

SATURATIONS OF POWERS OF CERTAIN DETERMINANTAL IDEALS

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ABSTRACT. In this paper we study certain determinantal ideals that extend the class of ideals of Herzog-Northcott type introduced by O’Carroll and Planas-Vilanova. As is well known, in a three-dimensional Cohen-Macaulay local ring, the second symbolic powers of ideals of Herzog-Northcott type can be controlled well. We aim to generalize this fact considering “saturation” instead of “symbolic power.” Furthermore, in order to compare the saturation with the symbolic power, we study the associated primes of the powers of certain determinantal ideals.

1. Introduction. Let R be a Noetherian ring and m an integer with $m \geq 2$. Let x_1, x_2, \dots, x_{m+1} be a sequence of elements of R generating a proper ideal of height $m + 1$, and let $\{\alpha_{ij}\}$ be a family of positive integers, where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m + 1$. We set

$$a_{ij} = \begin{cases} x_{i+j-1}^{\alpha_{ij}} & \text{if } i + j \leq m + 2 \\ x_{i+j-m-2}^{\alpha_{ij}} & \text{if } i + j > m + 2 \end{cases}$$

for any $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m + 1$, and consider the matrix $A = (a_{ij})$ of size $m \times (m + 1)$. If $\alpha_{ij} = 1$ for all i and j , the matrix A looks like

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_m & x_{m+1} \\ x_2 & \cdots & x_m & x_{m+1} & x_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x_m & x_{m+1} & x_1 & \cdots & x_{m-1} \end{pmatrix}.$$

However, we may put any exponents to each entry. In this paper, we study the ideal generated by the maximal minors of A . If $m = 2$, this kind of ideal is known as Herzog-Northcott type in the recent literature

The third author is supported by KAKENHI (23540042). The third author is the corresponding author.

Received by the editors on July 25, 2013, and in revised form on March 18, 2014.
DOI:10.1216/JCA-2015-7-2-167

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[18], and it is a well-known result of Herzog [12] that the defining ideal of a space monomial curves is Herzog-Northcott type. Because the ideals of Herzog-Northcott type provide interesting examples of symbolic Rees algebras, a lot of authors studied the symbolic powers of those ideals (cf., [5, 8, 9, 11, 13, 16, 21]). Although the symbolic powers of ideals usually behave very wildly, if the ideal is Herzog-Northcott type, its second symbolic power can be controlled well (cf., [10, 15, 17]). The purpose of this paper is to generalize this fact for ideals stated above replacing “symbolic power” by “saturation.” In order to explain our main result, let us recall the definitions of the symbolic power and the saturation of an ideal.

Let (R, \mathfrak{m}) be a local ring and I an ideal of R such that $\dim R/I > 0$. Let r be a positive integer. We set

$$(I^r)^{\text{sat}} = \{x \in R \mid \mathfrak{m}^i \cdot x \subseteq I^r \text{ for some integer } i \geq 0\}$$

and call it the saturation of I^r . As $(I^r)^{\text{sat}}/I^r \cong H_{\mathfrak{m}}^0(R/I^r)$, where $H_{\mathfrak{m}}^0(\cdot)$ denotes the 0-th local cohomology functor, we have $(I^r)^{\text{sat}} = I^r$ if and only if $\text{depth } R/I^r > 0$. Moreover, if J is an \mathfrak{m} -primary ideal such that $\text{depth } R/(I^r :_R J) > 0$, we have $(I^r)^{\text{sat}} = I^r :_R J$. On the other hand, the r -th symbolic power of I is defined by

$$I^{(r)} = \{x \in R \mid sx \in I^r \text{ for some } s \in R \text{ such that } s \notin \mathfrak{p} \text{ for any } \mathfrak{p} \in \text{Min}_R R/I\}.$$

In order to compare $(I^r)^{\text{sat}}$ and $I^{(r)}$, let us take a minimal primary decomposition of I^r ;

$$I^r = \bigcap_{\mathfrak{p} \in \text{Ass}_R R/I^r} \mathfrak{Q}(\mathfrak{p}),$$

where $\mathfrak{Q}(\mathfrak{p})$ denotes the \mathfrak{p} -primary component. It is easy to see that

$$(I^r)^{\text{sat}} = \bigcap_{\mathfrak{m} \neq \mathfrak{p} \in \text{Ass}_R R/I^r} \mathfrak{Q}(\mathfrak{p}) \quad \text{and} \quad I^{(r)} = \bigcap_{\mathfrak{p} \in \text{Min}_R R/I} \mathfrak{Q}(\mathfrak{p}).$$

Hence, we have $(I^r)^{\text{sat}} \subseteq I^{(r)}$ and the equality holds if and only if $\text{Ass}_R R/I^r$ is a subset of $\{\mathfrak{m}\} \cup \text{Min}_R R/I$. Therefore, if $\dim R/I = 1$, then $(I^r)^{\text{sat}} = I^{(r)}$. If $\dim R/I \geq 2$, $(I^r)^{\text{sat}}$ may be different from $I^{(r)}$, but even in that case, $(I^r)^{\text{sat}}$ has meaning as an approximation of $I^{(r)}$.

