JOURNAL OF COMMUTATIVE ALGEBRA
Volume 10, Number 3, Fall 2018

RATIONAL SINGULARITIES OF G-SATURATION

NHAM V. NGO

ABSTRACT. Let G be a semisimple algebraic group defined over an algebraically closed field of characteristic 0 and P a parabolic subgroup of G. Let M be a P-module and V a P-stable closed subvariety of M. We show in this paper that, if the varieties V and $G \cdot M$ have rational singularities, and the induction functor $R^i \operatorname{ind}_P^G(-)$ satisfies certain vanishing conditions, then the variety $G \cdot V$ has rational singularities. This generalizes a result of Kempf [**8**] on the collapsing of homogeneous bundles. As an application, we prove the property of having rational singularities for nilpotent commuting varieties over 3×3 matrices.

1. Introduction. The study of rational singularities for varieties of dimension higher than two dates back to the 1970's when Kempf [**7**] investigated the geometry of Riemann's theorem. One of the interesting questions which arose is when the G-saturation $G \cdot V$ preserves the property of having rational singularities of V. In particular, let M be a G-module and V a closed subvariety of M stabilized by a parabolic subgroup P of G. Kempf showed [**9**] that, if the action of P on V is completely reducible and V has rational singularities, then $G \cdot V$ has rational singularities. It is known that completely reducible actions rarely occur; thus, the usage of Kempf's result is rather restricted. We prove in the present paper that this condition can be relaxed, see Theorem 3.1. As an application, we show that nilpotent commuting varieties over 3×3 matrices have rational singularities. Our interest in nilpotent commuting varieties was motivated by their connection to the cohomology of Frobenius kernels of G in positive characteristic, see [**13**]. It is worth noting that it is interesting to study the converse of our question, see for example, [**1**].

2010 AMS *Mathematics subject classification.* Primary 13A50, 14L30, 14M20.

Keywords and phrases. Rational resolution (singularities), algebraic groups, G-saturation varieties, commuting varieties, homogeneous bundles, cohomology.

This research was supported by the EPSRC, grant No. EP/K022997/1.

Received by the editors on December 18, 2015, and in revised form on May 21, 2016.

DOI:10.1216/JCA-2018-10-3-375

The paper is organized as follows. Section 2 provides necessary notation and background. The main result is shown in Section 3. From Section 4 to the end of the paper, we assume $G = SL_3$. Before showing our applications, we prove in Section 4 a vanishing result of higher induction $R^i \operatorname{ind}_B^G(-)$ for certain modules. Computations in this section extend a recent result on the null-cone of Vilonen and Xue **[17]**. Next, in Section 5, these vanishing results are applied to prove the rational singularities of the nilpotent commuting varieties $C_r(\mathcal{N})$ and some related varieties.

2. Notation.

2.1. Algebraic groups and Lie algebras. Let k be an algebraically closed field of characteristic 0. Let G be a semisimple algebraic group defined over k, unless otherwise stated. Fix a maximal torus $T \subset G$, and let Φ be the root system of T in G. Let Φ^+ be the corresponding set of positive roots. Let B be the Borel subgroup of G containing T and corresponding to the set of negative roots Φ^-, and let U be the unipotent radical of B. Set $\mathfrak{g} = \operatorname{Lie}(G)$, the Lie algebra of G, $\mathfrak{b} = \operatorname{Lie}(B)$, $\mathfrak{u} = \operatorname{Lie}(U)$.

Given a vector space V, we denote by $S^n(V)$ and $\Lambda^n(V)$ the symmetric and exterior space of degree n over V. Then, the direct sums

$$S(V) = \bigoplus_{n=0}^{\infty} S^n(V),$$
$$\Lambda(V) = \bigoplus_{n=0}^{\infty} \Lambda^n(V)$$

denote the symmetric algebra and exterior algebra of V. Define $V^* = \operatorname{Hom}_k(V, k)$ the dual space of V. Throughout this paper, tensor products will be taken over k. Assume, for the rest of the paper, that every G-module is a rational module over G.

From now on, for an affine variety X, we write $k[X]$ for the coordinate algebra of X. This leads to the following.

2.2. Induction functor. Let M be a P-module where P is a parabolic subgroup of G. Then, the induced G-module can be defined as

$$\operatorname{ind}_P^G M = (k[G] \otimes M)^P.$$

The higher derived functor of $\operatorname{ind}_P^G(-)$ is denoted by $R^i \operatorname{ind}_P^G(-)$. Note that the definition of induction is not restricted to parabolic subgroups; the reader is referred to [**6,** Chapter I.3] for further details.

