf. also [**15, 22, 23, 8**]). In [**13**] it is proved that such kernels lead to compact solution operators. These results all indicate that, for these kernels, the decay rates of the modes increase to infinity with increasing frequency. So it is reasonable to expect that they yield also exponential stabilizability which is only limited by the decay of the kernel. Our paper gives the precise formulation and proof for this result. Our problem has been treated for the particular case of a viscoelastically modified wave equation on interval in [**14**]. They consider also the case of boundary control without a rigid mass attached to the boundary. This type of control shows significant qualitative differences to what we discuss below. The essential growth rate is no limit to the stabilizability by such methods. Also pure boundary stabilization exhibits extraordinary sensitivity to perturbation while control acting on rigid masses does not.

Let us make some remarks on the mathematical techniques of this paper. The velocity field and stress field in the flexibile parts are considered as vectors in suitable Hilbert spaces (reasonably normed by kinetic and potential energy). The equation of momentum is rewritten as an abstract differential equation, while the constitutive equation yields an abstract integral equation. The unbounded operators appearing in both equations turn out to be adjoint to each other. No more information is required about the shape of the mechanical system. To handle this abstract system, we introduce a semigroup setting on a state space which includes the history of the system as far as it is relevant for future stresses. Semigroup settings for Volterra equations have been discussed, e.g., in [**6, 19, 7, 29**]. (While writing this paper we have learned of ongoing work of R. Fabiano and K. Ito using a semigroup method to justify numerical techniques for solving integrodifferential equations with completely monotone kernels.) It seems that equations with singular kernels such as in [**7, 29**] require state spaces tailored to that particular kernel. Our setting makes use of the fact that the kernel is the Laplace-Stieltjes transform of a positive measure. We use this measure to construct an L^2-space. The physical meaning of its norm is unknown to us, but it turns out that the estimates that are to be expected for the solutions of the Volterra equations carry over to the semigroup (in spite of the fact that the state space has been augmented by a history part). The concept of an essential growth rate indicated in the beginning of our paper admits a precise formulation in the framework of semigroups. The use of this notion in applied work (namely population dynamics) goes back to [**20, 30**], where the main properties are developed. As a good reference we recommend [**2**]. Existence of the semigroup is proved by standard m-dissipativity arguments, the estimate of the essential growth rate is done by frequency domain methods. A theorem by Gearhart ([**11**], also [**21**]) allows us to obtain a uniform bound for the semigroup from uniform bounds for its Laplace transform. This is somewhat better for our purpose than the usual way of deriving L^2-estimates from L^2-estimates. Our technique can be generalized to treat problems of viscoelasticity with tensor valued kernels and can aslo be applied to discuss the wave propogation behavior and analyticity of the solution [**9**].

The paper is organized as follows: §2 starts from the abstract equations and introduces the state space and the semigroup. §3 proves the formula for the essential growth rate. As the proofs in both sections require some technical effort, we have stated the results first and delay the details of the proofs to the end of each section. §4 takes two model problems from [**27, 28**] to show how the results can be applied to actual mechanical problems. We discuss also how fractional derivative viscoelasticity fits into our framework.

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2. Existence and uniqueness. We consider the abstract equations

$$\begin{aligned}
u'(t) &= v(t),\\
v'(t) &= -D\sigma(t) + f(t),\\
\sigma(t) &= ED^*u(t) + \int_0^t a(t-s)D^*v(s)\,ds + h(t), \quad (t \geq 0),
\end{aligned}$$

with given initial data $u(0) = u_0, \quad v(0) = v_0$, and a suitable inhomogeneity h. Prime $'$ denotes derivative with respect to t. The functions u and v are supposed to take values in (subspaces of) some Hilbert space Y, σ takes values in another Hilbert space X.

We make the following assumptions:

Hypothesis D. D is a closed densely defined linear operator mapping $\operatorname{dom} D \subset X$ into Y. $D^* : \operatorname{dom} D^* \subset Y \to X$ is its adjoint. We assume furthermore that the embeddings of $\operatorname{dom} D$ and $\operatorname{dom} D^*$, endowed with the graph norms $(||x||^2 + ||Dx||^2)^{1/2}$, $(||y||^2 + ||D^* y||^2)^{1/2}$, in X or Y, respectively, are compact (or equivalently that DD^* has discrete point spectrum with finite dimensional eigenspaces, and that the kernels of D and D^* are finite dimensional). To rule out some trivial exceptional cases we assume that X and Y have infinite dimension and that D and D^* are unbounded operators.

(Notice that this hypothesis and [**16**, Theorem 3.24, p. 275] imply that DD^* and $D^* D$ are self-adjoint.)

Hypothesis a. $E > 0$ is a scalar, and $a : (0, \infty) \to (0, \infty)$ is completely monotone, such that $\lim_{t\to\infty} a(t) = 0$ and $\int_0^1 a(t)\, dt < \infty$. Let λ_0 denote the supremem of all $\lambda \in \mathbf{R}$ such that $e^{\lambda t} a(t)$ is bounded on $(1, \infty)$.

The inhomogeneity f will be considered as a control term, obtained from u and v by a feedback law

$$f(t) = C_1 u(t) + C_2 v(t), \tag{2.2}$$

where C_1 is a bounded linear operator mapping $\operatorname{dom} D^*$ into Y, and C_2 is a bounded linear operator from Y into Y. Moreover we assume that the ranges of both operators, C_1 and C_2, have finite dimension.

Our results hold as well for a more general type of dynamic feedback, such as

$$\begin{aligned} f(t) &= C_1 u(t) + C_2 v(t) + C_3 z(t), \\ z'(t) &= B_1 u(t) + B_2 v(t) + K z(t) + r(t), \quad (t \ge 0), \\ z(0) &= z_0. \end{aligned}$$

Here the control state $z(t)$ is a vector in $\mathbf{R}^n$; $r : [0, \infty) \to \mathbf{R}^n$ is some reference signal. The operators involved satisfy

Hypothesis C. K is an $n \times n$-matrix. The observers $B_1 : \operatorname{dom} D^* \to \mathbf{R}^n$ and $B_2 : Y \to \mathbf{R}^n$ are linear and bounded ($\operatorname{dom} D^*$ being endowed with the graph norm of D^*). $C_1 : \operatorname{dom} D^* \to Y$, $C_2 : Y \to Y$, and $C_3 : \mathbf{R}^n \to Y$ are bounded linear operators with finite dimensional range.

To prove existence and uniqueness of (generalized) solutions we treat the whole system (2.1), (2.2) as an abstract differential equation $\chi'(t) = \mathcal{A}\chi(t)$ in some big Hilbert space $\mathcal{H}$. For this purpose, the state $\chi(t)$ has to contain $u(t)$ and $v(t)$, but also all information about the history of the system that is forwarded by the convolution with the kernel a.

By Bernstein's Theorem [**31**, p. 160] there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ such that for any $t > 0$, $a(t) = \int_0^\infty e^{-\mu t} dg(\mu)$. As $\lim_{t\to\infty} a(t) = 0, g$ is continuous at 0. We may put $g(0) = 0$. Since $e^{-\lambda t}a(t)$ is integrable on $(0,\infty)$ for any $\lambda > 0$, we infer that $\int_0^\infty (\lambda+\mu)^{-1} dg(\mu) < \infty$. Consequently, for any $\varepsilon > 0, \int_\varepsilon^\infty \mu^{-1} dg(\mu)$ is finite.

Obviously, the support of g is contained in $[\lambda_0, \infty)$. For all $\lambda \in \mathbf{C}\backslash(-\infty, -\lambda_0]$, the analytic continuation of the Laplace transform of a is given by $\hat{a}(\lambda) = \int_0^\infty (\lambda+\mu)^{-1} dg(\mu)$.

Using this representation of a, we can rewrite the influence of the history of v on $\sigma(t)$ by

$$\int_0^t a(t-s)D^*v(s)\,ds = \int_0^\infty \int_0^t e^{-\mu(t-s)} D^*v(s)\,ds\,dg(\mu).$$

We assume that h (which usually describes the effect of the history before $t = 0$) can be written in the form

$$h(t) = \int_0^\infty e^{-\mu t} \phi_0(\mu)\,dg(\mu).$$

Then we obtain

$$\sigma(t) = ED^*u(t) + \int_0^\infty \phi(t,\mu)\,dg(\mu)$$

with

$$\phi(t,\mu) = e^{-\mu t}\phi_0(\mu) + \int_0^t e^{-\mu(t-s)} D^*v(s)\,ds.$$

The latter equation is just the variation-of-parameters solution to

$$\frac{\partial}{\partial t}\phi(t,\mu) = -\mu\phi(t,\mu) + D^*v(t),$$
$$\phi(0,\mu) = \phi_0(\mu).$$

With this notation, the system (2.1), (2.2) is at least formally transformed to an abstract differential equation for the state $\chi(t) = (u(t), v(t), \phi(t,.))$ in the Hilbert space $\mathcal{H} = \operatorname{dom} D^* \times Y \times L^2_g([0,\infty), X)$, where $L^2_g = L^2_g([0,\infty), X)$ denotes the space of X-valued L^2-functions with respect to the measure dg, and $\operatorname{dom} D^*$ is again equipped with the graph norm of D^*:

$$\chi'(t) = \mathcal{A}\chi(t),$$
$$\chi(0) = (u_0, v_0, \phi_0),$$

with an operator

$$\mathcal{A}(u, v, \phi) = (p, q, \psi),$$

where

$$p = v,$$
$$q = -D\Big(ED^*u + \int_0^\infty \phi(\mu)\, dg(\mu)\Big) + C_1 u + C_2 v,$$
$$\psi(\mu) = -\mu\phi(\mu) + D^*v,$$

defined on

$$\operatorname{dom}\mathcal{A} = \Big\{(u, v, \phi) \in \mathcal{H} : v \in \operatorname{dom} D^*,\ -\mu\phi + D^*v \in L^2_g,$$
$$ED^*u + \int_0^\infty \phi(\mu)\, dg(\mu) \in \operatorname{dom} D\Big\}.$$

REMARK 2.1. The existence of the integral $\int_0^\infty \phi(\mu)\, dg(\mu)$ is guaranteed for $(u, v, \phi) \in \operatorname{dom}\mathcal{A}$. In fact, if $\mu\phi - x$ and ϕ are in L^2_g, then $\int_0^\infty \phi(\mu) dg(\mu)$ exists and depends continuously on $x \in X$, ϕ and $\mu\phi - x \in L^2_g$.

PROOF.

$$\begin{aligned}
&\int_0^\infty ||\phi(\mu)||dg(\mu)\\
&\quad = \int_0^{1-} ||\phi(\mu)||\, dg(\mu) + \int_{1-}^\infty ||\phi(\mu)||\, dg(\mu)\\
&\quad \le \Big[\int_0^{1-} dg(\mu)\cdot \int_0^{1-} ||\phi(\mu)||^2\, dg(\mu)\Big]^{1/2}\\
&\qquad + \int_{1-}^\infty \mu^{-1}||\mu\phi(\mu)-x||dg(\mu) + \int_{1-}^\infty \mu^{-1}||x||dg(\mu)\\
&\quad \le g(1)^{1/2}\cdot ||\phi|| + \Big[\int_{1-}^\infty \mu^{-2}dg(\mu)\Big]^{1/2}\cdot ||\mu\phi - x||\\
&\qquad + \int_{1-}^\infty \mu^{-1}dg(\mu)\cdot ||x||.
\end{aligned}$$

□

REMARK 2.2. For the dynamic feedback case (2.3) the state of the system also contains $z(t)$, thus $\mathcal{H}_d = \operatorname{dom} D^* \times Y \times L^2_g([0,\infty),X) \times \mathbf{R}^n$, $\mathcal{A}_d(u,v,\phi,z) = (p,q,\psi,w)$, where

$$\begin{aligned}
p &= v,\\
q &= -D\Big(ED^*u + \int_0^\infty \phi(\mu)dg(\mu)\Big)\\
&\qquad + C_1u + C_2v + C_3z,\\
\psi(\mu) &= -\mu\phi(\mu) + D^*v,\\
w &= Kz + B_1u + B_2v,
\end{aligned}$$

defined on

$$\begin{aligned}
\operatorname{dom}\mathcal{A}_d = \Big\{(u,v,\phi,z) \in \mathcal{H}_d :& v \in \operatorname{dom} D^*, -\mu\phi + D^*v \in L^2_g,\\
& ED^*u + \int_0^\infty \phi(\mu)dg(\mu) \in \operatorname{dom} D\Big\}.
\end{aligned}$$

The reference signal enters as an inhomogeneity

$$\chi'(t) = \mathcal{A}_d\chi(t) + (0,0,0,r(t)).$$

While this operator looks a little more complicated than the previous one, the additional terms can all be treated as bounded perturbations, so that the equation is decoupled into the equation with $\mathcal{A}$ and a linear differential equation in $\mathbf{R}^n$. Thus all results we obtain for (2.1), (2.2) hold also for (2.1), (2.3).

