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UNIMODULAR ELEMENTS IN PROJECTIVE MODULES AND AN ANALOGUE OF A RESULT OF MANDAL

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ABSTRACT. (1) Let R be a commutative Noetherian ring of dimension n and P a projective $R[X_1,\dots,X_m]$-module of rank n. In this paper, we associate an obstruction for P to split off a free summand of rank one. (2) Let R be a local ring and $R[X]\subset A\subset R[X,X^{-1}]$. Let P and Q be two projective A-modules with $\operatorname{rank}(Q)<\operatorname{rank}(P)$. If Q_f is a direct summand of P_f for some special monic polynomial $f\in R[X]$, then Q is also a direct summand of P.

1. Introduction. *Throughout the paper, rings are commutative Noetherian, and projective modules are finitely generated and of constant rank.*

If R is a ring of dimension n, then Serre **[17]** proved that projective R-modules of rank $> n$ contain a unimodular element. Plumstead **[12]** generalized this result and proved that projective $R[X]=R[\mathbb{Z}_+]$-modules of rank $> n$ contain a unimodular element. Bhatwadekar and Roy **[4]** generalized this result and proved that projective $R[X_1,\dots,X_r]=R[\mathbb{Z}_+^r]$-modules of rank $> n$ contain a unimodular element.

In another direction, if A is a ring such that

$$R[X]\subset A\subset R[X,X^{-1}],$$

then Bhatwadekar and Roy **[3]** proved that projective A-modules of rank $> n$ contain a unimodular element. Rao **[14]** improved this result and proved that if B is a birational overring of $R[X]$, i.e.,

$$R[X]\subset B\subset S^{-1}R[X],$$

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where S is the set of non-zerodivisors of $R[X]$, then projective B-modules of rank $> n$ contain a unimodular element. Bhatwadekar, Lindel and Rao [**2,** Theorem 5.1, Remark 5.3] generalized this result and proved that projective $B[\mathbb{Z}_+^r]$-modules of rank $> n$ contain a unimodular element when B is seminormal. In [**1,** Theorem 3.5], Bhatwadekar removed the hypothesis of seminormality used in [**2**].

All of the above results are best possible in the sense that projective modules of rank n above rings need not have a unimodular element. Thus, it is natural to look for obstructions for a projective module of rank n over above rings to contain a unimodular element. We will prove some results in this direction.

Let P be a projective $R[\mathbb{Z}_+^r][T]$-module of rank $n = \dim R$ such that P_f and P/TP contain unimodular elements for some monic polynomial f in the variable T. Then, P contains a unimodular element. The proof of this result is implicit in [**2,** Theorem 5.1]. We will generalize this result to projective $R[M][T]$-modules of rank n, where $M \subset \mathbb{Z}_+^r$ is a Φ-simplicial monoid in the class $\mathcal{C}(\Phi)$. For this, we need the following result, the proof of which is similar to [**2,** Theorem 5.1].

Proposition 1.1. *Let R be a ring and P a projective $R[X]$-module. Let $J \subset R$ be an ideal such that P_s is extended from R_s for every $s \in J$. Suppose that*:

(a) *P/JP contains a unimodular element.*
(b) *If I is an ideal of $(R/J)[X]$ of height $\operatorname{rank}(P) - 1$, then there exist $\overline{\sigma} \in \operatorname{Aut}((R/J)[X])$ with $\overline{\sigma}(X) = X$ and $\sigma \in \operatorname{Aut}(R[X])$ with $\sigma(X) = X$, which is a lift of $\overline{\sigma}$ such that $\overline{\sigma}(I)$ contains a monic polynomial in the variable X.*
(c) *$EL(P/(X,J)P)$ acts transitively on $\operatorname{Um}(P/(X,J)P)$.*
(d) *There exists a monic polynomial $f \in R[X]$ such that P_f contains a unimodular element.*

Then, the natural map $\operatorname{Um}(P) \to \operatorname{Um}(P/XP)$ is surjective. In particular, if P/XP contains a unimodular element, then P contains a unimodular element.

We prove the following result as an application of Proposition 1.1.

Theorem 1.2. *Let R be a ring of dimension n and $M \subset \mathbb{Z}_+^r$ a Φ-simplicial monoid in the class $\mathcal{C}(\Phi)$. Let P be a projective $R[M][T]$-*

module of rank n whose determinant is extended from R. Assume P/TP and P_f contain unimodular elements for some monic polynomial f in the variable T. Then, the natural map $\mathrm{Um}(P) \to \mathrm{Um}(P/TP)$ is surjective. In particular, P contains a unimodular element.

Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$. If P is a projective $R[X]$-module of rank n, then Das and Zinna **[5]** obtained an obstruction for P to have a unimodular element. We fix an isomorphism

$$\chi : L \xrightarrow{\sim} \wedge^n P,$$

where L is the determinant of P. To the pair (P, χ), they associated an element $e(P, \chi)$ of the Euler class group $E(R[X], L)$ and proved that P has a unimodular element if and only if $e(P, \chi) = 0$ in $E(R[X], L)$ **[5]**.

It is desirable to have such an obstruction for projective $R[X, Y]$-module P of rank n. As an application of (1.2), we obtain such a result. Recall that $R(X)$ denotes the ring obtained from $R[X]$ by inverting all monic polynomials in X. Let L be the determinant of P and $\chi : L \overset{\sim}{\to} \wedge^n(P)$ an isomorphism. We define the Euler class group $E(R[X, Y], L)$ of $R[X, Y]$ as the product of Euler class groups $E(R(X)[Y], L \otimes R(X)[Y])$ of $R(X)[Y]$ and $E(R[Y], L \otimes R[Y])$ of $R[Y]$, defined by Das and Zinna **[5]**. To the pair (P, χ), we associate an element $e(P, \chi)$ in $E(R[X, Y], L)$ and prove the following result (Theorem 3.5).

Theorem 1.3. *Let the notation be as above. Then, $e(P, \chi) = 0$ in $E(R[X, Y], L)$ if and only if P has a unimodular element.*

Let R be a local ring and P a projective $R[T]$-module. Roitman **[15,** Lemma 10] proved that, if the projective $R[T]_f$-module P_f contains a unimodular element for some monic polynomial $f \in R[T]$, then P contains a unimodular element. Roy **[16,** Theorem 1.1] generalized this result and proved that, if P and Q are projective $R[T]$-modules with $\mathrm{rank}(Q) < \mathrm{rank}(P)$ such that Q_f is a direct summand of P_f for some monic polynomial $f \in R[T]$, then Q is a direct summand of P. Mandal **[11,** Theorem 2.1] extended Roy's result to Laurent polynomial rings.

We prove the following result (4.4), which gives Mandal's **[11]** in the case where $A = R[X, X^{-1}]$. Recall that a monic polynomial $f \in R[X]$ is called *special monic* if $f(0) = 1$.

Theorem 1.4. *Let R be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let P and Q be two projective A-modules with $\operatorname{rank}(Q) < \operatorname{rank}(P)$. If Q_f is a direct summand of P_f for some special monic polynomial $f \in R[X]$, then Q is also a direct summand of P.*

2. Preliminaries.

Definition 2.1. Let R be a ring and P a projective R-module. An element $p \in P$ is called *unimodular* if there is a surjective R-linear map

$$\varphi : P \twoheadrightarrow R$$

such that $\varphi(p) = 1$. Note that P has a unimodular element if and only if $P \simeq Q \oplus R$ for some R-module Q. The set of all unimodular elements of P is denoted by $\operatorname{Um}(P)$.

Definition 2.2. Let M be a finitely generated submonoid of $\mathbb{Z}_+^r$ of rank r such that $M \subset \mathbb{Z}_+^r$ is an integral extension, i.e. for any $x \in \mathbb{Z}_+^r$, $nx \in M$ for some integer $n > 0$. Such a monoid M is called a Φ-*simplicial* monoid of rank r [**8**].

Definition 2.3. Let $M \subset \mathbb{Z}_+^r$ be a Φ-simplicial monoid of rank r. We say that M belongs to the class $\mathcal{C}(\Phi)$ if M is seminormal, i.e., if $x \in gp(M)$ and $x^2, x^3 \in M$, then $x \in M$, and if we write

$$\mathbb{Z}_+^r = \{t_1^{s_1} \cdots t_r^{s_r} \mid s_i \geq 0\},$$

then, for $1 \leq m \leq r$,

$$M_m = M \cap \{t_1^{s_1} \cdots t_m^{s_m} \mid s_i \geq 0\}$$

satisfies the following properties: given a positive integer c, there exist integers $c_i > c$ for $i = 1, \ldots, m-1$ such that, for any ring R, the automorphism

$$\eta \in \operatorname{Aut}_{R[t_m]}(R[t_1, \ldots, t_m]),$$

defined by $\eta(t_i) = t_i + t_m^{c_i}$ for $i = 1, \ldots, m-1$, restricts to an R-automorphism of $R[M_m]$. It is easy to see that $M_m \in \mathcal{C}(\Phi)$ and rank $M_m = m$ for $1 \leq m \leq r$.