If (R, \mathfrak{m}) is a three-dimensional Cohen-Macaulay local ring and I is an ideal of Herzog-Northcott type, then $I^{(2)}/I^2$ is a cyclic R -module and its generator can be described precisely (cf., [10, (2.2), (2.3)]). This fact can be generalized as follows, which is the main result of this paper.

Theorem 1.1. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of $\dim R = m + 1$, where $m \geq 2$. Let x_1, x_2, \dots, x_{m+1} be an sop for R , and set I to be the ideal generated by the maximal minors of A . Then the following assertions hold.*

- (1) $(I^r)^{\text{sat}} = I^r$ for any $r = 1, \dots, m - 1$.
- (2) $(I^m)^{\text{sat}}/I^m$ is a cyclic R -module.

The proof of this theorem is given in Section 4. Moreover, for I of 1.1, we can describe a generator of $(I^m)^{\text{sat}}/I^m$ assuming suitable conditions on $\{\alpha_{ij}\}$. In order to compare $(I^r)^{\text{sat}}$ with $I^{(r)}$ for I of 1.1, we have to compare $\text{Ass}_R R/I^r$ with $\{\mathfrak{m}\} \cup \text{Min}_R R/I$. For that purpose, we study the associated primes of powers of ideals in a more general situation in Section 3. Our results are closely related to [3, Lemma 3.3, Corollary 3.5] and the frameworks for the proofs are similar. Anyway, as a corollary of the results stated in Section 3, we see that the ideal I of 1.1 satisfies $(I^r)^{\text{sat}} \subsetneq I^{(r)}$ if $r > m \geq 3$ and $\alpha_{ij} = 1$ for any i, j .

Throughout this paper, R is a commutative ring, and we often assume that R is a Noetherian local ring with the maximal ideal \mathfrak{m} . For positive integers m, n and an ideal \mathfrak{a} of R , we denote by $\text{Mat}(m, n; \mathfrak{a})$ the set of $m \times n$ matrices with entries in \mathfrak{a} . For any $A \in \text{Mat}(m, n; R)$ and any $k \in \mathbb{Z}$ we denote by $I_k(A)$ the ideal generated by the k -minors of A . In particular, $I_k(A)$ is defined to be R (respectively, (0)) for $k \leq 0$ (respectively, $k > \min\{m, n\}$). If $A, B \in \text{Mat}(m, n; R)$ and the (i, j) entries of A and B are congruent modulo a fixed ideal \mathfrak{a} for any (i, j) , we write $A \equiv B \pmod{\mathfrak{a}}$.

2. Preliminaries. In this section, we assume that R is just a commutative ring. Let m, n be positive integers with $m \leq n$ and $A = (a_{ij}) \in \text{Mat}(m, n; R)$. Let us recall the following rather well-known fact.

Lemma 2.1. *Suppose $I_m(A) \subseteq \mathfrak{p} \in \text{Spec}R$, and put $\ell = \max\{0 \leq k \in \mathbb{Z} \mid I_k(A) \not\subseteq \mathfrak{p}\}$. Then $\ell < m$ and there exists $B \in \text{Mat}(m-\ell, n-\ell; \mathfrak{p}R_{\mathfrak{p}})$ such that $I_k(A)_{\mathfrak{p}} = I_{k-\ell}(B)$ for any $k \in \mathbb{Z}$.*