In this section we prove

THEOREM 2.3. *$\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\mathcal{S}(t)$ in $\mathcal{H}$, similarly $\mathcal{A}_d$ is the infinitesimal generator of a C_0-semigroup $\mathcal{S}_d(t)$ in $\mathcal{H}_d$.*

Before we prove this theorem, we give an interpretation for the system (2.1), (2.2).

COROLLARY 2.4.

(a) *Let u_0, v_0, h be given such that $v_0 \in \operatorname{dom} D^*$, $h(t) = \int_0^\infty e^{-\mu t}\phi_0(\mu)dg(\mu)$* with ϕ_0 and $\mu\phi_0 - D^*v_0$ in L_g^2, and $u_0 \in \operatorname{dom} D^*$ such that $ED^*u_0 + \int_0^\infty \phi_0(\mu)dg(\mu)$ $(= ED^*u_0 + h(0)) \in \operatorname{dom} D$. *If*

$$(u(t), v(t), \phi(t, .)) = \mathcal{S}(t)(u_0, v_0, \phi_0),$$

then $u, v, \sigma(t) = ED^*u(t) + \int_0^\infty \phi(t, \mu)dg(\mu)$ yields the unique solution to (2.1), (2.2) in the sense that $u \in C^1([0,\infty), \operatorname{dom}D^*)$, $v \in C^1([0,\infty), Y)$, $\sigma \in C([0,\infty), X)$ and (2.1), (2.2) hold for all $t > 0$.

(b) The values $u(t) \in \operatorname{dom} D^*$ and $V(t) \in Y$ depend continuously on $u_0 \in \operatorname{dom} D^*$, $v_0 \in Y$, $\phi_0 \in L_g^2$, uniformly for t in compact intervals. Hence, if u_0, v_0, ϕ_0 are approximated by u_n, v_n, ϕ_n satisfying the conditions of (a) in $\operatorname{dom} D^*, Y$ and L_g^2, respectively, then the corresponding solutions converge to functions $u \in C([0,\infty), \operatorname{dom} D^*)$, $v \in C([0,\infty), Y)$, which can be regarded as generalized solutions to (2.1), (2.2).

(c) The generalized solutions are solutions in the sense of Laplace transforms, i.e., they are exponentially bounded, and for sufficiently large $\lambda \in \mathbf{R}$ their Laplace transforms satisfy

$$\begin{aligned}
&\lambda\hat{u}(\lambda) - u_0 = \hat{v}(\lambda),\\
&\lambda\hat{v}(\lambda) - v_0 = -D\hat{\sigma}(\lambda) + C_1\hat{u}(\lambda) + C_2\hat{v}(\lambda),
\end{aligned}$$

where $\hat{\sigma}(\lambda)$ is defined by

$$\hat{\sigma}(\lambda) = ED^*\hat{u}(\lambda) + \hat{a}(\lambda)D^*\hat{v}(\lambda) + \hat{h}(\lambda).$$

(Notice that, though $\hat{\sigma}(\lambda)$ is defined in some sense, there is no assertion about existence of any generalized solution $\sigma(t)$.)

PROOF OF COROLLARY 2.4.

(a). The conditions on the initial data u_0, v_0, h are constructed so that $(u_0, v_0, \phi_0) \in \mathrm{dom}\mathcal{A}$. Thus $\chi = (u, v, \phi)$ is the unique, continuously differentiable solution to $\chi'(t) = \mathcal{A}\chi(t)$ with these initial data in $\mathcal{H}$. We put $\sigma(t) = ED^*u(t) + \int_0^\infty e^{-\mu t}\phi(t,\mu)dg(\mu)$. The integral exists since $(u(t), v(t), \phi(t,.)) \in \mathrm{dom}\,\mathcal{A}$, cf. Remark 2.1. The equations

$$u'(t) = v(t), \quad v'(t) = -D\sigma(t) + C_1u(t) + C_2v(t)$$

are obvious from the definition of $\mathcal{A}$. Integrating $\frac{\partial}{\partial t}\phi(t,\mu) = -\mu\phi(t,\mu) + D^*v(t)$ from 0 to t we obtain

$$\begin{aligned}\sigma(t) &= ED^*u(t) + \int_0^\infty \left[e^{-\mu t}\phi_0(\mu) + \int_0^t e^{-\mu(t-s)}D^*v(s)ds\right]dg(\mu)\\ &= ED^*u(t) + h(t) + \int_0^t a(t-s)D^*v(s)\,ds.\end{aligned}$$

(b). This is an obvious consequence of the uniform boundedness of the operators $\mathcal{S}(t)$ for t in compact intervals.

(c). Let $\chi(t) = (u(t), v(t), \phi(t,.)) = \mathcal{S}(t)(u_0, v_0, \phi_0)$. As $||\mathcal{S}(t)||$ is exponentially bounded, $\chi(t)$ is also. The Laplace transform satisfies

$(\lambda - \mathcal{A})\hat{\chi}(t) = (u_0, v_0, \phi_0)$. We define $\hat{\sigma}(\lambda) = ED^*\hat{u}(\lambda) + \int_0^\infty \hat{\phi}(\lambda,\mu)dg(\mu)$. By definition of $\mathcal{A}$ we immediately have

$$\lambda\hat{u}(\lambda) - \hat{v}(\lambda) = u_0,$$

$$\lambda\hat{v}(\lambda) + D\hat{\sigma}(\lambda) - C_1\hat{u}(\lambda) - C_2\hat{v}(\lambda) = v_0$$

and

$$\lambda\hat{\phi}(\lambda,\mu) + \mu\hat{\phi}(\lambda,\mu) - D^*\hat{v}(\lambda) = \phi_0(\mu).$$

Integrating the last equation, we obtain

$$\begin{aligned}\hat{\sigma}(\lambda) &= ED^*\hat{u}(\lambda) + \int_0^\infty \hat{\phi}(\lambda,\mu)dg(\mu) \\ &= ED^*\hat{u}(\lambda) + \int_0^\infty \frac{1}{\lambda+\mu}\phi_0(\mu)dg(\mu) + \int_0^\infty \frac{1}{\lambda+\mu}dg(\mu)\cdot D^*\hat{v}(\lambda) \\ &= ED^*\hat{u}(\lambda) + \hat{h}(\lambda) + \hat{a}(\lambda)D^*\hat{v}(\lambda).\end{aligned}$$

□

Of course, a similar interpretation can be given for the dynamic feedback case (2.3).

The remainder of this section is devoted to the proof of Theorem 2.4. The proof is given by several lemmas and organized as follows. Being bounded perturbations, the feedback operators C_j can be omitted without loss of generality. We may also assume that D^* is one-to-one. Otherwise the finite dimensional kernel may be factored out. With these simplifications, we prove that $\mathcal{A}$ is densely defined and dissipative, moreover we compute $(\lambda - \mathcal{A})^{-1}$ by an explicit formula for suitable λ. By the Lumer-Phillips-Theorem, $\mathcal{A}$ generates a semigroup of contractions on $\mathcal{H}$.

LEMMA 2.5. *Let Y_1 be the closure of the range of D in Y. Evidently $Y = Y_1 \times \ker D^*$, $\operatorname{dom} D^* = (\operatorname{dom} D^* \cap Y_1) \times \ker D^*$. Let $\mathcal{A}_1 : \operatorname{dom}\mathcal{A} \to \mathcal{H}$ be defined by $\mathcal{A}_1(u,v,\phi) = (p,q,\psi)$ with*

$$\begin{aligned} p &= v \\ q &= -D\Big(ED^*u + \int_0^\infty \phi(\mu)dg(\mu)\Big), \\ \psi(\mu) &= -\mu\phi(\mu) + D^*v. \end{aligned}$$

Then $\mathcal{A} = \mathcal{A}_1 + \mathcal{K}$ with a compact continuous linear operator $\mathcal{K}$ in $\mathcal{H}$. Moreover, in the decomposition

$$\mathcal{H} = \ker D^* \times (\operatorname{dom} D^* \cap Y_1) \times \ker D^* \times Y_1 \times L^2_g,$$

the spaces $\ker D^* \times \ker D^* \times 0$ *and* $\mathcal{H}_1 = (\operatorname{dom} D^* \cap Y_1) \times Y_1 \times L^2_g$ *are $\mathcal{A}_1$-invariant subspaces.*

PROOF. From the definition of $\mathcal{A}$ and $\mathcal{A}_1$ it is clear that $\mathcal{A} = \mathcal{A}_1 + \mathcal{K}$, where $\mathcal{K}(u, v, \phi) = (0, C_1 u + C_2 v, 0)$ defines a compact linear operator on $\mathcal{H}$. For $u, v \in \ker D^*$, $\mathcal{A}_1(u, v, 0) = (v, 0, 0) \in \ker \mathrm{D}^* \times 0 \times 0$. If $\mathcal{A}_1(u, v, \phi) = (p, q, \psi)$ with u and v in Y_1, then evidently $p = v \in Y_1$, and q lies in the range of D which is contained in Y_1. □

REMARK 2.6. As $\dim(\ker D^* \times \ker D^*)$ is finite, it is obviously sufficient to show that the restriction of $\mathcal{A}_1$ to $\mathcal{H}_1$ generates a C_0-semigroup. We assume therefore without loss of generality that $\mathcal{A} = \mathcal{A}_1$, i.e., $C_j = 0$, and $\mathcal{H} = \mathcal{H}_1$, i.e., $\ker D^*$ is trivial. With the latter assumption we may introduce the following scalar product on $\mathcal{H}$:

$$\langle (u, v, \phi), (p, q, \psi) \rangle = E\langle D^* u, D^* p \rangle + \langle v, q \rangle + \int_0^\infty \langle \phi(\mu), \psi(\mu) \rangle dg(\mu).$$

LEMMA 2.7. *With the assumptions described in Remark 2.6, A is dissipative.*

PROOF. Let $(p, q, \psi) = \mathcal{A}(u, v, \phi)$. Then

$$\begin{aligned}
&\mathrm{Re}\langle (u, v, \phi), (p, q, \psi) \rangle \\
&\quad = \mathrm{Re}\Big[E\langle D^* u, D^* v \rangle - \Big\langle v, D(ED^* u + \int_0^\infty \phi(\mu) dg(\mu) \Big\rangle \\
&\qquad + \int_0^\infty \langle \phi(\mu), D^* v - \mu \phi(\mu) \rangle dg(\mu) \Big] \\
&\quad = \mathrm{Re}\Big[E\langle D^* u, D^* v \rangle - \langle D^* v, ED^* u \rangle - \Big\langle D^* v, \int_0^\infty \phi(\mu) dg(\mu) \Big\rangle \\
&\qquad + \int_0^\infty \langle \phi(\mu), D^* v \rangle dg(\mu) - \int_0^\infty \mu \langle \phi(\mu), \phi(\mu) \rangle dg(\mu) \Big] \\
&\quad = - \int_0^\infty \mu \langle \phi(\mu),\ \phi(\mu) \rangle dg(\mu) \le 0.
\end{aligned}$$