Example 2.4. The following monoids belong to $\mathcal{C}(\Phi)$ [**9**, Examples 3.5, 3.9, 3.10].

(i) If $M \subset \mathbb{Z}_+^2$ is a finitely generated and normal monoid (i.e., $x \in gp(M)$ and $x^n \in M$ for some $n > 1$, then $x \in M$) of rank 2, then $M \in \mathcal{C}(\Phi)$.

(ii) For a fixed integer $n > 0$, if $M \subset \mathbb{Z}_+^r$ is the monoid generated by all monomials in $t_1, \ldots, t_r$ of total degree n, then M is a normal monoid of rank r and $M \in \mathcal{C}(\Phi)$. In particular, $\mathbb{Z}_+^r \in \mathcal{C}(\Phi)$ and

$$\langle t_1^2, t_2^2, t_3^2, t_1t_2, t_1t_3, t_2t_3 \rangle \in \mathcal{C}(\Phi).$$

(iii) The submonoid M of $\mathbb{Z}_+^3$ is generated by $\langle t_1^2, t_2^2, t_3^2, t_1t_3, t_2t_3 \rangle \in \mathcal{C}(\Phi)$.

Remark 2.5. Let R be a ring and

$$M \subset \mathbb{Z}_+^r = \{t_1^{m_1} \cdots t_r^{m_r} \mid m_i \geq 0\}$$

a monoid of rank r in the class $\mathcal{C}(\Phi)$. Let I be an ideal of $R[M]$ of height $> \dim R$. Then, by [**8,** Lemma 6.5] and [**9,** Lemma 3.1], there exists an R-automorphism σ of $R[M]$ such that $\sigma(t_r) = t_r$, and $\sigma(I)$ contains a monic polynomial in t_r with coefficients in $R[M] \cap R[t_1, \ldots, t_{r-1}]$.

We now state some results for later use.

Theorem 2.6 ([**9,** Theorem 3.4])**.** *Let R be a ring and M a Φ-simplicial monoid such that $M \in \mathcal{C}(\Phi)$. Let P be a projective $R[M]$-module of rank $> \dim R$. Then, P has a unimodular element.*

Theorem 2.7 ([**6,** Theorem 4.5])**.** *Let R be a ring and M a Φ-simplicial monoid. Let P be a projective $R[M]$-module of rank $\geq \max\{\dim R + 1, 2\}$. Then, $EL(P \oplus R[M])$ acts transitively on $\mathrm{Um}(P \oplus R[M])$.*

The next result is proven in [**2,** Criterion-1 and Remark] in the case where $J = Q(P, R_0)$ is the Quillen ideal of P in R_0. The same proof works in our case.

Theorem 2.8. *Let $R = \oplus_{i \geq 0} R_i$ be a graded ring and P a projective R-module. Let J be an ideal of R_0 such that J is contained in the Quillen ideal $Q(P, R_0)$. Let $p \in P$ be such that $p_{1+R^+} \in \mathrm{Um}(P_{1+R^+})$ and $p_{1+J} \in \mathrm{Um}(P_{1+J})$, where $R^+ = \oplus_{i \geq 1} R_i$. Then, P contains a unimodular element p_1 such that $p = p_1$ modulo R^+P.*

The following result is a consequence of Eisenbud and Evans [**7**], as stated in [**12,** page 1420].

Lemma 2.9. *Let A be a ring and P a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then, there exists an element $\beta \in P^*$ such that* $\mathrm{ht}(I_a) \geq n$*, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$, then* $\mathrm{ht} I \geq n$*. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then* $\mathrm{ht} I = n$.

3. Proofs of Proposition 1.1, Theorem 1.2 and Theorem 1.3.

3.1. Proof of Proposition 1.1. Let $p_0 \in \mathrm{Um}(P/JP)$ and $p_1 \in \mathrm{Um}(P/XP)$. Let $\widetilde{p_0}$ and $\widetilde{p_1}$ be the images of p_0 and p_1 in $P/(X,J)P$. By hypothesis (c), there exists a $\widetilde{\delta} \in EL(P/(X,J)P)$ such that $\widetilde{\delta}(\widetilde{p_0}) = \widetilde{p_1}$. By [**4,** Proposition 4.1], $\widetilde{\delta}$ can be lifted to an automorphism δ of P/JP. Consider the fiber product diagrams for rings and modules:

$$\begin{array}{ccc} \frac{R[X]}{(XJ)} & \longrightarrow & \frac{R}{J}[X] \\ \big\downarrow & & \big\downarrow \\ \frac{R[X]}{(X)} & \longrightarrow & \frac{R[X]}{(X,J)} \end{array},$$

$$\begin{array}{ccc} \frac{P}{(XJ)P} & \longrightarrow & \frac{P}{JP} \\ \big\downarrow & & \big\downarrow \\ \frac{P}{XP} & \longrightarrow & \frac{P}{(X,J)P} \end{array}.$$

Since $\delta(p_0)$ and p_1 coincide over $P/(X,J)P$, we can patch $\delta(p_0)$ and p_1 to obtain a unimodular element $p \in \mathrm{Um}(P/XJP)$ such that $p = \delta(p_0)$ modulo JP and $p = p_1$ modulo XP. Writing $\delta(p_0)$ by p_0, we assume that $p = p_0$ modulo JP and $p = p_1$ modulo XP.