Proof. We prove by induction on ℓ . The assertion is obvious if $\ell = 0$. So, let us consider the case where $\ell > 0$. Then $I_1(A) \not\subseteq \mathfrak{p}$, and so some entry of A is a unit in $R_{\mathfrak{p}}$. Hence, applying elementary operations to A in $\text{Mat}(m, n; R_{\mathfrak{p}})$, we get a matrix of the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right),$$

where $A' \in \text{Mat}(m-1, n-1; R_{\mathfrak{p}})$. It is easy to see that $I_k(A)_{\mathfrak{p}} = I_{k-1}(A')$ for any $k \in \mathbb{Z}$. Hence $I_{m-1}(A') \subseteq \mathfrak{p}R_{\mathfrak{p}}$ and $\ell-1 = \max\{0 \leq k \in \mathbb{Z} \mid I_k(A') \not\subseteq \mathfrak{p}R_{\mathfrak{p}}\}$. By the induction hypothesis, there exists

$$B \in \text{Mat}((m-1)-(\ell-1), (n-1)-(\ell-1); \mathfrak{p}R_{\mathfrak{p}}) = \text{Mat}(m-\ell, n-\ell; \mathfrak{p}R_{\mathfrak{p}})$$

such that $I_t(A') = I_{t-(\ell-1)}(B)$ for any $t \in \mathbb{Z}$. Then we have $I_k(A)_{\mathfrak{p}} = I_{k-\ell}(B)$ for any $k \in \mathbb{Z}$. \square

In the rest of this section, we assume $n = m + 1$. For any $j = 1, 2, \dots, m+1$, A_j denotes the $m \times m$ submatrix of A determined by removing the j -th column. We set $d_j = (-1)^{j-1} \cdot \det A_j$ and $I = (d_1, d_2, \dots, d_{m+1})R = I_m(A)$. Let us take an indeterminate t over R and consider the Rees algebra of I ;

$$\mathcal{R}(I) := R[d_1t, d_2t, \dots, d_{m+1}t] \subseteq R[t],$$

which is a graded ring such that $\deg d_jt = 1$ for all $j = 1, 2, \dots, m+1$. On the other hand, let $S = R[T_1, T_2, \dots, T_{m+1}]$ be a polynomial ring over R with $m+1$ variables. We regard S as a graded ring by setting $\deg T_j = 1$ for all $j = 1, 2, \dots, m+1$. Let $\pi : S \rightarrow \mathcal{R}(I)$ be the homomorphism of R -algebras such that $\pi(T_j) = d_jt$ for any j . Then π is a surjective graded homomorphism. Now we set

$$f_i = \sum_{j=1}^{m+1} x_{ij}T_j \in S_1$$

for any $i = 1, \dots, m$. It is easy to see $(f_1, f_2, \dots, f_m)S \subseteq \text{Ker } \pi$. For our purposes, the following result due to Avramov [2] is very important (another elementary proof is given in [6]).

Proposition 2.2. *Suppose that R is a Noetherian ring. If $\text{grade } I_k(A) \geq m - k + 2$ for all $k = 1, \dots, m$, then $\text{Ker } \pi = (f_1, f_2, \dots, f_m)S$ and f_1, f_2, \dots, f_m is an S -regular sequence.*

As the last preliminary result, we describe a technique using determinants of matrices. Suppose that y_1, y_2, \dots, y_{m+1} are elements of R such that

$$A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We put $y = y_1 + y_2 + \dots + y_{m+1}$.

Lemma 2.3. *If y, y_k form a regular sequence for some $k = 1, 2, \dots, m+1$, then there exists $\delta \in R$ such that $y_j \cdot \delta = d_j$ for any $j = 1, 2, \dots, m+1$.*

Proof. We put $d = d_1 + d_2 + \dots + d_{m+1}$. Then the following assertion holds:

Claim. $y \cdot d_j = y_j \cdot d$ for all $j = 1, 2, \dots, m+1$.

In order to prove the claim above, let us consider the following $(m+1) \times (m+1)$ matrix:

$$B = \begin{pmatrix} 1 & 1 & \dots & 1 \\ & & A & \end{pmatrix}.$$

Expanding $\det B$ along the first row, we get $\det B = d$. Let us fix

$j = 1, 2, \dots, m + 1$. Multiplying the j -th column of B by y_j , we get

$$B' = \begin{pmatrix} 1 & \cdots & \overset{j}{y_j} & \cdots & 1 \\ a_{11} & \cdots & a_{1j}y_j & \cdots & a_{1,m+1} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj}y_j & \cdots & a_{m,m+1} \end{pmatrix}.$$