2.3. Adjoint action. Group G acts on the Lie algebra $\mathfrak{g}$ via the adjoint action denoted by "$\cdot$" called the *G-action*. Note that the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ is stable under this G-action and $\mathfrak{b}$ and $\mathfrak{u}$ are stable under the B-action, the restriction of the G-action to B. For every positive integer r, the G-action on the direct product $\mathfrak{g}^r$ is diagonally defined, i.e.,

$$g \cdot (x_1, \ldots, x_r) = (g \cdot x_1, \ldots, g \cdot x_r)$$

for all $g \in G$ and $x_i \in \mathfrak{g}$. It also restricts to the B-action on $\mathfrak{b}^r$ and $\mathfrak{u}^r$.

In general, let X be a variety and H a connected algebraic group acting on X. We call X an H-variety if the action map is a morphism from $H \times X$ to X. This action induces an action on the coordinate algebra $k[X]$; thus, we call it an H-algebra. A morphism between two H-varieties

$$f : X \longrightarrow Y$$

is called *H-equivariant* if it commutes with the actions of H on both varieties. For instance, the moment map

$$m : G \times^B \mathfrak{u}^r \longrightarrow G \cdot \mathfrak{u}^r,$$

defined by $(g, x_1, \ldots, x_r) \mapsto (g \cdot x_1, \ldots, g \cdot x_r)$ for all $g \in G$, $x_i \in \mathfrak{u}$, is G-equivariant.

2.4. Basic algebraic geometry conventions. Let X be a (not necessarily affine) variety. We also write $k[X]$ for the ring of global sections $\mathcal{O}_X(X)$ on X. In the case where X is affine, it coincides with the coordinate algebra of X.

For each variety X, a morphism $\pi : \widetilde{X} \to X$ is called a *resolution of singularities* if the variety $\widetilde{X}$ is non-singular and π is proper and birational. If, in addition, X is normal and the higher direct image of π vanishes, i.e., $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for all $i > 0$, then we call π a *rational resolution* and say that X has rational singularities. Note that this vanishing condition is equivalent to $\mathrm{H}^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for all $i > 0$ when X is affine [**4,** Proposition III.8.5]. This notion can also be applied to a commutative ring R if we replace X by $\mathrm{Spec}(R)$. Suppose further that π is H-equivariant. Then, the above resolution is called an H-equivariant resolution of singularities, respectively, H-equivariant rational resolution. The next proposition regarding the existence of equivariant rational resolutions of an H-variety should be well known; however, we have not seen it in the literature.

Proposition 2.1. *Let H be a connected algebraic group and X an H-variety. If X has rational singularities, then there exists an H-equivariant rational resolution of X.*

Proof. First, note that X has an H-equivariant resolution of singularities, namely,

$$\pi : \widetilde{X} \longrightarrow X,$$

see, for example, [**10,** Proposition 3.9.1]. On the other hand, the rational singularities of X and [**16,** Remark 4 (or Lemma 1)] imply that π must carry the property of having rational singularities. □

Next, let P be a parabolic subgroup of G. The associated bundle of a P-variety X over G/P is denoted by $G \times^P X$. It is known that the ring of global sections on $G \times^P X$ coincides with the ring of P-invariant global sections on $G \times X$. In particular, we have

$$k[G \times^P X] \cong k[G \times X]^P \cong (k[G] \otimes k[X])^P = \mathrm{ind}_P^G k[X].$$

Furthermore, we have, for all $i \geq 0$,

$$\mathrm{H}^i(G \times^P X, \mathcal{O}_{G \times^P X}) \cong R^i \mathrm{ind}_P^G(k[X]),$$

where the left-hand side is the sheaf cohomology of the scheme $G \times^P X$.

2.5. Determinantal varieties. Consider an $m \times n$ matrix

$$\mathcal{M} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix},$$

whose entries are independent indeterminates over the field k. Let

$$k[\mathcal{M}] := k[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n],$$

and let $I_t(\mathcal{M})$ be the ideal in $k[\mathcal{M}]$ generated by all $t \times t$ minors of $\mathcal{M}$. For each $t \geq 1$, the ring

$$R_t(\mathcal{M}) = \frac{k[\mathcal{M}]}{I_t(\mathcal{M})}$$

is called a *determinantal ring*. We denote by $D_t(\mathcal{M})$ the determinantal variety defined by $I_t(\mathcal{M})$. These rings (or varieties) are well known in commutative algebra. For convenience, we state some of their nice properties.

Proposition 2.2 ([**2,** 2.13, 11.23]). *For every $1 \leq t \leq \min(m, n)$, the ring $R_t(\mathcal{M})$ (or the variety $D_t(\mathcal{M})$) is a reduced, Cohen-Macaulay, normal domain of dimension $(t-1)(m+n-t+1)$. Furthermore, it has rational singularities.*

3. Equivariant rational resolution. We prove in this section the main result of the paper. Recall that G is a connected semisimple algebraic group. The argument is a combination of techniques in [**11,** Section 5].