Using hypothesis (d), we get an element $q \in P$ such that the order ideal

$$O_P(q) = \{\phi(q) \mid \phi \in \mathrm{Hom}_{R[X]}(P, R[X])\}$$

contains a power of f. We may assume that $f \in O_P(q)$.

Let "bar" denote reduction modulo the ideal (J). Write $\overline{P} = \overline{R[X]} p_0 \oplus Q$ for some projective $\overline{R[X]}$-module Q and $\overline{q} = (\overline{a p}_0, q')$ for some $q' \in Q$. By Eisenbud and Evans [**7**], there exists a $\overline{\tau} \in EL(\overline{P})$

such that

$$\overline{\tau}(\overline{q}) = (\overline{a}p_0, q'')$$

and

$$\mathrm{ht}(O_Q(q''))\overline{R[X]}_{\overline{a}} \geq \mathrm{rank}(P) - 1.$$

Since $\overline{\tau}$ can be lifted to $\tau \in \mathrm{Aut}\,(P)$, replacing P by $\tau(P)$, we may assume that $\mathrm{ht}(O_Q(q')) \geq \mathrm{rank}(P) - 1$ on the Zariski-open set $D(\overline{a})$ of $\mathrm{Spec}(\overline{R[X]})$.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be minimal prime ideals of $O_Q(q')$ in $\overline{R[X]}$ not containing $\overline{a}$. Then, $\mathrm{ht}(\cap_1^r \mathfrak{p}_i) \geq \mathrm{rank}(P) - 1$. By hypothesis (b), we can find $\overline{\sigma} \in \mathrm{Aut}\,(\overline{R[X]})$ with $\overline{\sigma}(X) = X$ and $\sigma \in \mathrm{Aut}\,(R[X])$ with $\sigma(X) = X$, which is a lift of $\overline{\sigma}$, such that $\overline{\sigma}(\cap_1^r \mathfrak{p}_i)$ contains a monic polynomial in $\overline{R[X]} = \overline{R}[X]$. Note that $\sigma(f)$ is a monic polynomial. Replacing $R[X]$ by $\sigma(R[X])$, we may assume that $\cap_1^r \mathfrak{p}_i$ contains a monic polynomial in $\overline{R}[X]$, and $f \in O_P(q)$ is a monic polynomial.

If $\mathfrak{p}$ is a minimal prime ideal of $O_Q(q')$ in $\overline{R[X]}$ containing $\overline{a}$, then $\mathfrak{p}$ contains $O_{\overline{P}}(\overline{q})$. Since $f \in O_P(q)$, $\mathfrak{p}$ contains the monic polynomial $\overline{f}$. Therefore, all minimal primes of $O_Q(q')$ contain a monic polynomial; hence, $O_Q(q')$ contains a monic polynomial, say $\overline{g} \in \overline{R}[X]$. Let $g \in R[X]$ be a monic polynomial which is a lift of $\overline{g}$.

Claim 3.1. *For large $N > 0$, $p_2 = p + X^N g^N q \in \mathrm{Um}(P_{1+JR})$.*

Proof. Choose $\phi \in P^*$ such that $\phi(q) = f$. Then, $\phi(p_2) = \phi(p) + X^N g^N f$ is a monic polynomial for large N. Since $p = p_0$ module JP, $\overline{p} = p_0$ and $\overline{q} = (\overline{ap}, q')$. Therefore,

$$\overline{p}_2 = \overline{p} + X^N \overline{g}^N (\overline{ap}, q') = ((1 + T^N \overline{g}^N \overline{a})\overline{p}, X^N \overline{g}^N q').$$

Since $\overline{g} \in O_Q(q') \subset O_{\overline{P}}(\overline{p}_2)$, we obtain $O_{\overline{P}}(\overline{p}) \subset O_{\overline{P}}(\overline{p}_2)$. In addition, since $\overline{p} \in \mathrm{Um}(\overline{P})$, we get $\overline{p}_2 \in \mathrm{Um}(\overline{P})$, and hence, $p_2 \in \mathrm{Um}(P_{1+JR[X]})$. Since $O_P(p_2)$ contains a monic polynomial, by [**10,** Lemma 1.1, p. 79], $p_2 \in \mathrm{Um}(P_{1+JR})$.