□

LEMMA 2.8. *$\mathcal{A}$ is densely defined in $\mathcal{H}$.*

PROOF. Given $(u, v, \phi) \in \mathcal{H}$ and $\varepsilon > 0$, we have to find $(p, q, \psi) \in \operatorname{dom} \mathcal{A}$ such that $||(u, v, \phi) - (p, q, \psi)|| \le \varepsilon$. We choose $p = u$ and some

$q \in \operatorname{dom} D^*$ such that $||q - v|| \leq \varepsilon/3$. Next we pick η sufficiently large so that $\int_\eta^\infty ||\phi(\mu)||^2 dg(\mu) < \varepsilon^2/18$ and $\int_\eta^\infty \mu^{-2} dg(\mu) \cdot ||D^* q|| < \varepsilon^2/18$. Putting

$$\pi(\mu) = \begin{cases} \phi(\mu) & \text{for } \mu < \eta, \\ \mu^{-1} D^* q & \text{for } \mu \geq \eta, \end{cases}$$

we obtain $\int_0^\infty ||\pi(\mu) - \phi(\mu)|| dg(\mu) < \varepsilon^2/9$, and $\mu\pi(\mu) - D^* v \in L_g^2$. Now choose an arbitrary scalar function α such that $\int_0^\infty \alpha(\mu) dg(\mu) = 1$ and $\int_0^\infty |\alpha(\mu)|^2 dg(\mu)$ as well as $\int_0^\infty |\mu\alpha(\mu)|^2 dg(\mu)$ are finite. As $\operatorname{dom} D$ is dense in X, there is some $h \in X$ with $||h||^2 < \varepsilon^2/(9 \int_0^\infty |\alpha(\mu)|^2 dg(\mu))$ such that $ED^* u + \int_0^\infty \psi(\mu) dg(\mu) + h \in \operatorname{dom} D$. Defining $\psi \in L_g^2$ by $\psi(\mu) = \pi(\mu) + \alpha(\mu) h$ we obtain that $\int_0^\infty ||\psi(\mu) - \pi(\mu)||^2 dg(\mu) < \varepsilon^2/9$, $\int_0^\infty ||\mu\psi(\mu) - \mu\pi(\mu)||^2 dg(\mu)$ is finite, and $ED^* u + \int_0^\infty \psi(\mu) dg(\mu) \in \operatorname{dom} D$. Then the triple (p, q, ψ) lies in $\operatorname{dom} \mathcal{A}$ and $||(u, v, \phi) - (p, q, \psi)||^2 \leq ||v - q||^2 + \int_0^\infty ||\phi(\mu) - \psi(\mu)||^2 dg(\mu) < \varepsilon^2$. □

In order to show that $\mathcal{A}$ is m-dissipative, it would be sufficient to show that $(\lambda - \mathcal{A})^{-1}$ exists for some large $\lambda \in \mathbf{R}$. For later use we prove a stronger result:

LEMMA 2.9. *We make the assumptions introduced in Remark* 2.6. *Let* $\lambda \in \mathbf{C} \backslash (-\infty, -\lambda_0]$ *be such that* $E + \hat{\lambda a}(\lambda) \neq 0$, *and* $\alpha(\lambda) = -\frac{\lambda^2}{E + \lambda\hat{a}(\lambda)}$ *is not contained in the spectrum of* DD^*. *Then* $(\lambda - \mathcal{A})^{-1}$ *exists as a bounded linear operator on* $\mathcal{H}$.

Moreover, putting $R(\lambda) = [\lambda^2 + (E + \lambda\hat{a}(\lambda)) DD^*]^{-1}$,

$$T(\lambda)x = \begin{cases} [\lambda^2 + (E + \lambda\hat{a}(\lambda)) D^* D]^{-1} x & \text{for } x \text{ in the range of } D^* D, \\ 0 & \text{for } x \text{ in the kernel of } D^* D, \end{cases}$$

extended to a linear operator on X, *and*

$$\tilde{\psi}(\lambda) = \int_0^\infty \frac{1}{\lambda + \mu} \psi(\mu) dg(\mu),$$

we obtain the following formula for $(u, v, \phi) = (\lambda - \mathcal{A})^{-1}(p, q, \psi)$:

$$\begin{aligned} u &= (\lambda + \hat{a}(\lambda) DD^*) R(\lambda) p + R(\lambda) q - DT(\lambda)\tilde{\psi}(\lambda), \\ v &= -EDD^* R(\lambda) p + \lambda R(\lambda) q - \lambda DT(\lambda)\tilde{\psi}(\lambda), \\ \phi(\mu) &= \frac{1}{\lambda + \mu} D^* v + \frac{1}{\lambda + \mu} \psi(\mu). \end{aligned} \tag{2.4}$$

PROOF. To prove injectivity of $\lambda - \mathcal{A}$ assume that $(\lambda - \mathcal{A})(u, v, \phi) = 0$, i.e.,

$$\lambda u - v = 0$$
$$\lambda v + D\Big(ED^*u + \int_0^\infty \phi(\mu)dg(\mu)\Big) = 0,$$
$$\lambda\phi(\mu) + \mu\phi(\mu) - D^*v = 0.$$

Given u we can solve explicitely for v and ϕ: $v = \lambda u$, $\phi(\mu) = \frac{\lambda}{\lambda+\mu}D^*u$. Hence $\lambda^2 u + D(ED^*u + \lambda\hat{a}(\lambda)D^*u) = 0$, which implies $u = 0$, since $[\lambda^2 + (E + \lambda\hat{a}(\lambda))DD^*]$ is injective. Now we prove that formula (2.4) yields in fact a solution to $(\lambda - \mathcal{A})(u, v, \phi) = (p, q, \psi)$.

Notice first that all expressions in (2.4) make sense, since $R(\lambda)$ maps Y into $\operatorname{dom} DD^*$ and $T(\lambda)$ maps X into $\operatorname{dom} D^*D$. Moreover, as $p \in \operatorname{dom} D^*$, we have $(E + \lambda\hat{a}(\lambda))DD^*R(\lambda)p = p - \lambda^2 R(\lambda)p \in \operatorname{dom} D^*$, thus $DD^*R(\lambda)p \in \operatorname{dom} D^*$. We infer that both, u and v, lie in $\operatorname{dom} D^*$. Since $\lambda \notin (-\infty, -\lambda_0], 1/\lambda + \mu$ is bounded and square integrable with respect to dg on the support of g. Now it is easy to check that $\phi \in L^2_g$. Explicit integration yields

$$\int_0^\infty \phi(\mu)dg(\mu) = \hat{a}(\lambda)D^*v + \tilde{\psi}(\lambda).$$

Now we evaluate the three components of $(\lambda - \mathcal{A})(u, v, \phi)$.

$$\begin{aligned}
\lambda u - v &= [\lambda^2 + (E + \lambda\hat{a}(\lambda))DD^*]R(\lambda)p = p,\\
\lambda v &+ D\Big(ED^*u + \int_0^\infty \phi(\mu)dg(\mu)\Big)\\
&= \lambda v + D(ED^*u + \hat{a}(\lambda)D^*v + \tilde{\psi}(\lambda))\\
&= -\lambda EDD^*R(\lambda)p + \lambda^2 R(\lambda)q - \lambda^2 DT(\lambda)\tilde{\psi}(\lambda)\\
&\quad + D[E\lambda D^*R(\lambda)p + (E + \lambda\hat{a}(\lambda))D^*R(\lambda)q\\
&\qquad - (\mathrm{E} + \lambda\hat{\mathrm{a}}(\lambda))\mathrm{D}^*\mathrm{D}\mathrm{T}(\lambda)\tilde{\psi}(\lambda) + \tilde{\psi}(\lambda)]\\
&= q - D[\lambda^2 T(\lambda)\tilde{\psi}(\lambda)\\
&\qquad + (\mathrm{E} + \lambda\hat{\mathrm{a}}(\lambda))\mathrm{D}^*\mathrm{D}\mathrm{T}(\lambda)\tilde{\psi}(\lambda) - \tilde{\psi}(\lambda)]\\
&= q
\end{aligned}$$

In the last line we have utilized the fact that $\tilde{\psi}(\lambda) - [\lambda^2 + (E + \lambda\hat{a}(\lambda))D^*D]T(\lambda)\tilde{\psi}(\lambda) \in \ker D$ by definition of $T(\lambda)$. Finally, $\lambda\phi(\mu) - D^*v + \mu\phi(\mu) = \psi(\mu)$. Thus $(\lambda - \mathcal{A})(u, v, \phi) = (p, q, \psi)$. □

Since, for $\lambda > 0$, the negative number $-\lambda^2/E + \lambda\hat{a}(\lambda)$ is never contained in the spectrum of the positive semidefinite operator DD^*, $(\lambda - \mathcal{A})^{-1}$ exists for positive λ. The Lumer-Phillips-Theorem implies now that $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. Thus the proof of Theorem 2.3 is complete.

3. Essential growth rate. Let $\{\mathcal{S}(t) : t \geq 0\}$ be a C_0-semigroup generated by an operator $\mathcal{A}$ in a Banach space $\mathcal{H}$. Then, for suitable $M > 0$, $\omega \in \mathbf{R}$, $\mathcal{S}(t)$ satisfies an estimate

$$||\mathcal{S}(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0. \tag{3.1}$$

By the growth rate $\omega_0(\mathcal{S})$ of $\mathcal{S}$ we mean the infimum of all ω that admit an estimate of type (3.1). In particular, $\omega_0(\mathcal{S}) < 0$ means precisely that 0 is an exponentially stable solution of $u'(t) = \mathcal{A}u(t)$. Unlike the finite dimensional case, ω_0 can in general not be estimated from the spectrum of the generator.

Suppose for a moment that $\mathcal{H}$ can be decomposed into the direct sum of two $\mathcal{S}(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_s$, such that $\mathcal{H}_1$ is finite dimensional and the restriction of $\mathcal{S}(t)$ to $\mathcal{H}_2$ has a growth rate less than some $\omega \in \mathbf{R}$. If then $\omega_0(\mathcal{S}) > \omega$, this is due to the finite dimensional part, hence to an eigenvalue λ of $\mathcal{A}$ with $\operatorname{Re}\lambda = \omega_0(\mathcal{S})$. Moreover, a growth rate less than ω can be achieved by a finite dimensional perturbation of $\mathcal{A}$.

We define the essential growth rate $\omega_1(\mathcal{S})$ to be the infimum of all ω such that a decomposition as in the preceding paragraph is possible. Equivalent definitions and detailed information on the essential growth rate can be found, e.g., in [**2**]. In particular, the essential growth rate has the following properties:

(a) $\omega_1(\mathcal{S}) \leq \omega_0(\mathcal{S})$. If the inequality is strict, then $\mathcal{A}$ has an eigenvalue λ with finite dimensional generalized eigenspace

$\cup_{n\in N}\ker(\lambda - \mathcal{A})^n$ such that $\operatorname{Re}\lambda = \omega_0(\mathcal{S})$.

(b) $(\lambda - \mathcal{A})^{-1}$ is meromorphic in the open half plane $\operatorname{Re}\lambda > \omega_1(\mathcal{S})$. The poles of $(\lambda - \mathcal{A})^{-1}$ are eigenvalues of $\mathcal{A}$ with finite dimensional generalized eigenspaces. For each $\omega > \omega_1(\mathcal{S})$, there exists at most finitely many poles with $\operatorname{Re}\lambda \geq \omega$.

(c) If $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ is a compact (continuous) linear oeprator and $\mathcal{T}(t)$ is the semigroup generated by $\mathcal{A}+\mathcal{K}$, then $\omega_1(\mathcal{S}) = \omega_1(\mathcal{T})$. In particular, no growth rate less than $\omega_1(\mathcal{S})$ can be achieved by perturbation by bounded linear operators of finite rank.

Thus $\omega_1(\mathcal{S})$ contains the following information about the stabilization problem:

If $\omega_1(\mathcal{S}) \geq 0$, then exponential stabilization by a finite rank continuous operator is impossible.

If $\omega(\mathcal{S}) < 0$, then exponential stability of the perturbed problem is guaranteed if all eigenvalues of the perturbed generator lie in the open left half plane.