Theorem 3.1. *Let V be a P-subvariety of a P-module M contained in a G-module N, possibly $M = N$. Let $I(V)$ be the defining ideal of V in $k[M] = S(M^*)$. Assume that the moment map*

$$m : G \times^P M \longrightarrow G \cdot M$$

is a rational resolution. If V is normal and, for all $i \geq 1$,

$$R^i \operatorname{ind}_P^G I(V) = 0,$$

then $G \cdot V$ is normal. Furthermore, suppose that V has rational singularities. If the map

$$m' : G \times^P V \longrightarrow G \cdot V,$$

the restriction of m, is a birational map, then the variety $G \cdot V$ has rational singularities.

The argument is split into several steps, as follows.

Lemma 3.2. *Let q be the embedding of V into M which induces a surjective homomorphism of P-algebras $q^* : S(M^*) \to k[V]$. Then, the map*

$$\phi := \operatorname{ind}_P^G(q^*) : \operatorname{ind}_P^G S(M^*) \longrightarrow \operatorname{ind}_P^G k[V]$$

is a surjective G-equivariant homomorphism of algebras.

Proof. Note first that

$$k[V] \cong \frac{S(M^*)}{I(V)}$$

is an isomorphism of P-algebras. We then have the following short exact sequence of P-modules

$$0 \longrightarrow I(V) \longrightarrow S(M^*) \xrightarrow{q^*} k[V] \longrightarrow 0. \tag{3.1}$$

Since we have, for all $i \geq 1$,

$$R^i \operatorname{ind}_P^G (I(V)) = 0,$$

the long exact sequence when applying the induction functor to the short exact sequence (3.1) deduces to the short exact sequence of G-modules

$$0 \longrightarrow \operatorname{ind}_P^G I(V) \longrightarrow \operatorname{ind}_P^G S(M^*) \xrightarrow{\phi} \operatorname{ind}_P^G(k[V]) \longrightarrow 0,$$

which implies the surjectivity of ϕ. ☐

We now prove the first statement in the theorem.

Lemma 3.3. *The variety $G \cdot V$ is normal.*

Proof. Since V is normal, the ring

$$\operatorname{ind}_P^G k[V] \cong k[G \times^P V] \cong k[G \times V]^P$$

is also normal. Hence, it suffices to show that the map

$$m'^* : k[G \cdot V] \longrightarrow k[G \times^P V]$$

is an isomorphism. Clearly, it is injective. In order to show the surjectivity of m'^*, we consider the commutative diagram of G-equivariant morphisms

$$\begin{tikzcd} G \times^P V \arrow[r, "m'"] \arrow[d, "\overline{q}"'] & G \cdot V \arrow[d, "e"] \\ G \times^P M \arrow[r, "m"'] & G \cdot M, \end{tikzcd}$$

where $\overline{q}$ is induced from the embedding $V \hookrightarrow M$, and e is the embedding from $G \cdot V$ into $G \cdot M$. The commutative diagram of G-algebras follows:

$$\begin{tikzcd} \mathrm{ind}_P^G k[V] & \cong & k[G \times^P V] & k[G \cdot V] \arrow[l, "m'^*"'] \\ \mathrm{ind}_P^G S(M^*) \arrow[u, two heads, "\phi"] & \cong & k[G \times^P M] \arrow[u, "\overline{q}^*"] & k[G \cdot M]. \arrow[l, "m^*"'] \arrow[u, "e^*"'] \end{tikzcd}$$

Since m^* is an isomorphism (as m is a rational resolution) and ϕ is onto (by Lemma 3.2), we have that m'^* is surjective, proving Lemma 3.3. $\square$

Before completing the proof, we need to set up a few things. By Proposition 2.1, V has a P-equivariant rational resolution, namely, $\pi : \widetilde{V} \to V$. This morphism can be extended to the birational map

$$\widetilde{\pi} : G \times^P \widetilde{V} \longrightarrow G \times^P V$$

by setting $(g, v) \mapsto (g, \pi(v))$ for all $g \in G, v \in V$. Then, composing with the map

$$m' : G \times^P V \longrightarrow G \cdot V,$$

we have a G-equivariant resolution of singularities

$$m' \circ \widetilde{\pi} : G \times^P \widetilde{V} \longrightarrow G \cdot V. \tag{3.2}$$

Proof of Theorem 3.1. We show that (3.2) is a rational resolution of $G \cdot V$. From Lemma 3.3, we only need show that

$$R^i(m' \circ \widetilde{\pi})_* \mathcal{O}_{G \times^P \widetilde{V}} = 0$$

for all $i \geq 1$. Using [**4,** Proposition III.8.5], we have, for all $i \geq 1$, the following:

$$\begin{aligned} R^i(m' \circ \widetilde{\pi})_* \mathcal{O}_{G\times^P \widetilde{V}} &\cong \mathrm{H}^i(G \times^P \widetilde{V}, \mathcal{O}_{G\times^P \widetilde{V}})^\sim \\ &\cong \left(R^i \operatorname{ind}_P^G k[\widetilde{V}]\right)^\sim \\ &\cong \left(R^i \operatorname{ind}_P^G k[V]\right)^\sim . \end{aligned}$$