Now, $p_2 = p = p_1$ modulo XP, and we obtain $p_2 \in \mathrm{Um}(P/XP)$. By (2.8), there exists a $p_3 \in \mathrm{Um}(P)$ such that $p_3 = p_2 = p_1$ modulo XP. This completes the proof. □

3.2. Proof of Theorem 1.2. Without loss of generality, we may assume that R is reduced. When $n = 1$, the result follows from the well-known [**13, 18**]. When $n = 2$, the result follows from [**1,** Proposition 3.3], where it is proven that, if P is a projective $R[T]$-module of rank 2 such that P_f contains a unimodular element for some monic polynomial $f \in R[T]$, then P contains a unimodular element. Thus, now, we assume $n \geq 3$.

Write $A = R[M]$. Let

$$J(A,P) = \{s \in A \mid P_s \text{ is extended from } A_s\}$$

be the *Quillen ideal* of P in A. Let $\widetilde{J} = J(A,P) \cap R$ be the ideal of R and $J = \widetilde{J}R[M]$. We will show that J satisfies the properties of Proposition 1.1.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht}(\mathfrak{p}) = 1$ and $S = R - \mathfrak{p}$. Then, $S^{-1}P$ is a projective module over $S^{-1}A[T] = R_{\mathfrak{p}}[M][T]$. Since $\dim(R_{\mathfrak{p}}) = 1$, by (2.6),

$$S^{-1}P = \wedge^n P_S \oplus S^{-1}A[T]^{n-1}.$$

Since the determinant of P is extended from R, $\wedge^n P_S = A[T]_S$, and hence, $S^{-1}P$ is free. Therefore, there exists an $s \in R - \mathfrak{p}$ such that P_s is free. Hence, $s \in \widetilde{J}$, and thus, $\operatorname{ht}(\widetilde{J}) \geq 2$.

Since $\dim(R/\widetilde{J}) \leq n-2$ and $A[T]/(J) = (R/\widetilde{J})[M][T]$, by (2.6), P/JP contains a unimodular element. If I is an ideal of $(A/J)[T] = (R/\widetilde{J})[M][T]$ of height $\geq n-1$, then, by (2.5), there exists an $R[T]$-automorphism $\sigma \in \operatorname{Aut}_{R[T]}(A[T])$ such that, if $\overline{\sigma}$ denotes the induced automorphism of $(A/J)[T]$, then $\overline{\sigma}(I)$ contains a monic polynomial in T. By Theorem 2.7, $EL(P/(T,J)P)$ acts transitively on $\operatorname{Um}(P/(J,T)P)$. Therefore, the result now follows from (1.1). $\square$

Corollary 3.2. *Let R be a ring of dimension n,*

$$A = R[X_1, \ldots, X_m]$$

a polynomial ring over R and P a projective $A[T]$-module of rank n. Assume that P/TP and P_f both contain a unimodular element for some monic polynomial $f(T) \in A[T]$. Then, P has a unimodular element.

Proof. If $n = 1$, the result follows from the well-known Quillen and Suslin theorem [**13, 18**]. When $n = 2$, the result follows from [**1,** Proposition 3.3]. Assume $n \geq 3$. Let L be the determinant of P. If $\widetilde{R}$ is the seminormalization of R, then, by Swan **[19]**, $L \otimes \widetilde{R}[X_1, \ldots, X_m]$ is extended from $\widetilde{R}$. By Theorem 1.2, $P \otimes \widetilde{R}[X_1, \ldots, X_m]$ has a unimodular element. Since

$$\widetilde{R}[X_1, \ldots, X_n]$$

is the seminormalization of A, by **[1,** Lemma 3.1], P has a unimodular element. □

3.3. Obstruction for projective modules to have a unimodular element. Let R be a ring of dimension $n \geq 2$ containing $\mathbb{Q}$, and let P be a projective $R[X, Y]$-module of rank n with determinant L. Let

$$\chi : L \xrightarrow{\sim} \wedge^n(P)$$

be an isomorphism. We call χ an *orientation* of P. In general, we shall use 'hat' when we move to $R(X)[Y]$ and 'bar' when we move modulo the ideal (X). For instance, we have:

(1) $L \otimes R(X)[Y] = \widehat{L}$ and $L/XL = \overline{L}$,
(2) $P \otimes R(X)[Y] = \widehat{P}$ and $P/XP = \overline{P}$.