Then $\det B' = y_j \cdot \det B = y_j \cdot d$. Next, for any $\ell \in \{1, 2, \dots, m+1\} \setminus \{j\}$, we add the ℓ -th column of B' multiplied by y_ℓ to the j -th column, and get

$$B'' = \begin{pmatrix} 1 & \cdots & \overset{j}{y} & \cdots & 1 \\ a_{11} & \cdots & 0 & \cdots & a_{1,m+1} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & 0 & \cdots & a_{m,m+1} \end{pmatrix},$$

since our assumption means

$$a_{i1}y_1 + \cdots + a_{ij}y_j + \cdots + a_{i,m+1}y_{m+1} = 0$$

for all $i = 1, \dots, m$. Then $\det B'' = \det B' = y_j \cdot d$. Finally, replacing the first j columns of B'' , we get

$$B''' = \left(\begin{array}{c|ccc} y & 1 & \cdots & 1 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & A_j & \end{array} \right).$$

Then $y \cdot d_j = y \cdot (-1)^{j-1} \cdot \det A_j = (-1)^{j-1} \cdot \det B''' = \det B'' = y_j \cdot d$. Thus, we get the equalities of the claim.

Now we take $k = 1, 2, \dots, m+1$ so that y, y_k form a regular sequence. Because $y \cdot d_k = y_k \cdot d$, there exists $\delta \in R$ such that $d = y\delta$. Then $y \cdot d_j = y_j \cdot y\delta$ for any $j = 1, 2, \dots, m+1$. As y is an R -NZD, we get $d_j = y_j \cdot \delta$ for any $j = 1, 2, \dots, m+1$, and the proof is complete. \square

Lemma 2.4. *If R is a Cohen-Macaulay local ring and y_1, y_2, \dots, y_{m+1} is an ssop for R , then y, y_k form a regular sequence for any $k = 1, 2, \dots, m+1$.*

Proof. It is enough to show for $k = 1$. Because $(y_1, y_2, \dots, y_{m+1})R = (y, y_1, \dots, y_m)R$, it follows that y, y_1, \dots, y_m is an ssp for R , too. Hence y, y_1 is R -regular. \square

Lemma 2.5. *Suppose that \mathfrak{a} is an ideal of R and $a_{ij} \in \mathfrak{a}$ for all i, j . We put $Q = (y_1, y_2, \dots, y_{m+1})R$. Then δ of Lemma 2.3 is an element of $\mathfrak{a}^m :_R Q$.*

Proof. We get this assertion since $d_j \in \mathfrak{a}^m$ for any $j = 1, 2, \dots, m+1$. \square

3. Associated primes of R/I^n . Let R be a Noetherian ring and $A = (a_{ij}) \in \text{Mat}(m, m + 1; R)$, where $1 \leq m \in \mathbb{Z}$. Let $I = I_m(A)$. Throughout this section, we assume that I is a proper ideal and $\text{grade } I_k(A) \geq m - k + 2$ for all $k = 1, \dots, m$. Let us keep the notation of Section 2.

Let K_\bullet be the Koszul complex of f_1, f_2, \dots, f_m , which is a complex of graded free S -modules. We denote its boundary map by ∂_\bullet . Let e_1, e_2, \dots, e_m be an S -basis of K_1 consisting of homogeneous elements of degree 1 such that $\partial_1(e_i) = f_i$ for any $i = 1, \dots, m$. Then, for any $s = 1, \dots, m$,

$$\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_s} \mid 1 \leq i_1 < i_2 < \dots < i_s \leq m\}$$

is an S -basis of K_s consisting of homogeneous elements of degree s , and we have

$$\partial_s(e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_s}) = \sum_{p=1}^s (-1)^{p-1} \cdot f_{i_p} \cdot e_{i_1} \wedge \dots \wedge \widehat{e_{i_p}} \wedge \dots \wedge e_{i_s},$$

where $\widehat{e_{i_p}}$ means that e_{i_p} is omitted from the exterior product. Let $1 \leq r \in \mathbb{Z}$. Taking the homogeneous part of degree r of K_\bullet , we get a complex

$$[K_\bullet]_r : 0 \longrightarrow [K_m]_r \xrightarrow{\partial_m} [K_{m-1}]_r \longrightarrow \dots \longrightarrow [K_1]_r \xrightarrow{\partial_1} [K_0]_r \longrightarrow 0$$

of finitely generated free R -modules. It is obvious that $[K_s]_r = 0$ if $r < s$. On the other hand, if $r \geq s$, then

$$\left\{ T_1^{\alpha_1} T_2^{\alpha_2} \dots T_{m+1}^{\alpha_{m+1}} \cdot e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_s} \mid \begin{array}{l} 0 \leq \alpha_1, \alpha_2, \dots, \alpha_{m+1} \in \mathbb{Z}, \\ \alpha_1 + \alpha_2 + \dots + \alpha_{m+1} = r - s, \\ 1 \leq i_1 < i_2 < \dots < i_s \leq m \end{array} \right\}$$

is an R -basis of $[K_s]_r$.