Now, applying the induction functor to the short exact sequence (3.1) and using the vanishing

$$R^i \operatorname{ind}_P^G I(V) = R^i \operatorname{ind}_P^G S(M^*) = 0$$

for all $i > 0$, we obtain $R^i \operatorname{ind}_P^G k[V] = 0$ for all $i > 0$. Hence, the theorem is proved. □

Remark 3.4. When $M = N$, it automatically induces the isomorphism

$$m^* : k[G \cdot N] \longrightarrow k[G \times^P N].$$

Indeed, since N is a G-module, we have $G \cdot N = N$. We then have the commutative diagram

$$\begin{tikzcd} G \times^P N \arrow[r, "m"] & N , \\ G \times^G N \arrow[u, "i"] \arrow[ur, "\overline{m}"'] & \end{tikzcd}$$

where $\overline{m}(g, n) = g \cdot n$ and $i(g, n) = (g, n)$ for all $g \in G$, $n \in N$. Note further that $\overline{m}$ is an isomorphism of varieties, see [**6,** page 88, I.5.14], hence giving us an isomorphism

$$\overline{m}^* : k[N] \longrightarrow k[G \times^G N].$$

On the other hand, the induced homomorphism

$$i^* : k[G \times^G N] \longrightarrow k[G \times^P N]$$

is also an isomorphism, since

$$k[G \times^P N] = \operatorname{ind}_P^G k[N] \cong \operatorname{ind}_P^G(k) \otimes k[N] \cong k[N] \cong k[G \times^G N],$$

by the tensor identity (see [**6,** I.4.8]) and the fact that $\operatorname{ind}_P^G(k) \cong k$.

Hence, the commutativity of the diagram

$$\begin{tikzcd} k[G \times^P N] \arrow[d, "i^*"'] & k[N] \arrow[l, "m^*"'] \arrow[dl, "\overline{m}^*"] \\ k[G \times^G N] & \end{tikzcd}$$

implies the isomorphism

$$m^* : k[N] \longrightarrow k[G \times^P N].$$

Next, we explain how our work generalizes the results in Kempf's papers [**8**].[1] His papers focus on studying singularities of the G-saturation $G \cdot V$, where V is a P-subvariety of a P-module W contained in a G-module N. Explicitly, Kempf studied geometric properties of the homogeneous bundle $G \times^P W \to G/P$ and the collapsing $m : G \times^P W \to N$, which is simply the moment map in our context. In his 1976 paper, he was able to prove that, if P acts completely reducibly on V, then $G \cdot V$ is normal and Cohen-Macaulay. If, in addition,

$$G \times^P V \longrightarrow G \cdot V$$

is birational, then $G \cdot V$ has rational singularities. The proof of this statement totally rests on [**8,** Theorem 2] which is strengthened as follows.

Theorem 3.5. *Let W be a P-module, and let $N = \operatorname{ind}_P^G(W)$, which is a subspace of $k[G \times^P W]$. Consider the bundle*

$$\pi : G \times^P W \longrightarrow G/P$$

and collapsing

$$m : G \times^P W \longrightarrow N.$$

If $R^i \operatorname{ind}_P^G I(W) = 0$ for all $i > 0$ where $I(W)$ is the defining ideal of W in $S(N^*)$, then we have

(i) *m is a projective morphism*;
(ii) *the homomorphism $\mathcal{O}_N \to m_* \mathcal{O}_{G \times^P W}$ is surjective*;
(iii) *$R^i m_* \mathcal{O}_{G \times^P W} = 0$ for all $i > 0$.*

Proof. Statement (i) follows from the identical argument in [**8,** Theorem 2]. The second is equivalent to the surjectivity of the map $S(N^*) \to \operatorname{ind}_P^G S(W^*)$. This is accomplished by the use of Lemma 3.3 with $M = N$ and $V = W$. Lastly, (iii) is proven by the same argument for the proof of Theorem 3.1. $\square$

4. Vanishing of the induction functor. Before giving applications of the main theorem in the previous section, we prove some vanishing results for the induction functor in certain cases. The calculations in this section are of independent interest since they extend a result in [**17**]. The strategy is repeatedly making use of Koszul resolutions for various vector spaces.