Similarly, $\widehat{\chi}$ denotes the induced isomorphism $\widehat{L} \xrightarrow{\sim} \wedge^n \widehat{P}$ and $\overline{\chi}$ denotes the induced isomorphism $\overline{L} \xrightarrow{\sim} \wedge^n \overline{P}$.

We now define the *Euler class* of (P, χ).

Definition 3.3. First, we consider the case $n \geq 2$ and $n \neq 3$. Let $E(R(X)[Y], \widehat{L})$ be the nth Euler class group of $R(X)[Y]$ with respect to the line bundle $\widehat{L}$ over $R(X)[Y]$, and let $E(R[Y], \overline{L})$ be the nth Euler class group of $R[Y]$ with respect to the line bundle $\overline{L}$ over $R[Y]$ (see **[5,** Section 6] for the definition). We define the nth *Euler class group* of $R[X, Y]$, denoted by $E(R[X, Y], L)$, as the product $E(R(X)[Y], \widehat{L}) \times E(R[Y], \overline{L})$.

To the pair (P, χ) we associate an element $e(P, \chi)$ of $E(R[X, Y], L)$, called the *Euler class* of (P, χ), as follows:

$$e(P, \chi) = (e(\widehat{P}, \widehat{\chi}), e(\overline{P}, \overline{\chi}))$$

where $e(\widehat{P},\widehat{\chi}) \in E(R(X)[Y],\widehat{L})$ is the Euler class of $(\widehat{P},\widehat{\chi})$, and $e(\overline{P},\overline{\chi}) \in E(R[Y],\overline{L})$ is the Euler class of $(\overline{P},\overline{\chi})$, defined in [**5,** Section 6].

Now, we treat the case when $n = 3$. Let $\widetilde{E}(R(X)[Y],\widehat{L})$ be the nth restricted Euler class group of $R(X)[Y]$ with respect to the line bundle $\widehat{L}$ over $R(X)[Y]$ and $\widetilde{E}(R[Y],\overline{L})$ the nth restricted Euler class group of $R[Y]$ with respect to the line bundle $\overline{L}$ over $R[Y]$ (see [**5,** Section 7] for the definition). We define the *Euler class group* of $R[X,Y]$, again denoted $E(R[X,Y],L)$, as the product $\widetilde{E}(R(X)[Y],\widehat{L}) \times \widetilde{E}(R[Y],\overline{L})$.

To the pair (P,χ) we associate an element $e(P,\chi)$ of $E(R[X,Y],L)$, called the *Euler class* of (P,χ), as follows:

$$e(P,\chi) = (e(\widehat{P},\widehat{\chi}), e(\overline{P},\overline{\chi})),$$

where $e(\widehat{P},\widehat{\chi}) \in \widetilde{E}(R(X)[Y],\widehat{L})$ is the Euler class of $(\widehat{P},\widehat{\chi})$ and $e(\overline{P},\overline{\chi}) \in \widetilde{E}(R[Y],\overline{L})$ is the Euler class of $(\overline{P},\overline{\chi})$, defined in [**5,** Section 7].

Remark 3.4. Note that, when $n = 2$, the definition of the Euler class group $E(R[T],L)$ is slightly different from the case $n \geq 4$. See [**5,** Remark 7.8] for details.

Theorem 3.5. *Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$, and let P be a projective $R[X,Y]$-module of rank n with determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Then, $e(P,\chi) = 0$ in $E(R[X,Y],L)$ if and only if P has a unimodular element.*

Proof. First, we assume that P has a unimodular element. Therefore, $\widehat{P}$ and $\overline{P}$ also have unimodular elements. If $n \geq 4$, by [**5,** Theorem 6.12], we have $e(\widehat{P},\widehat{\chi}) = 0$ in $E(R(X)[Y],\widehat{L})$ and $e(\overline{P},\overline{\chi}) = 0$ in $E(R[Y],\overline{L})$. The case $n = 2$ is taken care of by [**5,** Remark 7.8]. Now, if $n = 3$, it follows from [**5,** Theorem 7.4] that $e(\widehat{P},\widehat{\chi}) = 0$ in $E(R(X)[Y],\widehat{L})$ and $e(\overline{P}],\overline{\chi}) = 0$ in $\widetilde{E}(R[Y],\overline{L})$. Consequently, $e(P,\chi) = 0$.