Proposition 3.1. *If (R, \mathfrak{m}) is a local ring and $A \in \text{Mat}(m, m + 1; \mathfrak{m})$, we have*

$$\text{proj. dim}_R R/I^r = \begin{cases} r + 1 & \text{if } r < m, \\ m + 1 & \text{if } r \geq m. \end{cases}$$

Proof. By Proposition 2.2 and [4, 1.6.17], we see that

$$0 \longrightarrow K_m \xrightarrow{\partial_m} K_{m-1} \longrightarrow \cdots \longrightarrow K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\pi} \mathcal{R}(I) \longrightarrow 0$$

is a graded S -free resolution of $\mathcal{R}(I)$. Hence, for any integer $r \geq 0$,

$$0 \longrightarrow [K_m]_r \xrightarrow{\partial_m} [K_{m-1}]_r \longrightarrow \cdots \longrightarrow [K_1]_r \xrightarrow{\partial_1} [K_0]_r \xrightarrow{\pi} I^r t^r \longrightarrow 0$$

is an R -free resolution of the R -module $I^r t^r$. Let us notice $I^r t^r \cong I^r$ as R -modules. Suppose $1 \leq s \leq m$ and $r \geq s$. Then, for any non-negative integers $\alpha_1, \alpha_2, \dots, \alpha_{m+1}$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_{m+1} = r - s$ and positive integers i_1, i_2, \dots, i_s with $1 \leq i_1 < i_2 < \cdots < i_s \leq m$, we have

$$\begin{aligned} & \partial_s(T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{m+1}^{\alpha_{m+1}} \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s}) \\ &= T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{m+1}^{\alpha_{m+1}} \cdot \sum_{p=1}^s (-1)^{p-1} \\ & \quad \cdot \left(\sum_{j=1}^{m+1} a_{i_p, j} T_j \right) \cdot e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s} \\ &= \sum_{p=1}^s \sum_{j=1}^{m+1} (-1)^{p-1} a_{i_p, j} \cdot T_1^{\alpha_1} \cdots T_j^{1+\alpha_j} \cdots T_{m+1}^{\alpha_{m+1}} \cdot e_{i_1} \\ & \quad \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s} \\ & \in \mathfrak{m} \cdot [K_{s-1}]_r. \end{aligned}$$

Hence, $[K_\bullet]_r$ gives a minimal R -free resolution of I^r . If $r < m$, we have $[K_r]_r \neq 0$ and $[K_s]_r = 0$ for any $s > r$, and so $\text{proj. dim}_R I^r = r$. On the other hand, if $r \geq m$, we have $[K_m]_r \neq 0$ and $[K_s]_r = 0$ for any $s > m$, and so $\text{proj. dim}_R I^r = m$. Thus, we get the required equality as $\text{proj. dim}_R R/I^r = \text{proj. dim}_R I^r + 1$. \square

By the Auslander-Buchsbaum formula (cf., [4, 1.3.3]), we get the following.

Corollary 3.2. *If (R, \mathfrak{m}) is local and $A \in \text{Mat}(m, m + 1; \mathfrak{m})$, we have*

$$\text{depth } R/I^r = \begin{cases} \text{depth } R - r - 1 & \text{if } r < m, \\ \text{depth } R - m - 1 & \text{if } r \geq m. \end{cases}$$

Here we remark that $\text{depth } R \geq \text{grade } I_1(A) \geq m + 1$ by our assumption of this section. As a consequence of Corollary 3.2, we see that the next assertion holds.

Corollary 3.3. *Suppose that (R, \mathfrak{m}) is a local ring and $A \in \text{Mat}(m, m + 1; \mathfrak{m})$. Then we have $\mathfrak{m} \in \text{Ass}_R R/I^r$ if and only if $r \geq m$ and $\text{depth } R = m + 1$.*

The next result is a generalization of Corollary 3.3.

Proposition 3.4. *Let $I \subseteq \mathfrak{p} \in \text{Spec } R$ and $1 \leq r \in \mathbb{Z}$. We put*

$$\ell = \max\{0 \leq k < m \mid I_k(A) \not\subseteq \mathfrak{p}\}.$$

Then the following conditions are equivalent.

- (1) $\mathfrak{p} \in \text{Ass}_R R/I^r$.
- (2) $r \geq m - \ell$ and $\text{depth } R_{\mathfrak{p}} = m - \ell + 1$.