We assume, for the rest of the paper, that $G = SL_3$, unless otherwise stated. Let

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$$

be the set of positive roots of the root system Φ of G. Denote by $X(T)$ the weight lattice of T. We further denote by

$$X(T)^+ = \{\lambda \in X(T) : (\lambda, \gamma^\vee) \geq 0 \quad \text{for all } \gamma \in \Phi^+\},$$

the set of dominant weights in $X(T)$.

First note that Vilonen and Xue recently showed [**17**] that, for all $i, r \geq 1$,

$$\mathrm{H}^i(G \times^B \mathfrak{u}^r, \mathcal{O}_{G\times^B \mathfrak{u}^r}) \cong R^i \operatorname{ind}_B^G S(\mathfrak{u}^{*r}) = 0. \tag{4.1}$$

It follows that the resolution

$$G \times^B \mathfrak{u}^r \longrightarrow G \cdot \mathfrak{u}^r,$$
$$(g, x_1, \ldots, x_r) \longmapsto (g\cdot x_1, \ldots, g \cdot x_r)$$

is a (G-equivariant) rational resolution.

Now, for γ, either α or β, we let

$$\mathcal{A}_\gamma = X^+ \cup \{\mu \in X(T) : (\mu, \gamma^\vee) = -1 \text{ and } \mu + \gamma \in X(T)^+\}.$$

For each simple root, say α, we denote by $\mathfrak{u}_\alpha$ the Lie algebra of the unipotent radical of the parabolic subgroup of G generated by $\{\alpha\}$. It then follows that

$$0 \longrightarrow \mathfrak{u}_\alpha \longrightarrow \mathfrak{u} \longrightarrow -\alpha \longrightarrow 0.$$

For convenience, we will write $\mathfrak{u}^{*r}$ instead of $(\mathfrak{u}^*)^r$, and $\mathrm{H}^i(M)$ instead of $R^i \operatorname{ind}_B^G(M)$ for all $i \geq 0$. We first prove a lemma.

Lemma 4.1. *For $r, i \geq 1$ and $\mu \in \mathcal{A}_\gamma$,*

$$\mathrm{H}^i\left(S(\mathfrak{u}_\gamma^{*r}) \otimes \mu\right) = 0.^2$$

Proof. By symmetry, it suffices to prove the lemma for $\gamma = \alpha$. If $(\mu, \alpha^\vee) = -1$, then we have the vanishing of $\mathrm{H}^i(S(\mathfrak{u}_\alpha^{*r}) \otimes \mu)$ for all $i \geq 0$ by the 'easy' lemma in [**3,** Section 2]. Hence, we assume that μ is dominant. We prove by induction on r. First, consider $r = 1$. We have:

$$0 \longrightarrow \alpha \longrightarrow \mathfrak{u}^* \longrightarrow \mathfrak{u}_\alpha^* \longrightarrow 0.$$

Tensoring the Koszul resolution of this short exact sequence with μ, we obtain

$$0 \longrightarrow S^{n-1}(\mathfrak{u}^*) \otimes (\alpha + \mu) \longrightarrow S^n(\mathfrak{u}^*) \otimes \mu \longrightarrow S^n(\mathfrak{u}_\alpha^*) \otimes \mu \longrightarrow 0.$$

Since μ is dominant and $\alpha + \mu$ is in $\mathcal{A}_\alpha$, [**11,** Theorem 2] gives us, for all $i > 0$,

$$\mathrm{H}^i(S^n(\mathfrak{u}^*) \otimes \mu) = \mathrm{H}^i(S^{n-1}(\mathfrak{u}^*) \otimes (\alpha + \mu)) = 0,$$

which implies $\mathrm{H}^i(S^n(\mathfrak{u}_\alpha^*) \otimes \mu) = 0$.

Suppose this holds for $r - 1$ for some positive integer r. We consider

$$0 \longrightarrow \mathfrak{u}_\alpha^* \longrightarrow \mathfrak{u}_\alpha^{*r} \longrightarrow \mathfrak{u}_\alpha^{*(r-1)} \longrightarrow 0.$$

Tensoring the Koszul resolution of this short exact sequence with μ, we obtain

(4.2)
$$0 \longrightarrow S^{n-2}\mathfrak{u}_\alpha^{*r} \otimes \Lambda^2(\mathfrak{u}_\alpha^*) \otimes \mu \longrightarrow S^{n-1}\mathfrak{u}_\alpha^{*r} \otimes \mathfrak{u}_\alpha^* \otimes \mu \longrightarrow S^n\mathfrak{u}_\alpha^{*r} \otimes \mu \longrightarrow S^n\mathfrak{u}_\alpha^{*(r-1)} \otimes \mu \longrightarrow 0$$

for all $n \geq 0$. Observe that $\mu + \beta$, $\mu + (\alpha + \beta)$ and $\mu + (\alpha + 2\beta)$ are in $\mathcal{A}_\alpha$. Induction on n then implies that, for all $i \geq 1$,

$$\mathrm{H}^i(S^{n-2}\mathfrak{u}_\alpha^{*r} \otimes \Lambda^2(\mathfrak{u}_\alpha^*) \otimes \mu) = \mathrm{H}^i(S^{n-1}\mathfrak{u}_\alpha^{*r} \otimes \mathfrak{u}_\alpha^* \otimes \mu) = 0.$$

Now, breaking up (4.2) into short exact sequences and applying inductive hypotheses, we obtain

$$\mathrm{H}^i(S^n\mathfrak{u}_\alpha^{*r}\otimes\mu)=0 \quad \text{for all } i\geq 1,$$

which inductively proves our lemma. □

We further extend the above result as follows.