Let us now return to the semigroup $\mathcal{S}(t)$ defined in §2. We prove for this semigroup:

THEOREM 3.1. *There is at most one solution λ to $E + \lambda\hat{a}(\lambda) = 0$ in $\mathbf{C}\backslash(-\infty, -\lambda_0]$, which is real and negative if it exists. In this case we put $\kappa = -\lambda$, otherwise $\kappa = \lambda_0$.*

The essential growth rate of $\mathcal{S}(t)$ satisfies the following conditions.

(a) *If $\lim_{t\to 0+} a'(t) = -\infty$, then $\omega_1(\mathcal{S}) = -\kappa$.*

(b) *If $\lim_{t\to 0+} a'(t) = -F > -\infty$, then $\omega_1(\mathcal{S}) = \max(-\kappa, -\frac{F}{2(a(0)+E)})$.*

(c) *In particular, $\omega_1(\mathcal{S}) < 0$ if and only if $\lambda_0 > 0$.*

The proof of this theorem is again performed by several lemmas. As bounded perturbations of finite rank have no influence on the essential growth rate, we may again make the simplifying assumptions listed in Remark 2.6 without loss of generality.

LEMMA 3.2. Let the assumptions listed in Remark 2.6 be satisfied. If $\lambda \in \mathbf{C}\backslash(-\infty, -\lambda_0]$ is such that $E + \hat{\lambda a}(\lambda) \neq 0$ and $\alpha(\lambda) = -\lambda^2/E + \lambda\hat{a}(\lambda)$ is contained in the spectrum of DD^*, then λ is an eigenvalue of $\mathcal{A}$ with a finite dimensional generalized eigenspace $\mathcal{E}$, and there is a decomposition of $\mathcal{H}$ into invariant closed subspaces $\mathcal{E}$ and $\mathcal{F}$ such that $(\lambda - \mathcal{A})^{-1}$ exists on $\mathcal{F}$.

PROOF. Since α is holomorphic at λ and $(\rho - DD^*)^{-1}$ and $(\rho - D^*D)^{-1}$ have at most poles at $\rho = \alpha(\lambda)$, we infer that $R(\lambda)$ and $T(\lambda)$ as defined in Lemma 2.9 are meromorphic at λ. Hence $(\lambda - \mathcal{A})^{-1}$ is also. Using the techniques of [**16**, p. 178ff] we can decompose $\mathcal{H}$ into the generalized eigenspace $\mathcal{E}$ and some complement $\mathcal{F}$ where $(\lambda - \mathcal{A})^{-1}$ exists. To prove that the generalized eigenspace is nontrivial, but finite dimensional, it is sufficient to show the same for the eigenspace, as $\mathcal{E} = \ker(\lambda - \mathcal{A})^m$ where m is the order of the pole λ of $(\lambda - \mathcal{A})^{-1}$. Thus we consider the equation $(\lambda - \mathcal{A})(u, v, \phi) = 0$. Following the lines of the proof of Lemma 2.9 we infer that u determines v and ϕ uniquely, and u solves $\lambda^2 u + D(ED^*u + \lambda\hat{a}(\lambda)D^*u) = 0$, i.e., u is an eigenvector of DD^* with the eigenvalue $\alpha(\lambda)$. As the corresponding eigenspace of DD^* is nontrivial and finite dimensional, the same holds for the eigenspace of $\mathcal{A}$. □

LEMMA 3.3. *Assume that the assumptions listed in Remark* 2.6 *hold. For some* $\lambda \in \mathbf{C}\backslash(-\infty, -\lambda_0]$ *let* $E + \lambda\hat{a}(\lambda) = 0$. *Then* λ *is real and negative, and* $(\lambda - \mathcal{A})^{-1}$ *has an essential singularity at* λ. *Moreover, there is at most one such* λ.

PROOF. Put $\lambda = \rho + i\sigma$. If $\sigma \neq 0$, then

$$\begin{aligned}\operatorname{Im}(E + \lambda\hat{a}(\lambda)) &= \operatorname{Im} \int_{\lambda_0}^{\infty} \frac{\rho + i\sigma}{\rho + i\sigma + \mu} dg(\mu) \\ &= \int_{\lambda_0}^{\infty} \frac{\sigma\mu}{(\rho + \mu)^2 + \sigma^2} dg(\mu) \neq 0.\end{aligned}$$

Thus $\sigma = 0$. As for positive $\lambda, E + \lambda\hat{a}(\lambda) > E$, we infer that λ is negative. On $(-\lambda_0, 0)$ $\hat{a}(\lambda)$ is decreasing and $-E/\lambda$ is increasing, hence there is at most one solution λ with $-E/\lambda = \hat{a}(\lambda)$. If ν approaches λ from the left, $\alpha(\nu)$ converges to ∞ along the real line. Thus there is a sequence ν_n converging to λ such that $\alpha(\nu_n)$ lies in the spectrum of DD^*. Thus λ is a cluster point of poles of $(\lambda - \mathcal{A})^{-1}$. □

LEMMA 3.4. *With the conditions of Remark* 2.6, $(\lambda - \mathcal{A})^{-1}$ *is not meromorphic in any neighborhood of* $[-\lambda_0, \infty)$.

PROOF. By the lemma above, this assertion is clearly true if there exists some $\lambda > -\lambda_0$ with $E+\lambda\hat{a}(\lambda) = 0$. Thus assume the contrary and let $(\lambda - \mathcal{A})^{-1}$ be meromorphic in some neighborhood of $-\lambda_0$. We take an arbitrary eigenvalue $\xi \neq 0$ of DD^* and a corresponding eigenvector q. Putting $p = 0$ and $\psi = 0$ in (2.4) (in Lemma 2.9) we infer that

$$\lambda R(\lambda)q = \frac{\lambda}{\lambda^2 + (E + \lambda\hat{a}(\lambda))\xi} q$$

is meromorphic at $-\lambda_0$, hence $\hat{a}(\lambda)$ is also. We show that $\hat{a}$ is even analytic.

Consider first the case $\lambda_0 = 0$. As $\lim_{t\to\infty} a(t) = 0$, we have $\lim_{\lambda\to 0} \lambda\hat{a}(\lambda) = 0$, thus $\hat{a}(\lambda)$ has no pole at 0. If $\lambda_0 \neq 0$, we make use of the assumption that $E + \lambda\hat{a}(\lambda) \neq 0$ for $\lambda > -\lambda_0$. This implies $E > -\lambda\hat{a}(\lambda)$ for $\lambda > -\lambda_0$, which excludes that $-\lambda_0$ is a pole of $\hat{a}$. From [**31**, p. 58., Theorem 5b] we infer now that, for each λ in a sufficiently small neighborhood of $-\lambda_0$, the integral $\int_0^\infty e^{-\lambda t}a(t)dt$ is finite. Using the boundedness of a' on $[1, \infty)$, it is now easy to infer that, for such $\lambda, e^{-\lambda t}a(t)$ is bounded on $[1, \infty)$, in contradiction to the definition of λ_0. □

REMARK 3.5. Lemmas 3.3 and 3.4 ensure that the essential growth rate of $\mathcal{S}$ is not less than κ. Intuitively, it is appealing that no better decay rate is to be expected than the memory of the kernel decays. It is somewhat surprising that the bound obtained in Lemma 3.3 is due to the real part of the spectrum. This means that the high-frequency modes of DD^* also give rise to non-oscillating exponentially decaying solutions.

We begin now to derive estimates for $(\lambda - \mathcal{A})^{-1}$ for λ with large imaginary parts.

LEMMA 3.6. *Let $\psi \in L^2_g$, $x \in X$, $\lambda \in \mathbf{C}\backslash\mathbf{R}$, and let $\tilde{\psi}$ be defined as in Lemma 2.9. Then the following estimates hold:*

$$\int_0^\infty \left|\left|\frac{1}{\lambda+\mu}x\right|\right|^2 dg(\mu) = -\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}||x||^2, \tag{3.2}$$

$$\int_0^\infty \left|\left|\frac{1}{\lambda+\mu}\psi(\mu)\right|\right|^2 dg(\mu) \leq \frac{1}{|\operatorname{Im}\lambda|^2}||\psi||^2, \tag{3.3}$$

$$\|\tilde{\psi}(\lambda)\|^2 \leq -\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\|\psi\|^2. \tag{3.4}$$

PROOF. For (3.2):

$$\int_0^\infty \frac{1}{|\lambda+\mu|^2}\,dg(\mu)$$
$$= \frac{1}{\operatorname{Im}\lambda}\operatorname{Im}\Big[\int_0^\infty \frac{i\operatorname{Im}\lambda}{|\lambda+\mu|^2}\,dg(\mu) = -\frac{1}{\operatorname{Im}\lambda}\operatorname{Im}\Big[\int_0^\infty \frac{1}{\lambda+\mu}\,dg(\mu)\Big]\Big].$$

For (3.3):

$$\int_0^\infty \Big\|\frac{1}{\lambda+\mu}\psi(\mu)\Big\|^2 dg(\mu) \leq \int_0^\infty \frac{1}{|\operatorname{Im}\lambda|^2}\|\psi(\mu)\|^2\,dg(\mu).$$

For (3.4): take an arbitrary $x \in X$ with $\|x\| \leq 1$. Then

$$\begin{aligned}|\Big\langle x, \tilde{\psi}(\lambda)\Big\rangle| &= \Big|\int_0^\infty \Big\langle x, \frac{1}{\lambda+\mu}\psi(\mu)\Big\rangle\,dg(\mu)\Big| \\ &\leq \Big[\int_0^\infty \Big\|\frac{1}{\lambda+\mu}x\Big\|^2 dg(\mu)\Big]^{1/2}\Big[\int_0^\infty \|\psi(\mu)\|^2\,dg(\mu)\Big]^{1/2} \\ &\leq \Big[-\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\Big]^{1/2}\|\psi\|.\end{aligned}$$

□

LEMMA 3.7. *For some $\lambda \in \mathbf{C}\backslash\mathbf{R}$ such that $\alpha(\lambda) = -\lambda^2/E + \lambda\hat{a}(\lambda)$ is not contained in the spectrum of DD^*, let $R(\lambda)$, $T(\lambda)$ be defined as in Lemma 2.9. We define the quantities*

$$\beta(\lambda) = \sup_{\nu\in\sigma(DD^*)} \frac{1}{|\lambda^2 + [E+\lambda\hat{a}(\lambda)]\nu|},$$
$$\gamma(\lambda) = \sup_{\nu\in\sigma(DD^*)} \frac{\nu}{|\lambda^2 + [E+\lambda\hat{a}(\lambda)]\nu|}.$$

Then the following estimates hold:

(3.5)

$$\|R(\lambda)\| = \beta(\lambda),\ \|DD^*R(\lambda)\| = \gamma(\lambda),\ \|D^*R(\lambda)\| \leq [\beta(\lambda)\gamma(\lambda)]^{1/2},$$

(3.6)
$$||T(\lambda)|| = \beta(\lambda),\ ||D^*DT(\lambda)|| = \gamma(\lambda),\ ||DT(\lambda)|| \le [\beta(\lambda)\gamma(\lambda)]^{1/2}.$$

Moreover we have the identities

$$T(\lambda)D^* = D^*R(\lambda),\ R(\lambda)D = DT(\lambda). \tag{3.7}$$

PROOF. First consider (3.7): For $y \in \operatorname{dom} D^*$, $z = T(\lambda)D^*y$ is the unique solution to $[\lambda^2 + (E + \lambda\hat{a}(\lambda))D^*D]z = D^*y$ orthogonal to $\ker D$. Clearly $D^*R(\lambda)y$ is orthogonal to $\ker D$, and satisfies

$$\begin{aligned}[\lambda^2 + (E + \lambda\hat{a}(\lambda))D^*D]D^*R(\lambda)y &= D^*[\lambda^2 + (E + \lambda\hat{a}(\lambda))DD^*]R(\lambda)y \\ &= D^*y.\end{aligned}$$