When this is the case, $\text{grade } I_{\ell+1}(A) = m - \ell + 1$.

Proof. By Lemma 2.1, there exists $B \in \text{Mat}(m - \ell, m - \ell + 1; \mathfrak{p}R_{\mathfrak{p}})$ such that $I_k(B) = I_{k+\ell}(A)_{\mathfrak{p}}$ for any $k \in \mathbb{Z}$. Hence, for any $k = 1, \dots, m - \ell$, we have

$$\text{grade } I_k(B) = \text{grade } I_{k+\ell}(A)_{\mathfrak{p}} \geq m - (k + \ell) + 2 = (m - \ell) - k + 2.$$

Therefore, by Corollary 3.3, we see that $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/I_{m-\ell}(B)^r$ if and only if $r \geq m - \ell$ and $\text{depth } R_{\mathfrak{p}} = m - \ell + 1$. Let us notice $I_{m-\ell}(B) = I_{\mathfrak{p}}$. Because $\mathfrak{p} \in \text{Ass}_R R/I^r$ if and only if $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/I_{\mathfrak{p}}^r$, we see (1) \Leftrightarrow (2). Furthermore, as $I_{\ell+1}(A) \subseteq \mathfrak{p}$, we have $\text{grade } I_{\ell+1}(A) \leq \text{depth } R_{\mathfrak{p}}$, and so we get $\text{grade } I_{\ell+1}(A) = m - \ell + 1$ if condition (2) is satisfied. \square

For any positive integer r , let Λ_A^r be the set of integers i such that $\max\{1, m - r + 1\} \leq i \leq m$ and $\text{grade } I_i(A) = m - i + 2$. We denote by $\text{Assh}_R R/I_i(A)$ the set of $\mathfrak{p} \in \text{Ass}_R R/I_i(A)$ such that $\dim R/\mathfrak{p} = \dim R/I_i(A)$. Then the following assertion holds.

Proposition 3.5. *Let R be a Cohen-Macaulay ring. Then, for any positive integer r , we have*

$$\text{Ass}_R R/I^r = \bigcup_{i \in \Lambda_A^r} \text{Assh}_R R/I_i(A).$$

Proof. Let us take any $\mathfrak{p} \in \text{Ass}_R R/I^r$ and put $\ell = \max\{0 \leq k < m \mid I_k(A) \not\subseteq \mathfrak{p}\}$. Then $I_{\ell+1}(A) \subseteq \mathfrak{p}$. Moreover, by Proposition 3.4, we have $r \geq m - \ell$, $\text{depth } R_{\mathfrak{p}} = m - \ell + 1$ and $\text{grade } I_{\ell+1}(A) = m - \ell + 1$. Hence $\ell + 1 \in \Lambda_A^r$. Let us notice that $\text{ht } \mathfrak{p} = \text{depth } R_{\mathfrak{p}}$ and $\text{ht } I_{\ell+1}(A) = \text{grade } I_{\ell+1}(A)$ as R is Cohen-Macaulay. Therefore $\text{ht } \mathfrak{p} = \text{ht } I_{\ell+1}(A)$, which means $\mathfrak{p} \in \text{Assh}_R R/I_{\ell+1}(A)$.

Conversely, let us take any $i \in \Lambda_A^r$ and $\mathfrak{q} \in \text{Assh}_R R/I_i(A)$. Then $\text{ht } \mathfrak{q} = \text{ht } I_i(A) = \text{grade } I_i(A) = m - i + 2$. As our assumption implies $\text{ht } I_{i-1}(A) \geq m - i + 3$, it follows that $i - 1 = \max\{0 \leq k < m \mid I_k(A) \not\subseteq \mathfrak{q}\}$. Let us notice $r \geq m - (i - 1)$ as $m - r + 1 \leq i$, which is one of the conditions for $i \in \Lambda_A^r$. Moreover, we have $\text{depth } R_{\mathfrak{q}} = \text{ht } \mathfrak{q} = m - (i - 1) + 1$. Thus we get $\mathfrak{q} \in \text{Ass}_R R/I^r$ by Proposition 3.4, and the proof is complete. \square

As a natural question, one may ask whether the results stated above can be extended to the case where $A = (a_{ij}) \in \text{Mat}(m, n; R)$ with $m < n$. As far as the authors know, the following two kinds of generalizations seem to be possible.