Theorem 4.2. *For all $i\geq 1$, $r,s\geq 0$ and $\lambda\in\mathcal{A}_\gamma$,*

$$\mathrm{H}^i\left(S(\mathfrak{u}^{*r}\times\mathfrak{u}_\gamma^{*s})\otimes\lambda\right)=0.$$

Proof. Again, we only need argue for the case when $\gamma=\alpha$. By Lemma 4.1, the theorem holds for $r=0$. Suppose it holds for $r-1$ (with $r\geq 1$) and all $s\geq 0$. Proceeding inductively for n, we only need prove that, for all $i\geq 1$,

$$\mathrm{H}^i\left(S^n(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})\otimes\lambda\right)=0. \tag{4.3}$$

Assume that $(\lambda,\alpha^\vee)=-1$. Then, consider the short exact sequence

$$0\longrightarrow\alpha\longrightarrow\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s}\longrightarrow\mathfrak{u}^{*(r-1)}\times\mathfrak{u}_\alpha^{*(s+1)}\longrightarrow 0.$$

Tensoring the Koszul resolution of this short exact sequence with λ, we obtain

$$\begin{aligned}0\longrightarrow S^{n-1}(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})\otimes(\alpha+\lambda)\longrightarrow S^n(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})&\\ \otimes\lambda\longrightarrow S^n(\mathfrak{u}^{*(r-1)}\times\mathfrak{u}_\alpha^{*(s+1)})\otimes\lambda\longrightarrow 0.&\end{aligned}$$

By inductive hypotheses, the vanishing of $\mathrm{H}^i(S(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})\otimes(\alpha+\lambda))$ implies that of $\mathrm{H}^i(S(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})\otimes\lambda)$ for all $i\geq 1$. Hence, we only need verify (4.3) for $\lambda\in X^+$. Consider:

$$0\longrightarrow\mathfrak{u}_\alpha^*\longrightarrow\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s}\longrightarrow\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*(s-1)}\longrightarrow 0.$$

Tensoring the Koszul resolution of this short exact sequence with λ, we get for all $n\geq 0$,

$$\begin{aligned}0\longrightarrow S^{n-2}(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})\otimes\Lambda^2(\mathfrak{u}_\alpha^*)&\\ \otimes\lambda\longrightarrow S^{n-1}(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})&\\ \otimes\mathfrak{u}_\alpha^*\otimes\lambda\longrightarrow S^n(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*s})&\\ \otimes\lambda\longrightarrow S^n(\mathfrak{u}^{*r}\times\mathfrak{u}_\alpha^{*(s-1)})\otimes\lambda\longrightarrow 0.&\end{aligned} \tag{4.4}$$

Since we are assuming λ is dominant, $\lambda+\beta, \lambda+(\alpha+\beta)$ and $\lambda+(\alpha+2\beta)$ are in $\mathcal{A}_\gamma$. Induction on n then implies that, for all $i \geq 1$,

$$\mathrm{H}^i(S^{n-2}(\mathfrak{u}^{*r} \times \mathfrak{u}_\alpha^{*s}) \otimes \Lambda^2(\mathfrak{u}_\alpha^*) \otimes \lambda) = \mathrm{H}^i(S^{n-1}(\mathfrak{u}^{*r} \times \mathfrak{u}_\alpha^{*s}) \otimes \mathfrak{u}_\alpha^* \otimes \lambda) = 0.$$

Now, breaking up (4.4) into short exact sequences and applying inductive hypotheses, we obtain $\mathrm{H}^i(S^n(\mathfrak{u}^{*r} \times \mathfrak{u}_\alpha^{*s}) \otimes \lambda) = 0$ for all $i \geq 1$, which inductively proves our lemma. $\square$

Setting $s = 0$ in Theorem 4.2, we obtain the following.

Corollary 4.3. *For all $i \geq 1$, $r \geq 0$ and $\lambda \in \mathcal{A}_\gamma$, we have*

$$\mathrm{H}^i(S(\mathfrak{u}^{*r}) \otimes \lambda) = 0.$$

Remark 4.4. Our vanishing results in this section hold for all characteristics greater than 3, see [**17,** Remark 6.2]. Note also that analogous vanishings for $S(\mathfrak{b}^{*r})$ do not hold for $r > 1$, see the counterexamples in [**17,** subsection 5.2].