The second identity is proved similarly. For (3.5) and (3.6): $R(\lambda)$ and $DD^*R(\lambda)$ are normal operators with eigenvalues $1/\lambda^2 + [E + \hat{\lambda a}(\lambda)]\nu$ and $\nu/\lambda^2 + [E + \hat{\lambda a}(\lambda)]\nu$, respectively, where ν takes all values from the spectrum of DD^*. Therefore the first two estimates in (3.5) are obvious. For any $y \in Y$ with $||y|| \le 1$ we have

$$\langle D^*R(\lambda)y,\ D^*R(\lambda)y\rangle = \langle R(\lambda)y, DD^*R(\lambda)y\rangle \le \beta(\lambda)\gamma(\lambda).$$

The proof for the estimates (3.6) is the same. □

LEMMA 3.8. *Let the assumptions listed in Remark 2.6 be satisfied. Let $\theta > 0$ be arbitrary. There is some consant M which satisfies the following estimate for all λ with $|\operatorname{Im}\lambda| \ge 1$, $|\operatorname{Re}\lambda| \le \theta$, such that $(\lambda - \mathcal{A})^{-1}$ exists:*

$$||(\lambda - \mathcal{A})^{-1}|| \le M\Big[|\lambda|\beta(\lambda) + \gamma(\lambda)|\hat{a}(\lambda)| + \Big(-\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2}\Big].$$

PROOF. Let $(\lambda - \mathcal{A})(u, v, \phi) = (p, q, \psi)$ where $||D^*p|| \le 1$, $||q|| \le 1$, $||\psi|| \le 1$. We have to estimate $||D^*u||$, $||v||$, and $||\phi||$. Utilizing the two previous Lemmas and Equation (2.4) in Lemma 2.9 we obtain

$$\begin{aligned}
||D^*u|| &= ||[\lambda + \hat{a}(\lambda)D^*D]T(\lambda)D^*p + D^*R(\lambda)q - D^*DT(\lambda)\tilde{\psi}(\lambda)|| \\
&\le |\lambda|\beta(\lambda) + |\hat{a}(\lambda)|\gamma(\lambda) + [\beta(\lambda)\gamma(\lambda)]^{1/2} + \gamma(\lambda)\Big(-\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2} \\
&\le |\lambda|\beta(\lambda) + |\hat{a}(\lambda)|\gamma(\lambda) + \Big(|\lambda\beta(\lambda)\gamma(\lambda)\hat{a}(\lambda)|\frac{1}{|\lambda\hat{a}(\lambda)|}\Big)^{1/2} \\
&\quad + \frac{\gamma(\lambda)|\hat{a}(\lambda)|}{|\hat{a}(\lambda)\operatorname{Im}\lambda|^{1/2}} \\
&\le M_1[|\lambda|\beta(\lambda) + \gamma(\lambda)|\hat{a}(\lambda)|]
\end{aligned}$$

with a suitable constant. Here we have used the fact that $\lambda\hat{a}(\lambda)$ is bounded away from 0, see Lemma 3.9c below, and that

$$\begin{aligned}
|\lambda| &= O(|\operatorname{Im}\lambda|.) \\
||v|| &= ||-EDT(\lambda)D^*p + \lambda R(\lambda)q - \lambda DT(\lambda)\tilde{\psi}(\lambda)|| \\
&\le E[\beta(\lambda)\gamma(\lambda)]^{1/2} + |\lambda|\beta(\lambda) + |\lambda|[\beta(\lambda)\gamma(\lambda)]^{1/2}\Big(-\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2} \\
&\le M_2[|\lambda|\beta(\lambda) + \gamma(\lambda)|\hat{a}(\lambda)|]
\end{aligned}$$

by the same reasoning. For the estimate on ϕ we need also

$$\begin{aligned}
||D^*v|| &= ||-ED^*DT(\lambda)D^*p + \lambda D^*R(\lambda)q - \lambda D^*DT(\lambda)\tilde{\psi}(\lambda)|| \\
&\le E\gamma(\lambda) + |\lambda|[\beta(\lambda)\gamma(\lambda)]^{1/2} + |\lambda|\gamma(\lambda)\Big(-\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
||\phi|| &\le \Big[E\gamma(\lambda) + |\lambda|[\beta(\lambda)\gamma(\lambda)]^{1/2} \\
&\text{kern.2in} + |\lambda|\gamma(\lambda)\Big(-\frac{\operatorname{Im}\hat{\mathrm{a}}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2}\Big]\Big(-\frac{\operatorname{Im}\hat{\mathrm{a}}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2} + \frac{1}{|\operatorname{Im}\lambda|} \\
&\le \gamma(\lambda)|\hat{a}(\lambda)|\frac{E}{|\hat{a}(\lambda)\operatorname{Im}(\lambda)|^{1/2}} + |\lambda\beta(\lambda)\gamma(\lambda)\hat{a}(\lambda)|^{1/2}\Big|\frac{\lambda}{\operatorname{Im}\lambda}\Big|^{1/2} \\
&\quad + |\gamma(\lambda)\hat{a}(\lambda)|\Big|\frac{\lambda}{\operatorname{Im}\lambda}\Big| + \Big(-\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2}\Big|\frac{1}{\operatorname{Im}\lambda\operatorname{Im}\hat{a}(\lambda)}\Big|^{1/2} \\
&\le M_3\Big[|\lambda|\beta(\lambda) + \gamma(\lambda)|\hat{a}(\lambda)| + \Big(-\frac{\operatorname{Im}\hat{a}(\lambda)}{\operatorname{Im}\lambda}\Big)^{1/2}\Big].
\end{aligned}$$

(We have again used Lemma 3.9c.) □

LEMMA 3.9. *Put* $\mathcal{H}(\lambda) = |\lambda| \mathrm{Im}\,[\lambda\hat{a}(\lambda)]/E + \mathrm{Re}[\lambda\hat{a}(\lambda\hat{a}(\lambda)]$.

a) *If* $\lim_{t\to 0+} a'(t) = -\infty$, *then* $\lim_{\sigma\to\infty} \kappa(\rho+i\sigma) = \infty$, *uniformly for* ρ *in compact intervals.*

b) *If* $\lim_{t\to 0+} a'(t) = -F > -\infty$, *then* $\lim_{\sigma\to\infty} \kappa(\rho+i\sigma) = \frac{F}{E+a(0)}$, *uniformly for* ρ *in compact intervals.*

c) $\liminf_{\sigma\to\infty} \mathrm{Re}[(\rho+i\sigma)\hat{a}(\rho+i\sigma)] > 0$, *uniformly for* ρ *in compact intervals. (In particular,* $\sigma \mathit{Im}\,\hat{a}(\rho+i\sigma)$ *is bounded away from 0 for* ρ *in compact intervals and* $|\sigma| \geq 1$.)

PROOF.

a) Let $a'(0) = -\infty$, i.e., $\int_0^\infty \mu dg(\mu) = \infty$. Note that $(\rho+i\sigma)\hat{a}(\rho+i\sigma) = \int_0^\infty \frac{\rho^2+\rho\mu+\sigma^2+i\mu\sigma}{(\rho+\mu)^2+\sigma^2} dg(\mu)$. As $\lim_{\sigma\to\infty} \frac{|\rho+i\sigma|}{\sigma} = 1$, it is sufficient to show that, for any given N, the following inequality holds for sufficiently large σ:

$$\int_0^\infty \frac{\mu\sigma^2}{(\rho+\mu)^2+\sigma^2} dg(\mu) > NE + N\int_0^\infty \frac{\rho^2+\rho\mu+\sigma^2}{(\rho+\mu)^2+\sigma^2} dg(\mu).$$

Since $\int_0^\infty \frac{\mu\sigma^2}{(\rho+\mu)^2+\sigma^2} dg(\mu) = \int_0^\infty \frac{\mu}{(\rho+\mu)^2/\sigma^2+1} dg(\mu) \to \infty$, the contribution of NE may be ignored. Now, for any $\sigma > 0$,

$$\Big| \int_0^{2N} \frac{(\mu-N)\sigma^2 - N\rho\mu - N\rho^2}{(\rho+\mu)^2+\sigma^2} dg(\mu) \Big| \leq N\int_0^{2N} dg(\mu).$$

Assume without loss of generality that $N > \rho$, and let σ be sufficiently large such that $\sigma^2 \geq 8N\rho$. Then

$$\begin{aligned} \int_{2N}^\infty \frac{(\mu-N)\sigma^2 - N\rho\mu - N\rho^2}{(\rho+\mu)^2+\sigma^2} dg(\mu) &\geq \int_{2N}^\infty \frac{\mu\sigma^2/2 - \mu\sigma^2/8 - \mu\sigma^2/8}{(\rho+\mu)^2+\sigma^2} dg(\mu) \\ &= \frac{1}{4}\int_{2N}^\infty \frac{\mu}{(\rho+\mu)^2/\sigma^2+1} dg(\mu) \to \infty, \end{aligned}$$

thus

$$\int_0^\infty \frac{(\mu-n)\sigma^2 - N\rho\mu - N\rho^2}{(\rho+\mu)^2+\sigma^2} dg(\mu) \to \infty \quad \text{as } \sigma\to\infty.$$

b)

$$\begin{aligned}\sigma \mathrm{Im}\,[(\rho+i\sigma)\hat{a}(\rho+i\sigma)] &= \int_0^\infty \frac{\mu\sigma^2}{(\rho+\mu)^2+\sigma^2} dg(\mu)\\ &= \int_0^\infty \frac{\mu}{(\rho+\mu)^2/\sigma^2+1} dg(\mu) \to \int_0^\infty \mu dg(\mu)\\ &= F,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Re}[(\rho+i\sigma)\hat{a}(\rho+i\sigma)]\\ &= \int_0^\infty \frac{\rho^2+\rho\mu+\sigma^2}{(\rho+\mu)^2+\sigma^2} dg(\mu)\\ &= \int_0^\infty \frac{(\rho^2+\rho\mu)/\sigma^2+1}{(\rho+\mu)^2/\sigma^2+1} dg(\mu) \to \int_0^\infty dg(\mu)\\ &= a(0).\end{aligned}$$

Thus $\kappa\mathcal{H}(\rho+i\sigma) \to F/E + a(0)$.

c) $\mathrm{Re}[(\rho+i\sigma)\hat{a}(\rho+i\sigma)] = \int_0^\infty \frac{\rho^2+\sigma^2+\rho\mu}{(\rho+\mu)^2+\sigma^2} dg(\mu) = \int_0^\infty \frac{\rho^2+\sigma^2}{(\rho+\mu)^2+\sigma^2} dg(\mu) + \int_0^\infty \frac{\rho\mu}{(\rho+\mu)^2+\sigma^2} dg(\mu)$. The first integral converges to $\int_0^\infty dg(\mu)$ (whether this is finite or infinite) as $\sigma \to \infty$, while the second integral goes to zero by dominated convergence. □

LEMMA 3.10. *Let the assumptions listed in Remark* 2.6 *be satisfied.*

a) *If* $\lim a'(t) = -\infty$, *then* $\lim_{\sigma\to\infty} ||(\rho+i\sigma-\mathcal{A})^{-1}|| = 0$, *uniformly for* ρ *in compact intervals.*

b) *If* $\lim_{t\to 0+} a'(t) = -F > -\infty$ *and* $\rho_0 > -F/2(E+a(0))$, *then there exists some* $\sigma_0 > 0$ *such that* $||(\rho+i\sigma-\mathcal{A})^{-1}||$ *is uniformly bounded for* $\sigma > \sigma_0, \rho > \rho_0$.

c) *If* $\lim_{t\to 0+} a'(t) = -F > -\infty$, *then* $\limsup_{\sigma\to\infty} ||(\rho+i\sigma-\mathcal{A})^{-1}||$

$= \infty$ *for* $\rho = -F/2(E+a(0))$.