Conversely, assume that $e(P,\chi) = 0$. Then, $e(\widehat{P},\widehat{\chi}) = 0$ in $E(R(X)[Y],\widehat{L})$ and $e(\overline{P},\overline{\chi}) = 0$ in $E(R[Y],\overline{L})$. If $n \neq 3$, by [**5,** Theorem 6.12, Remark 7.8], $\widehat{P}$ and $\overline{P}$ have unimodular elements. If $n = 3$, by [**5,** Theorem 7.4], $\widehat{P}$ and $\overline{P}$ have unimodular elements. Since $\widehat{P}$ has a unimodular element, we can find a monic polynomial $f \in R[X]$ such that P_f contains a unimodular element. Therefore, then, by Theorem 3.2, P has a unimodular element. $\square$

Remark 3.6. Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$, and let P be a projective $R[X_1, \ldots, X_r]$-module, $r \geq 3$, of rank n with determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^r(P)$ be an isomorphism. By induction on r, we can define the Euler class group of $R[X_1, \ldots, X_r]$ with respect to the line bundle L, denoted by $E(R[X_1, \ldots, X_r], L)$, as the product of $E(R(X_r)[X_1, \ldots, X_{r-1}], \widehat{L})$ and $E(R[X_1, \ldots, X_{r-1}], \overline{L})$.

To the pair (P, χ) we can associate an invariant $e(P, \chi)$ in $E(R[X_1, \ldots, X_r], L)$ as follows:

$$e(P, \chi) = (e(\widehat{P}, \widehat{\chi}), e(\overline{P}, \overline{\chi}))$$

where

$$e(\widehat{P}, \widehat{\chi}) \in E(R(X_r)[X_1, \ldots, X_{r-1}], \widehat{L})$$

is the Euler class of $(\widehat{P}, \widehat{\chi})$ and

$$e(\overline{P}, \overline{\chi}) \in E(R[X_1, \ldots, X_{r-1}], \overline{L})$$

is the Euler class of $(\overline{P}, \overline{\chi})$. Finally, we have the following result.

Theorem 3.7. *Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$, and let P be a projective $R[X_1, \ldots, X_r]$-module of rank n with determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Then, $e(P, \chi) = 0$ in $E(R[X_1, \ldots, X_r], L)$ if and only if P has a unimodular element.*

4. Analogue of Roy and Mandal.

In this section, we will prove Theorem 1.4. We begin with the following result from [**16,** Lemma 2.1].

Lemma 4.1. *Let R be a ring and P, Q two projective R-modules. Suppose that*

$$\phi : Q \longrightarrow P$$

is an R-linear map. For an ideal I of R, if ϕ is a split monomorphism modulo I, then

$$\phi_{1+I} : Q_{1+I} \longrightarrow P_{1+I}$$

is also a split monomorphism.

Lemma 4.2. *Let $(R, \mathcal{M})$ be a local ring and A a ring such that*

$$R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}].$$

Let P and Q be two projective A-modules and

$$\phi : Q \longrightarrow P$$

an R-linear map. If ϕ is a split monomorphism modulo $\mathcal{M}$, and, if ϕ_f is a split monomorphism for some special monic polynomial $f \in R[X]$, then ϕ is also a split monomorphism.

Proof. By Lemma 4.1 $\phi_{1+\mathcal{M}A}$ is a split monomorphism. Thus, there is an element h in $1+\mathcal{M}A$ such that ϕ_h is a split monomorphism. Since f is a special monic polynomial, $R \hookrightarrow A/f$ is an integral extension, and hence, h and f are comaximal. As ϕ_f is also a split monomorphism, it follows that ϕ is a split monomorphism. □

Lemma 4.3. *Let R be a local ring, and let A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]$. Let P and Q be two projective A-modules and*

$$\phi, \psi : Q \longrightarrow P$$

A-linear maps. Furthermore, assume that

$$\gamma : P \longrightarrow Q$$

is an A-linear map such that $\gamma\psi = f1_Q$ for some special monic polynomial $f \in R[X]$. For large m, there exists a special monic polynomial $g_m \in A$ such that $X\phi + (1+X^m)\psi$ becomes a split monomorphism after inverting g_m.

Proof. As in **[11, 16]**, first, we assume that Q is free. We have

$$\gamma(X\phi + (1 + X^m)\psi) = X\gamma\phi + (1 + X^m)f1_Q.$$

Since Q is free, $X\gamma\phi + (1 + X^m)f1_Q$ is a matrix. Clearly, for large integer m, $\det(X\gamma\phi + (1 + X^m)f1_Q)$ is a special monic polynomial which can be taken for g_m.