5. Nilpotent commuting varieties. Let $\mathfrak{g}$ be a Lie algebra defined over k and X a closed subvariety of $\mathfrak{g}$. For each $r \geq 1$, the commuting variety (of r-tuples) over X is defined by

$$C_r(X) = \{(x_1, \ldots, x_r) \in X^r \mid [x_i, x_j] = 0\}$$

for $r \geq 2$ and $C_1(X) = X$. When $V = \mathcal{N}$, we call $C_r(\mathcal{N})$ the *nilpotent commuting variety*. The study of nilpotent commuting varieties was begun not long ago. The pioneering work of Premet [**15**] showed that $C_2(\mathcal{N})$ has pure dimension[3] $\dim \mathfrak{g}$. In [**13**], I proved that the result does not hold for arbitrary r. In joint work with Šivic [**14**], we determined the (ir)reducibility of the variety $C_r(\mathcal{N})$ with $\mathfrak{g} = \mathfrak{sl}_n$ for various values of n and r. Explicitly, it is reducible for all $n, r \geq 4$. Moreover, for $r = 3$, it is irreducible for all $n \leq 6$.

In this section, we continue with the assumption that $G = SL_3$ is defined over k. Recall from [**12**] that various commuting varieties over 2×2 matrices were proven to be Cohen-Macaulay and have rational singularities. While these properties easily follow from determinantal varieties for $C_r(\mathfrak{sl}_2)$ and $C_r(\mathfrak{gl}_2)$, the proof for the nilpotent commuting varieties requires deep methods in commutative algebra, due to the difficulty in computing their defining ideals. Note also that, in [**12,** Section 7], the author showed that the singular locus of $C_r(\mathcal{N})$ is of codimension 2, which is strong evidence for the normality of $C_r(\mathcal{N})$. In this section, we verify the normality of $C_r(\mathcal{N})$ and further prove that it has rational singularities.

Note that, for each $r \geq 1$, we have $C_r(\mathcal{N}) = G \cdot C_r(\mathfrak{u})$ and the moment map $m : G \times^B C_r(\mathfrak{u}) \to C_r(\mathcal{N})$ is proper birational map, see [**12,** Proposition 3.4.3].

We first analyze some properties of $C_r(\mathfrak{u})$. Let f_α, f_β and $f_{\alpha+\beta}$ be root vectors in $\mathfrak{u}$ corresponding to weights $-\alpha$, $-\beta$ and $-\alpha-\beta$. Then, each element v in $\mathfrak{u}$ can be written as

$$v = af_\alpha + bf_\beta + cf_{\alpha+\beta}$$

for some $a, b, c \in k$. Now, suppose that $(v_i) = (a_i f_\alpha + b_i f_\beta + c_i f_{\alpha+\beta} : 1 \leq i \leq r)$ is an r-tuple in $\mathfrak{u}^r$. Analyzing the commutator $[v_i, v_j]$ for all $i \neq j$, we obtain

$$(v_i) \in C_r(\mathfrak{u}) \Longleftrightarrow a_i b_j - a_j b_i = 0.$$

These equations are exactly all 2×2 minors of the matrix

$$\mathcal{M} = \begin{pmatrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_r \end{pmatrix}.$$

It follows that $C_r(\mathfrak{u})$ is a product of an affine space and determinantal variety $D_2(\mathcal{M})$. Therefore, from Proposition 2.2, we have obtained the following.

Proposition 5.1. *For all $r \geq 1$, we have*:

(a) $C_r(\mathfrak{u})$ *has rational singularities*;
(b) *the defining ideal of $C_r(\mathfrak{u})$ in $S(\mathfrak{u}^{*r})$ is $I(C_r(\mathfrak{u})) = \langle f_{i,j} : 1 \leq i \neq j \leq r \rangle$, where all $f_{ij} = a_i b_j - a_j b_i \in S(\mathfrak{u}^{*r})$ are of weight $\alpha + \beta$. Moreover, since U acts trivially on $I(C_r(\mathfrak{u}))$, we have*

$$I(C_r(\mathfrak{u})) \cong \bigoplus_{1 \leq i \neq j \leq r} (\alpha + \beta) \otimes S(\mathfrak{u}^{*r})$$

as a B-module.

Next, we state the main result of this section.