PROOF. By Lemma 3.8, assertions (a) and (b) can be proved by finding upper estimates for $|\rho+i\sigma|\beta(\rho+i\sigma)$, $|\hat{a}(\rho+i\sigma)|\gamma(\rho+i\sigma)$, and $-\mathrm{Im}\,\hat{a}(\rho+i\sigma)/\sigma$. Using formula (2.4) (for v) and $||R(\lambda)|| = |\lambda|\beta(\lambda)$, one can prove (c) by deriving a lower estimate for $|\rho + i\sigma|\beta(\rho + i\sigma)$. For simplicity, we will write λ for $\rho + i\sigma$. It is obvious that $-\mathrm{Im}\,\hat{a}(\rho + i\sigma)/\sigma$ converges to 0 if $\sigma \to \infty$, uniformly for ρ in compact intervals. Let us now derive estimates for $|\lambda|\beta(\lambda)$. By its definition,

$$
\begin{aligned}
\frac{1}{\beta(\lambda)} &\geq \mathrm{inf}_{\nu\in\mathbf{R}}|\lambda^2(E + \lambda\hat{a}(\lambda))\nu| \\
&= |E + \lambda\hat{a}(\lambda)|\mathrm{inf}_{\nu\in\mathbf{R}}\left|\frac{\lambda^2}{E + \lambda\hat{a}(\lambda)} - \nu\right| \\
&= |E + \lambda\hat{a}(\lambda)|\left|\mathrm{Im}\,\frac{\lambda^2}{E + \lambda\hat{a}(\lambda)}\right| \\
&= \left|\frac{2\rho\sigma\mathrm{Re}(E + \lambda\hat{a}(\lambda)) + (\sigma^2 - \rho^2)\mathrm{Im}\,(\lambda\hat{a}(\lambda))}{E + \lambda\hat{a}(\lambda)}\right| \\
&= \left|\frac{2\rho\sigma + \kappa(\lambda)(\sigma^2 - \rho^2)/|\lambda|}{[1 + \kappa^2(\lambda)/|\lambda|^2]^{1/2}}\right|
\end{aligned}
$$

Thus

$$
\frac{1}{|\lambda|\beta(\lambda)} \geq \left|\frac{2\rho\sigma/|\lambda| + \mathcal{H}(\lambda)(\sigma^2 - \rho^2)/|\lambda|^2}{[1 + \mathcal{H}^2(\lambda)/|\lambda|^2]^{1/2}}\right|. \tag{3.8}
$$

In case (a), $\kappa(\rho + i\sigma)$ goes to infinity if $\sigma \to \infty$. Using $|\sigma/\lambda| \to 1$, we infer that $1/|\lambda|\beta(\lambda) \to \infty$, thus $\lambda\beta(\lambda)$ converges to 0. In case (b), $\kappa(\rho+i\sigma)$ converges to $F/(E + a(0))$, thus the denominator in (3.8) goes to 1 while the numerator converges to $2\rho + F/(E + a(0))$. Therefore, $\lambda\beta(\lambda)$ stays bounded. To check (c) put $\rho = -F/2(E + a(0))$. Then the numerator in (3.8) tends to 0. From the computations leading to (3.8) we see that $\frac{1}{|\lambda|}\mathrm{inf}_{\nu\in\mathbf{R}}|\lambda^2 + (E + \lambda\hat{a}(\lambda))\nu| \to 0$. The infimum is obtained at

$$
\nu = -\mathrm{Re}\frac{\lambda^2}{E + \lambda\hat{a}(\lambda)} = \frac{\sigma^2 - \rho^2 + 2\kappa(\lambda)\rho\sigma/|\lambda|}{[1 + \kappa^2(\lambda)/|\lambda|^2][E + \mathrm{Re}\lambda\hat{a}(\lambda)]} \to \infty.
$$

Consequently we may pick a sequence $\sigma_n \to \infty$ such that the corresponding ν_n are precisely the eigenvalues of DD^*. For these σ_n, $1/|\lambda|\beta(\lambda) = \frac{1}{|\lambda|}\mathrm{inf}_{\nu\in\mathbf{R}}|\lambda^2 + (E + \lambda\hat{a}(\lambda))\nu| \to 0$. Therefore, $\lambda\beta(\lambda)$ is unbounded. This completes the proof of (c). To finish the proof of (a) and (b) we need similar estimates for $|\hat{a}(\lambda)|\gamma(\lambda)$. Proceeding as above we obtain

$$\begin{aligned}\frac{1}{\gamma(\lambda)} &\geq \inf_{\nu\in\mathbf{R}}\Big|\frac{\lambda^2}{\nu}+E+\lambda\hat{a}(\lambda)\Big| \\ &= |\lambda|^2\Big|\mathrm{Im}\,\frac{E+\lambda\hat{a}(\lambda)}{\lambda^2}\Big| \\ &= \Big|\frac{(\rho^2-\sigma^2)\mathrm{Im}\,(\lambda\hat{a}(\lambda))-2\rho\sigma\mathrm{Re}(E+\lambda\hat{a}(\lambda))}{\lambda^2}\Big| \\ &= |\mathrm{Re}(E+\lambda\hat{a}(\lambda))|\Big|\frac{2\rho\sigma}{|\lambda|^2}+\frac{(\sigma^2-\rho^2)\kappa(\lambda)}{|\lambda|^3}\Big|.\end{aligned}$$

By Lemma 3.9 (c) we infer that $|\lambda\hat{a}(\lambda)|\leq|E+\lambda\hat{a}(\lambda)|$ for sufficiently large σ. Thus

$$|\hat{a}(\lambda)|\leq\Big|\frac{E+\lambda\hat{a}(\lambda)}{\lambda}\Big|=|\mathrm{Re}(E+\lambda\hat{a}(\lambda))|\Big|\frac{(1+\kappa^2(\lambda)/|\lambda|^2)^{1/2}}{\lambda}\Big|.$$

Hence $\frac{1}{|\hat{a}(\lambda)|\gamma(\lambda)}\geq\Big|\frac{2\rho\sigma/|\lambda|+\kappa(\lambda)(\sigma^2-\rho^2)/|\lambda|^2}{(1+\kappa^2(\lambda)/|\lambda|^2)^{1/2}}\Big|$. Again, in case (a), this expression goes to infinity, and in case (b) it is bounded away from 0. Now we have proved all estimates to show that $||(\lambda-\mathcal{A})^{-1}||$ converges to 0 in case (a) and is bounded in case (b). □

PROOF OF THEOREM 3.1. We may again make the simplifying assumptions listed in Remark 2.6. First we estimate the essential growth rate of $\mathcal{S}$ from above. Choose some ρ_0 such that $\rho_0>-\kappa$, and in case (b) also $\rho_0>-F/2(E+a(0))$. From Lemma 3.10 we infer that, for sufficiently large σ_0, $||(\rho+i\sigma-\mathcal{A})^{-1}||$ is uniformly bounded for $|\rho|\leq|\rho_0$, $|\sigma|\geq\sigma_0$. By dissipativity of $\mathcal{A}$, uniform boundedness holds generally for $\rho\geq|\rho_0|$. On the other hand, $(\rho+i\sigma-\mathcal{A})^{-1}$ is meromorphic for $\rho>-\kappa$, thus there are only finitely many poles in the compact set $\{\rho+i\sigma:\rho_0\leq\rho\leq 0,\ |\sigma|\leq\sigma_0\}$. By Lemma 3.2 we may factor off the corresponding generalized eigenspaces, thus decomposing $\mathcal{H}$ into a finite dimensional $\mathcal{S}$-invariant subspace $\mathcal{H}_1$ and an invariant complement $\mathcal{H}_2$ such that the restriction of $(\rho+i\sigma-\mathcal{A})^{-1}$ to $\mathcal{H}_2$ is uniformly bounded on $\{\rho+i\sigma:\rho\geq\rho_0\}$. By Gearhart's Theorem (see, e.g., [**21**, Proposition 2] or [**2**, p. 96]) this implies that the growth rate of $\mathcal{S}$ restricted to $\mathcal{H}_s$ is not greater than ρ_0, thus the essential growth rate of $\mathcal{S}$ on $\mathcal{H}$ does not exceed ρ_0. We have proved that $\omega_1(\mathcal{S}) \leq -\kappa$ in case (a) and $\omega_1(\mathcal{S}) \leq \max(-\kappa, \frac{-F}{2(E+a(0))})$ in case (b). To prove the converse inequality, notice that $(\rho+i\sigma-\mathcal{A})^{-1}$ is meromorphic for $\rho > \omega_1(\mathcal{S})$, thus, by Lemmas 3.3 and 3.4, $-\kappa \leq \omega_1(\mathcal{S})$. This settles case (a). Moreover, if $\rho > \omega_1(\mathcal{S})$, κ can be decomposed into a finite dimensional invariant subspace $\mathcal{H}_1$ and an invariant complement $\mathcal{H}_2$ where the growth rate is less than ρ. On both spaces, $\lim_{\sigma\to\infty} ||(\rho+i\sigma-\mathcal{A})^{-1}|| = 0$. Hence by Lemma 3.10 (c), $\rho > -F/2(E+a(0))$ in case (b). This completes the proof of Theorem 3.1. □

4. Applications. In this section we apply the theory developed above to some mechanical stabilization problems. The basic situation is that a rigid body is connected to some flexible members, which are assumed to be linearly viscoelastic. A feedback law controlling a force or torque acting on the rigid part is implemented in order to stabilize the vibrations of the system exponentially. Whether this is possible or not depends on the constitutive equation of the flexible material, not on the particular structure of the system, as our theory shows. We will discuss two simple model problems to show how the abstract setting can be adapted to mechanical problems. Finally we rewrite constitutive equations of fractional derivative type to fit into our assumptions.

The following two mechanical models are taken from [**27, 28**]. The case of smooth relaxation moduli has been treated in [**10**].

EXAMPLE 4.1. Consider a rigid body glued to the face of a cylindrical flexible rod, subject to motion in axial direction only. The other end of the rod is supposed to be free (Figure 1). Let $u_r(t)$ be the position of the rigid mass at time t, $u_f(t,\zeta)$ the displacement of the cross section with (Lagrangian) body coordinates ζ of the flexible rod, $v_r(t)$ and $v_f(t,\zeta)$ the corresponding velocities. $\sigma(t,\zeta)$ denotes the stress at time t and Lagrangian coordinates ζ in the flexible part. Let l be the length of the rod, for simplicity let its cross sectional area equal 1, let m be the mass of the rigid body and ρ the mass density per unit length in the stress free reference configuration of the rod. A control force f acts on the rigid body, obtained by a feedback law from the position and velocity of the rigid part. The equation of momentum is then

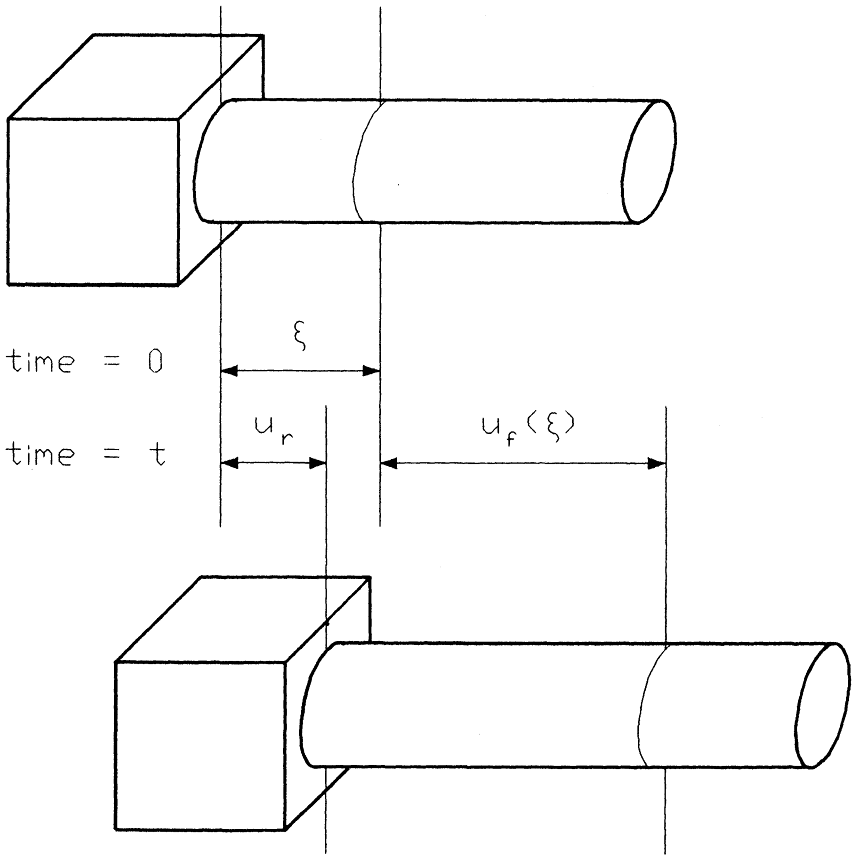

FIGURE 1. Axially Flexible Appendage

$$m\frac{\partial}{\partial t}v_r(t) = \sigma(t,0) + f(t), \tag{4.1}$$

$$\rho\frac{\partial}{\partial t}v_f(t,\zeta) = \frac{\partial}{\partial\zeta}\sigma(t,\zeta). \tag{4.2}$$