In the general case, find projective A-module Q' such that $Q \oplus Q'$ is free. Define maps

$$\phi', \psi' : Q \oplus Q' \longrightarrow P \oplus Q'$$

and

$$\gamma' : P \oplus Q' \longrightarrow Q \oplus Q'$$

as $\phi' = \phi \oplus 0$, $\psi' = \psi \oplus f1_{Q'}$ and $\gamma' = \gamma \oplus 1_{Q'}$. By the previous case, we can find a special monic polynomial g_m for some large m such that $(X\phi' + (1 + X^m)\psi')_{g_m}$ becomes a split monomorphism. Hence, $X\phi + (1+X^m)\psi$ becomes a split monomorphism after inverting g_m. $\square$

The next result generalizes Mandal's **[11]**.

Theorem 4.4. *Let* $(R, \mathcal{M})$ *be a local ring and* $R[X] \subset A \subset R[X, X^{-1}]$. *Let* P *and* Q *be two projective* A*-modules with* $\operatorname{rank}(Q) < \operatorname{rank}(P)$. *If* Q_f *is a direct summand of* P_f *for some special monic polynomial* $f \in R[X]$, *then* Q *is also a direct summand of* P.

Proof. The method of proof is similar to **[16,** Theorem 1.1]; hence, we merely give an outline of the proof.

Since Q_f is a direct summand of P_f, we can find A-linear maps $\psi : Q \to P$ and $\gamma : P \to Q$ such that $\gamma\psi = f1_Q$ (possibly after replacing f by a power of f).

Let 'bar' denote reduction modulo $\mathcal{M}$. Then we have $\overline{\gamma}\overline{\psi} = \overline{f}1_{\overline{Q}}$. As f is special monic, $\overline{\psi}$ is a monomorphism.

We may assume that $A = R[X, f_1/X^t, \ldots, f_n/X^t]$ with $f_i \in R[X]$. If $f_i \in \mathcal{M}\, R[X]$, then $\overline{R}[X, f_i/X^t] = \overline{R}[X, Y]/(X^t Y)$. If $f_i \in R[X] - \mathcal{M}\, R[X]$, then $\overline{R}[X, f_i/X^t]$ is either $\overline{R}[X]$ or $\overline{R}[X, X^{-1}]$, depending upon whether $\overline{f}_i/X^t$ is a polynomial in $\overline{R}[X]$ or $\overline{F}_i/X^s$ with $\overline{F}_i(0) \neq 0$ and $s > 0$.

In general, $\overline{A}$ is one of $\overline{R}[X]$, $\overline{R}[X, X^{-1}]$ or

$$\overline{R}[X, Y_1, \ldots, Y_m]/(X^t(Y_1, \ldots, Y_m))$$

for some $m > 0$. By **[20,** Theorem 3.2], any projective $\overline{R}[X, Y_1, \ldots, Y_m]/(X^t(Y_1, \ldots, Y_m))$-module is free. Therefore, in all cases, projective $\overline{A}$-modules are free and hence extended from $\overline{R}[X]$. In particular, $\overline{P}$ and $\overline{Q}$ are extended from $\overline{R}[X]$, which is a PID.

Let $\operatorname{rank}(P) = r$ and $\operatorname{rank}(Q) = s$. Therefore, using the elementary divisors theorem, we can find bases $\{\overline{p}_1, \dots, \overline{p}_r\}$ and $\{\overline{q}_1, \dots, \overline{q}_s\}$ for $\overline{P}$ and $\overline{Q}$, respectively, such that $\overline{\psi}(\overline{q}_i) = \overline{f}_i \overline{p}_i$ for some $f_i \in R[X]$ and $1 \le i \le s$.

For the remainder of the proof, we can follow the proof of **[16,** Theorem 1.1]. □

Now, we have the following consequence of Theorem 4.4.

Corollary 4.5. *Let R be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let P and Q be two projective A-modules such that P_f is isomorphic to Q_f for some special monic polynomial $f \in R[X]$. Then*:

(i) *Q is a direct summand of $P \oplus L$ for any projective A-module L*;

(ii) *P is isomorphic to Q if P or Q has a direct summand of rank one*;

(iii) *$P \oplus L$ is isomorphic to $Q \oplus L$ for all rank one projective A-modules L*;

(iv) *P and Q have same number of generators.*

Proof.

(i) Follows trivially from Theorem 4.4, and (iii) follows from (ii).

The proof of (iv) is the same as **[16,** Proposition 3.1 (4)].

For (ii), we can follow the proof of **[11,** Theorem 2.2 (ii)] by replacing doubly monic polynomial by special monic polynomial in his arguments. □

Corollary 4.6. *Let R be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let P be a projective A-module such that P_f is free for some special monic polynomial $f \in R[X]$. Then, P is free.*

Proof. Follows from the second part of Corollary 4.5. □

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