Theorem 5.2. *For all $r \geq 1$, the nilpotent commuting variety $C_r(\mathcal{N})$ has rational singularities. Consequently, it is Cohen-Macaulay.*

Proof. Note first that $C_r(\mathfrak{u})$ is a B-subvariety of $\mathfrak{u}^r$. Moreover, $G \cdot \mathfrak{u}^r$ is shown to have rational singularities in [**17,** 6.1], in particular, we have

$$m : G \times^B \mathfrak{u}^r \longrightarrow G \cdot \mathfrak{u}^r$$

is a rational resolution. Hence, our theorem follows immediately from Proposition 2.1, Theorem 3.1, Corollary 4.3 and Proposition 5.1. □

Remark 5.3. As pointed out earlier, the result cannot be extended further for higher rank groups for all r, as it was shown by Šivic and the author [**14**] that $C_r(\mathcal{N})$ is reducible for all $r \geq 4$ and $\mathrm{rank}(G) \geq 3$ for type A (also see [**13**] for other classical types).

The above result can be strengthened as follows.

Theorem 5.4. *Let V be a B-subvariety of $\mathfrak{u}^r$ for some $r \geq 1$, whose defining ideal $I(V)$ contains polynomials of weight $\alpha + \beta$. If V has rational singularities and*

$$G \times^B V \longrightarrow G \cdot V$$

is a birational map, then $G \cdot V$ has rational singularities.

Proof. Recall from the last section that $G \times^B \mathfrak{u}^r \to G \cdot \mathfrak{u}^r$ is a rational resolution. The assertion then follows from Theorem 3.1 if we show that $R^i \operatorname{ind}_B^G I(V) = 0$ for all $i \geq 1$. This can be done by the same argument as in Proposition 5.1. Indeed, set $I(V) = \langle f_1, \ldots, f_s \rangle$ with all f_i of weight $\alpha + \beta$. This implies that U acts trivially on $I(V)$ so that, as a B-module,

$$I(V) \cong \bigoplus_{i=1}^{s} f_i \otimes S(\mathfrak{u}^{*r}) \cong \bigoplus_{i=1}^{s} (\alpha + \beta) \otimes S(\mathfrak{u}^{*r}).$$

Now, Corollary 4.3 gives us the desired vanishing of $R^i \operatorname{ind}_B^G I(V)$. □

Acknowledgments. A part of the paper was written when the author worked at Lancaster University as a Senior Research Associate. The author acknowledges the support of the EPSRC. The author would like to thank Eric Sommers for discussions about using Koszul resolution and thank Jean-Yves Charbonnel for informing us about work of Vilonen and Xue. He also wants to thank the referee for useful comments and suggestions on earlier manuscripts.

ENDNOTES

1. One may find it difficult to follow the notation in this paper. We refer the reader to [**5,** Section 3], which is more useful in our context.

2. This result also holds for $r = 0$ by [**11,** Theorem 2].

3. This means all irreducible components are of the same dimension.

REFERENCES

1. M. Brion, *Multiplicity-free subvarieties of flag varieties*, Contemp. Math. **331** (2003), 13–23.

2. W. Bruns and U. Vetter, *Determinantal rings*, Lect. Notes Math. **1327** (1988).

3. M. Demazure, *A very simple proof of Bott's theorem*, Invent. Math. **33** (1976), 271–272.

4. R. Hartshorne, *Algebraic geometry*, Grad. Texts Math. **52** (1977).

5. W. Hesselink, *The normality of closures of orbits in a Lie algebra*, Comment. Math. Helv. **54** (1979), 105–110.

6. J.C. Jantzen, *Representations of algebraic groups*, Math. Surv. Mono. **107** (2003).

7. G. Kempf, *On the geometry of a theorem of Riemann*, Ann. Math. **98** (1973), 178–185.

8. ______, *On the collapsing of homogeneous vector bundles*, Invent. Math. **37** (1976), 229–239.

9. ______, *Varieties with rational singularities*, Contemp. Math. **58** (1986), 179–182.

10. J. Kollar, *Lectures on resolution of singularities*, Ann. Math. Stud. **166** (2007).

11. S. Kumar, N. Lauritzen and J.F. Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math. **136** (1999), 603–621.

12. N.V. Ngo, *Commuting varieties of r-tuples over Lie algebras*, J. Pure Appl. Alg. **218** (2014), 1400–1417.

13. ______, *On nilpotent commuting varieties and cohomology of Frobenius kernels*, J. Algebra **425** (2015), 65–84.

14. N.V. Ngo and K. Šivic, *On varieties of commuting nilpotent matrices*, Lin. Alg. Appl. **452** (2014), 237–262.

15. A. Premet, *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. **154** (2003), 653–683.

16. E. Viehweg, *Rational singularities of higher dimensional schemes*, Proc. Amer. Math. Soc. **63** (1977), 6–8.

17. K. Vilonen and T. Xue, *The null-cone and cohomology of vector bundles on flag manifolds*, Represent. Th. **20** (2016), 482–498.

University of North Georgia, Department of Mathematics, Oakwood, GA 30566

Email address: **nvngo@ung.edu**