For the flexible material we assume a constitutive equation of the following kind

$$\sigma(t,\zeta) = E\frac{\partial}{\partial\zeta}u_f(t,\zeta) + \int_0^t a(t-s)\frac{\partial}{\partial\zeta}v_f(s,\zeta)\,ds \tag{4.3}$$

with a completely monotone kernel a and some $E > 0$ ($E + a$ is the bulk relaxation modulus of the material). The problem fits into the abstract setting as follows: Let the velocity state $v(t)$ be the pair $(v_r(t), v_f(t,.))$ in $\mathbf{R} \times L^2([0,l],\mathbf{R})$, normed by kinetic energy $||(v_r, v_f)||^2 = mv_r^2 + \rho \int_0^l v_f^2(\zeta)\,d\zeta$. The displacement state is the pair $u(t) = (u_r(t), u_f(t,.))$ in a suitable subspace of $\mathbf{R} \times L^2([0,l],\mathbf{R})$ to be specified below. $\sigma(t)$ is just the function $\sigma(t,.)$ in $L^2([0,l],\mathbf{R})$ with the usual L^2-norm (which would be proportional to the potential strain energy in a purely elastic case). The equation of momentum and the free end condition are now translated to

$$\frac{\partial}{\partial t}(v_r, v_f) = \Big(\frac{1}{m} f(t), 0\Big) - D\sigma(t), \tag{4.4}$$

where

$$D\sigma = -\Big(\frac{1}{m}\sigma(0),\ \frac{1}{\rho}\frac{\partial}{\partial\zeta}\sigma(\zeta)\Big),$$

defined on

$$\operatorname{dom} D = \{\sigma \in W^{1,2} : \sigma(l) = 0\}.$$

Evidently,

$$\frac{\partial}{\partial t}(u_r, u_f) = (v_r, v_f). \tag{4.5}$$

The constitutive equation yields

$$\sigma(t) = ED^* u(t) + \int_0^t a(t-s) D^* v(s) ds, \tag{4.6}$$

where $D^*(v_r, v_f) = \frac{\partial}{\partial \xi} v_f$, defined on $\operatorname{dom} D^* = \{(v_r, v_f) \in \mathbf{R} \times W^{1,2} :$

$v_r = v_f(0)\}$. It can now be easily checked that with the norms introduced above D and D^* are in fact adjoint to each other. Equations (4.4), (4.5), (4.6) are precisely the abstract equations of system (2.1). We see also that the natural space for displacements is $\{(u_f, u_f) \in \mathbf{R} \times W^{1,2} : u_r = u_f(0)\}$.

The results of the previous section imply that exponential stabilization is possible if and only if a satisfies an exponential bound $a(t) \leq Me^{-\omega t}$ with some $\omega > 0$ for $t \geq 1$. Whether or not the problem is actually exponentially stable in the latter case, depends of course on the specified feedback law. If the feedback law is given by $f(t) = -c_1 u_r(t) - c_2 v_r(t)$ with $c_k > 0$, then it can be proved in an analogous manner to **[10]** that the Laplace transform of the solution admits no poles in the closed positive half plane, hence the system is exponentially stable.

The same equations arise (in a manner which is more rigorous from the viewpoint of continuum mechanics) if one considers torsional motion of a rod with a rigid mass fixed to its end. This system has been investigated in detail in **[14]**.

EXAMPLE 4.2. We consider a simple model of a rotating satellite, consisting of a rigid hub and four flexible rods attached to it in the radial direction (Figure 2). We treat only rotation about the axis of the hub. A control torque f is applied to the hub to stabilize the motion of the system. The flexible rods are modelled as Euler-Bernoulli beams. In particular we assume that the deflections of the spokes are small. Let $u_r(t)$ and $v_r(t)$ be the angular position and velocity of the hub. R_I denotes the radius of the hub, $R_I - R_0$ is the length of the spokes. $y(t,\zeta)$ denotes the deflection of the spoke from the radial direction at time t and distance ζ from the axis, $\sigma(t,\zeta)$ denotes the bending moment in the rods at distance ζ from the origin. It is convenient to introduce the variable $u_f(t,\zeta) = \zeta u_r(t) + y(t,\zeta)$ and its time derivative $v_f(t,\zeta)$. With suitable physical constants I_H (moment of inertial of the hub), I (cross sectional moment of inertia of the rods), ρ (mass density per unit length of the rods) one obtains the following equations of momentum:

$$I_H \frac{d}{dt} v_r(t) = -4R_I \frac{\partial}{\partial \zeta}\sigma(t,R_I) + 4\sigma(t,R_I) + f(t), \tag{4.7}$$

$$\rho \frac{\partial}{\partial t} v_f(t,\zeta) = -\frac{\partial^2 f}{\partial \zeta^2}\sigma(t,\zeta). \tag{4.8}$$

Assuming a linear viscoelastic constitutive equation as in the previous example, we are led to

$$\sigma(t,\zeta) = IE\frac{\partial^2}{\partial \zeta^2}u_f(t,\zeta) + \int_0^t Ia(t-s)\frac{\partial^2}{\partial \zeta^2}v_f(s,\zeta)\,ds. \tag{4.9}$$

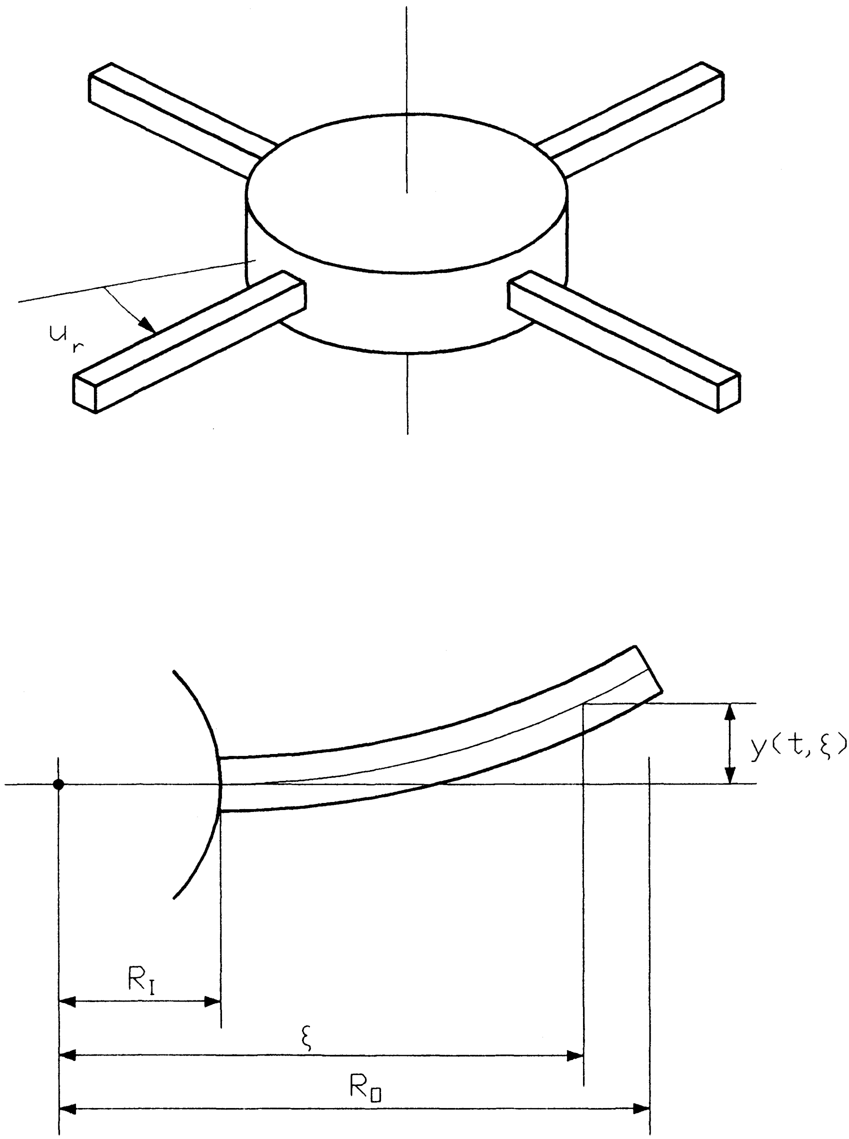

FIGURE 2. Rotating Satellite Model

The boundary conditions are

$$
\begin{aligned}
&u_f(t,R_I) = R_I u_r(t), \quad \frac{\partial}{\partial\zeta}u_f(t,R_I) = u_r(t),\\
(4.10)\qquad &v_f(t,R_I) = R_I v_r(t), \quad \frac{\partial}{\partial\zeta}v_f(t,R_I) = v_r(t),\\
&\sigma(t,R_0) = 0, \quad \frac{\partial}{\partial\zeta}\sigma(t,R_0) = 0.
\end{aligned}
$$

(These are the equations [**10**, (4.15)-(4.17)] rewritten as a first order system.) Again let the velocity state be $v = (v_r, v_f) \in \mathbf{R} \times L^2([R_I,R_0],\mathbf{R})$, normed by kinetic energy $||(v_r,v_f)||^2 = \frac{H}{16}v_r^2 + \int_{R_I}^{R_0} \rho v_f^2(\zeta)\,d\zeta$, and $u = (u_r,u_f)$ in a suitable subspace (namely domD^* to be specified below), $\sigma \in L^2([R_I,R_0],\mathbf{R})$ with the norm $||\sigma||^2 = \int_{R_I}^{R_0} \sigma^2(\zeta)/I\,d\zeta$. Then the equations (4.7)-(4.10) may be rewritten in astract form:

$$
\begin{aligned}
u' &= v(t),\\
v'(t) &= -D\sigma(t) + \Big(\frac{1}{I_H}f(t),0\Big),\\
\sigma(t) &= ED^*u(t) + \int_0^t a(t-s)D^*v(s)\,ds
\end{aligned}
$$

with $D(\sigma) = (\frac{4}{I_H}(-R_I\frac{\partial}{\partial\zeta}\sigma(R_I)+\sigma(R_I)),\ -\frac{1}{I}\frac{\partial^2}{\partial\zeta^2}\sigma)$, defined on $\operatorname{dom} D = \{\sigma \in W^{2,2} : \sigma(R_0) = \frac{\partial}{\partial\zeta}\sigma(R_0) = 0\}$, and $D^*(v_r,v_f) = -I\frac{\partial^2}{\partial\zeta^2}v_f$, defined on dom$D^* = \{(v_r,v_f) \in \mathbf{R}\times W^{2,2} : v_f(R_I) = R_I v_r,\ \ \frac{\partial}{\partial\zeta}v_f(R_I) = v_r\}$. A somewhat lengthy but straightforward computation shows again that D and D^* are adjoint to each other, so that the results of the previous sections can be applied. In particular, exponential stabilization is possible if and only if the relaxation kernel a decays exponentially. A calculation similar to that in [**10**] yields that in this case a feedback of type $f(t) = -c_1u_r(t) - c_2v_r(t)$ with $c_1 > 0$ and $c_2 \geq 0$ stabilizes the system exponentially.