First, we would like to suggest considering the powers of modules. We set M to be the cokernel of the R -linear map $R^m \rightarrow R^n$ defined by ${}^t A$. Let us assume $\text{grade } I_k(A) \geq m - k + 2$ for all $k = 1, \dots, m$. Then, by [2, Proposition 4], M has rank $n - m$ and the r -th symmetric power $\mathcal{S}^r(M)$ is torsion-free over R for any $r \geq 1$. In this case, M can be embedded into a finitely generated free R -module F and the r -th power M^r of M is defined to be the image of $\mathcal{S}^r(M)$ in $\mathcal{S}^r(F)$. Similarly as the case of $m \times (m + 1)$ matrix, we consider the polynomial ring $S = R[T_1, T_2, \dots, T_n]$ and set $f_i = a_{i1}T_1 + a_{i2}T_2 + \dots + a_{in}T_n \in S$ for all $i = 1, \dots, m$. Then, by [2, Proposition 1 and Proposition 4], we

get an R -free resolution of $\mathcal{S}^r(F)/M^r$ by taking the homogeneous part of the Koszul complex of f_1, f_2, \dots, f_m over S . In this way, detailed information on the associated primes of $\mathcal{S}^r(F)/M^r$ could be deduced.

On the other hand, by using the free resolution of Akin-Buchsbaum-Weyman [1], another generalization seems to be possible. This idea was suggested by the referee. Similarly as in the case of the $m \times (m + 1)$ matrix, we set I to be the ideal generated by the maximal minors of A . Let us assume $\text{grade } I_k(A) \geq (m - k + 1)(n - m) + 1$ for any $k = 1, \dots, m$. Then, by [1, Theorem 5.4], we get an R -free resolution of R/I^r for any $r \geq 1$, and it could be used in place of Proposition 3.1 to deduce the $m \times n$ matrix version of Propositions 3.4 and 3.5.

Proposition 3.6. *Let R be an $(m + 1)$ -dimensional Cohen-Macaulay local ring, where $2 \leq m \in \mathbb{Z}$. Let A be the matrix given in the introduction. Then the following assertions hold:*

- (1) $\text{ht } I_k(A) \geq m - k + 2$ for all $k = 1, 2, \dots, m$.
- (2) $\text{proj. dim}_R R/I^r = \begin{cases} r + 1 & \text{if } r < m, \\ m + 1 & \text{if } r \geq m. \end{cases}$
- (3) $\text{depth } R/I^r = \begin{cases} m - r & \text{if } r < m, \\ 0 & \text{if } r \geq m. \end{cases}$

Furthermore, if $\alpha_{ij} = 1$ for all i and j , the following assertions hold:

- (4) $\text{ht } I_2(A) = m$ and $\text{Assh}_R R/I_2(A) \subseteq \text{Ass}_R R/I^r$ for any $r \geq m - 1$.
- (5) If m is an odd integer with $m \geq 3$, then $\text{ht } I_3(A) = m - 1$ and $\text{Assh}_R R/I_3(A) \subseteq \text{Ass}_R R/I^r$ for any $r \geq m - 2$.
- (6) If $m \geq 3$, then $(I^r)^{\text{sat}} \subsetneq I^{(r)}$ for any $r \geq m - 1$.

Proof. (1) We aim to prove the following.

Claim. $J_{k-1} + I_k(A)$ is an \mathfrak{m} -primary ideal for any $k = 1, 2, \dots, m$, where $J_{k-1} = (x_1, x_2, \dots, x_{k-1})R$.

If this is true, we have $\dim R/I_k(A) \leq k - 1$, and so $\text{ht } I_k(A) \geq \dim R - (k - 1) = m - k + 2$, which is the required inequality.

In order to prove the claim, we take any $\mathfrak{p} \in \text{Spec } R$ containing $J_{k-1} + \mathbf{I}_k(A)$. It is enough to show $J_{m+1} = (x_1, x_2, \dots, x_{m+1})R \subseteq \mathfrak{p}$. For that purpose, we prove $J_\ell \subseteq \mathfrak{p}$ for any $\ell = k - 1, k, \dots, m + 1$ by induction on ℓ . As we obviously have $J_{k-1} \subseteq \mathfrak{p}$, let us assume $k \leq \ell \leq m + 1$ and $J_{\ell-1} \subseteq \mathfrak{p}$. Because the k -minor of A with respect to the first k rows and the columns $\ell - k + 1, \dots, \ell - 1, \ell$ is congruent with

$$\det \begin{pmatrix} & & & & x_\ell^{\alpha_{1,\ell}} \\ & 0 & & & \\ & & x_\ell^{\alpha_{2,\ell-1}} & & \\ & & \ddots & & * \\ x_\ell^{\alpha_{k,\ell-k+1}} & & & & \end{pmatrix}$$

mod $J_{\ell-1}$, it follows that $J_{\ell-1} + \mathbf{I}_k(A)$ includes some power of x_ℓ . Hence $x_\ell \in \mathfrak{p}$, and so we get $J_\ell \subseteq \mathfrak{p}$.