REMARK 4.3. The last example can be criticized on the following grounds. This paper concerns the behavior of the high-frequency models of the system. It is known that, for short wavelengths, the

Euler-Bernoullie model of the beam deviates significantly from reality, i.e., to obtain a reasonably realistic model, one has to abandon the assumption that the deformation of the beam is orthotropic [**17**]. Better models (e.g. Timoshenko's beam equation) include also shear stresses. As a Timoshenko model deals with two different relaxation moduli, it does not fit immediately into a theory based on scalar kernels. However, the methods of this paper may be extended to operator valued kernels with some technical effort [**9**]. In this generality, one can also treat genuine three-dimensional problems without isotropy or symmetry properties.

Fractional derivative laws. The constitutive equation relating strain ε to stress σ in the examples above is always of the form

$$\sigma(t) = E\varepsilon(t) + \int_0^t a(t-s)\frac{\partial}{\partial s}\varepsilon(s)\,ds, \tag{4.11}$$

with a positive constant E and a completely monotone kernel a. For example, $a(t)$ could be the usual finite sum of exponential terms. It is known that a completely monotone kernel a is also obtained from certain fractional derivative models (see, e.g., [**3**]). For sake of completeness we show below how this is accomplished. Equations with fractional derivatives have been proposed and successfully fitted to experimental data [**4,5**]. For such models the constitutive equation has the form

$$\sigma(t) + \beta\frac{\partial^\nu}{\partial t^\nu}\sigma(t) = E\Big(\varepsilon(t) + \alpha\frac{\partial^\nu}{\partial t^\nu}\varepsilon(t)\Big), \tag{4.12}$$

with positive constants α and E, $0 \le \beta < \alpha$, $0 < \nu < 1$. The fractional derivative is defined by the relation

$$\frac{\partial^\nu}{\partial t^\nu}\sigma(t) = \frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\nu}\sigma(s)\,ds.$$

(This formula is correct for functions σ which vanish on $(-\infty, 0]$. For other functions the integral must be taken from $-\infty$ to t. In the discussion below we assume that stress and strain vanish for negative t.)

We show that the constitutive equation (4.12) can be rewritten in form (4.11). Inserting the definition of the fractional derivative in (4.12) we have

$$
\begin{aligned}
&\sigma(t) + \frac{\beta}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\nu}\sigma(s)\,ds \\
&\qquad = E\Big(\varepsilon(t) + \frac{\alpha}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\nu}\varepsilon(s)\,ds\Big).
\end{aligned}
\tag{4.13}
$$

If $\beta = 0$, then this equation is simply

$$\sigma(t) = E\varepsilon(t) + \frac{E\alpha}{\Gamma(1-\nu)}\int_0^t (t-s)^{-\nu}\frac{\partial}{\partial t}\varepsilon(s)\,ds.$$

This is (4.11) with $a(t) = \frac{E\alpha}{\Gamma(1-\nu)}t^{-\nu}$. This is a singular kernel with $a(0) = \infty, a(0) = -\infty$. However, since a does not decay exponentially, exponential stabilization is impossible for systems with this kernel.

We remark that some authors (e.g., [**14**]) consider modified fractional derivative laws where $t^{-\nu}$ is multiplied by some exponential factor which insures exponential decay. Consider for instance the kernel $a(t) = E\alpha(\pi t)^{-1/2}e^{-\mu t}$ with some $\mu > 0$, which corresponds to a modified fractional derivative law with $\nu = 1/2$. The derivative of a near 0 is unbounded, therefore the real parts of the eigenvalues tend to $-\infty$ as the imaginary parts tend to ∞. The essential growth rate is determined by the solution to

$$0 = E + \kappa\hat{a}(\kappa) = E[1 + \alpha\kappa(\kappa+\mu)^{-1/2}],$$

i.e.,

$$\kappa = -\frac{(1+4\alpha^2\mu)^{1/2} - 1}{2\alpha^2}.$$

In particular, exponential stabilization is possible.

Some more work is in order if $\beta \neq 0$. In (4.13) we take convolutions with the kernel $\frac{1}{\beta\Gamma(\nu)}t^{\nu-1}$ (which is inverse to $\beta\frac{\partial^\nu}{\partial t^\nu}\cdot$ with respect to convolution) and obtain

$$
\begin{aligned}
&\sigma(t) + \int_0^t \frac{1}{\beta\Gamma(\nu)}(t-s)^{\nu-1}\sigma(s)\,ds \\
&\quad = \frac{E\alpha}{\beta}\varepsilon(t) + \int_0^t \frac{E}{\beta\Gamma(\nu)}(t-s)^{\nu-1}\varepsilon(s)\,ds \\
&\quad = E\Big[\varepsilon(t) + \int_0^t \frac{1}{\beta\Gamma(\nu)}(t-s)^{\nu-1}\varepsilon(s)\,ds\Big] + E\Big(\frac{\alpha}{\beta} - 1\Big)\varepsilon(t).
\end{aligned}
\tag{4.14}
$$

Let r be the solution of

$$(\delta - r) * \left(\delta + \frac{1}{\beta\Gamma(\nu)} t^{\nu-1}\right) = \delta. \tag{4.15}$$

(This is convolution notation for

$$r(t) + \int_0^t \frac{1}{\beta\Gamma(\nu)} (t-s)^{\nu-1} r(s)\, ds = \frac{1}{\beta\Gamma(\nu)} t^{\nu-1},$$

where δ denotes the Dirac delta distribution.) Taking convolutions with $(\delta - r)$ in (4.14) yields

$$\sigma(t) = \frac{E\alpha}{\beta}\varepsilon(t) - E\Big(\frac{\alpha}{\beta} - 1\Big) \int_0^t r(t-s)\varepsilon(s)\, ds. \tag{4.16}$$

From [**18**, p. 221] we get that r is positive and $\int_0^\infty r(s)ds = 1$. Putting $R(t) = \int_t^\infty r(s)ds = 1 - \int_0^t r(s)ds$ we can integrate the right hand side of (4.16) by parts, thus

$$\begin{aligned}
\sigma(t) &= \frac{E\alpha}{\beta}\varepsilon(t) + E\Big(\frac{\alpha}{\beta} - 1\Big) \int_0^t R(t-s)\frac{\partial}{\partial s}\varepsilon(s)\, ds - E\Big(\frac{\alpha}{\beta} - 1\Big)\varepsilon(t) \\
&= E\varepsilon(t) + E\Big(\frac{\alpha}{\beta} - 1\Big) \int_0^t R(t-s)\frac{\partial}{\partial s}\varepsilon(s)\, ds \\
&= E\varepsilon(t) + \int_0^t a(t-s)\frac{\partial}{\partial s}\varepsilon(s)\, ds,
\end{aligned}$$

with $a(t) = E(\frac{\alpha}{\beta} - 1)R(t)$. Integrating (4.15) from 0 to t we infer that

$$R * \left(\delta + \frac{1}{\beta\Gamma(\nu)} t^{\nu-1}\right) = 1,$$

i.e.,

$$R(t) + \int_0^t \frac{1}{\beta\Gamma(\nu)} (t-s)^{\nu-1} R(s)\, ds = 1.$$

From [**26**, Satz 1] we infer that R is completely monotone. One can easily derive from (4.15) that $r = -R'$ is unbounded near 0, while R of course is bounded. So in this case the kernel is "less singular" than in the case $\beta = 0$. STill the singularity is strong enough to push the real parts of poles with large imaginary parts arbitrarily far to the left. Again the kernel does not decay exponentially (which can be concluded, e.g., from [**10**, Corollary 2.8]), thus once more exponential stabilization is impossible, unless the kernel is modified by an exponential factor.

REFERENCES

1. J. Achenback and D. Reddy, *Note on wave propagation in linearly viscoelastic media*, Z. Angew. Math. Phys. **18** (1967), 141-144.

2. W. Arendt et al., *One parameter semigroups of positive operators*, R. Nagel, ed., Lecture Notes in Math. **1184**, Springer 1986.

3. R.L. Bagley, *On relationships between the power law model and fractional calculus model of viscoelastic behavior*, Preprint, Dept. of Aeronautics and Astronautics, Air Force Institute of Technology, Wright-Patterson A.F.B., 1987.

4. ——— and P.J. Torvik, *Fractional calculus, a different approach to viscoelastically damped structures*, AIAA Journal **21** (1983), 741-748.

5. M. Caputo and F. Mainardi, *Linear models of dissipation in anelastic solids*, Riv. Nuovo Cimento ser. II, **1** (1971), 161-198.

6. G. Chen and R.C. Grimmer, *Semigroups and integral equations*, J. Integral Equations 2 (1980), 133-154.

7. C.M. Dafermos, *Contraction semigroups and trend to equilibrium in continuum mechanics*, IUTAM/IMU Symp. on Applications of Methods of Functional Analysis to Problems in Mechanics, P. Germain and P. Nayroles (Eds.), Lecture Notes in Math. **503**, Springer, Berlin 1976.

8. W. Desch and R.C. Grimmer, *Smoothing properties of linear Volterra integrodifferential equations*, to appear.

9. ——— and ———, *Singular relaxation moduli and smoothing in three-dimensional viscoelasticity*, in preparation

10. ——— and R.K. Miller, *Exponential stabilization of Volterra integrodifferential equations in Hilbert space*, J. Differential Equations **70** (1987), 366-389.

11. L. Gearhart, *Spectral theory for contraction semigroups on Hilbert space*, Trans. Amer. Math. Soc. **236** (1978), 385-394.

12. J.S. Gibson, *A note on stabilization of infinite dimensional linear oscillators by compact linear feedback*, SIAM J. Control Optim. **18** (1980), 311-316.

13. K. Kannsgen and R.L. Wheeler, *Behavior of the solution of a Volterra equation as a parameter tends to infinity*, J. Integral Equations **7** (1984), 229-237.

14. ———, Y. Renardy and R.L. Wheeler, *Effectiveness and robustness with respect to time delays of boundary feedback stabilization in one-dimensional viscoelasticity*, to appear.

15. W.J. Hrusa and M. Renardy, *On wave propagation in linear viscoelasticity*, Q. Appl. Math. **43** (1985), 237-254.

16. T. Kato, Perturbation Theory for Linear Operators, Springer 1966.

17. H. Kolsky, Stress Waves in Solids, Dover Publications Inc., New York, 1963.

18. R.K. Miller, Nonlinear Volterra Integral Equations, Benjamin, Menlo Park, 1971.

19. ———, *Volterra integral equations in a Banach space*, Funkc. Edv. **18** (1975), 163-194.

20. J. Prüss, *Stability analysis for equilibria in age-dependent population dynamics*, Nonlinear Analsysis T.M.A. **7** (1983), 1291-1313.

21. ———, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc. **284** (1984), 847-857.

22. ———, *Positivity and regularity of hyperbolic Volterra equations in Banach spaces*, to appear in Math. Ann..

23. ———, *Regularity and integrability of resolvents of linear Volterra equations*, to appear in Proceedings of the Conference "Volterra Integral Equations in Banach Spaces and Applications", Trento, 1987.

24. M. Renardy, *Some remarks on the propagation and nonpropagation of discontinuities in linearly viscoelastic liquids*, Rheol. Acta **21** (1982), 251-254.

25. ———, W.J. Hrusa and J.A. Nohel, *Mathematical Problems in Viscoelasticity*, Pitman Monographs and Surveys in Pure and Applied Mathematics **35**, Longman Scientific & Technical, Burnt Mill, 1987.

26. G.E.H. Reuter, *Über eine Volterrasche Integralgleichung mit totalmonotonem Kern*, Arch. Math. **7** (1956), 59-66.

27. S. Skaar, *Closed form otpimal control for continuous linear elastic systems*, J. Aeronaut. Sciences **32** (1984), 447-461.

28. ———, L. Tang, and I. Yalda-Mooshabad, *Observations regarding on-off attitude control of flexible satellites*, Preprint 86062, Engineering Research Institute, Iowa State University, Ames.

29. M. Slemrod, *A hereditary partial differential equation with applications in the theory of simple fluids*, Arch. Rat. Mech. Anal. **62** (1976), 303-322.

30. G. Webb, *Theory of nonlinear age-dependent population dynamics*, Monographs and Textbooks in Pure and Applied mathematics Series **89**, Marcel Dekker Inc., 1985.

31. D.V. Widder, The Laplace Transform, Princeton University Press, 1946.

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