(2) and (3) follow from Proposition 3.1 and Corollary 3.2, respectively.

In the rest of this proof, we assume $\alpha_{ij} = 1$ for any i and j .

(4) Let $\mathfrak{q} = (x_1 - x_2, x_2 - x_3, \dots, x_m - x_{m+1})R$. Then $x_1 \equiv x_j \pmod{\mathfrak{q}}$ for any $j = 1, 2, \dots, m + 1$. Hence, any 2-minor of A is congruent with

$$\det \begin{pmatrix} x_1 & x_1 \\ x_1 & x_1 \end{pmatrix} = 0$$

mod \mathfrak{q} . This means $\mathbf{I}_2(A) \subseteq \mathfrak{q}$, and so $\text{ht } \mathbf{I}_2(A) \leq \mu_R(\mathfrak{q}) = m$. On the other hand, $\text{ht } \mathbf{I}_2(A) \geq m$ by (1). Thus, we get $\text{ht } \mathbf{I}_2(A) = m$. Then, for any $r \geq m - 1$, we have $2 \in \Lambda_A^r$, and so $\text{Assh}_R R/\mathbf{I}_2(A) \subseteq \text{Ass}_R R/I^r$ by Proposition 3.5.

(5) Let \mathfrak{p} be the ideal of R generated by $\{x_i - x_{i+2}\}$, where i runs all odd integers with $1 \leq i \leq m - 2$. Similarly, we set \mathfrak{q} to be the ideal of R generated by $\{x_j - x_{j+2}\}$, where j runs all even integers with $2 \leq j \leq m - 1$. Let A' be the submatrix of A with the rows i_1, i_2, i_3 and the columns j_1, j_2, j_3 , where $1 \leq i_1 < i_2 < i_3 \leq m$ and $1 \leq j_1 < j_2 < j_3 \leq m + 1$. We can choose p, q with $1 \leq p < q \leq 3$ so that $i_p \equiv i_q \pmod{2}$. Then, for any $t = 1, 2, 3$, we have $i_p + j_t \equiv i_q + j_t \pmod{2}$, and so, if $i_p + j_t$ is odd (respectively, even), it follows that $a_{i_p, j_t} \equiv a_{i_q, j_t} \pmod{\mathfrak{q}}$ (respectively, \mathfrak{p}). Hence, we see that the p -th row of A' is congruent with the q -th row of A' mod $\mathfrak{p} + \mathfrak{q}$, which means $\det A' \equiv 0 \pmod{\mathfrak{p} + \mathfrak{q}}$. As a consequence, we get $\mathbf{I}_3(A) \subseteq \mathfrak{p} + \mathfrak{q}$.

or

$$\left(\begin{array}{c|ccc} & & & x_{m+1}^{\alpha_{1,m+1}} \\ & & & x_{m+1}^{\alpha_{2,m}} \\ & & \ddots & \\ & & & \\ x_{m+1}^{\alpha_{m,2}} & & & \end{array} \right)$$

if $k = m$ or $m + 1$, respectively. So, we have

$$\beta_k = \begin{cases} \min\{\alpha_{i,k-i+1}\}_{1 \leq i \leq k} \cup \{\alpha_{i,k-i+m+2}\}_{k < i \leq m} & \text{if } 1 \leq k < m, \\ \min\{\alpha_{i,k-i+1}\}_{1 \leq i \leq m} & \text{if } k = m \text{ or } m + 1. \end{cases}$$

Then, for any $k = 1, 2, \dots, m + 1$, we can choose i_k so that one of the following conditions is satisfied:

(i) $1 \leq i_k \leq k$ and $\beta_k = \alpha_{i_k, k-i_k+1}$

or

(ii) $k < i_k \leq m$ and $\beta_k = \alpha_{i_k, k-i_k+m+2}$.

Now, for any $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m + 1$, we set

$$a'_{ik} = \begin{cases} x_k^{\alpha_{i,k-i+1}-\beta_k} & \text{if } i \leq k, \\ x_k^{\alpha_{i,k-i+m+2}-\beta_k} & \text{if } i > k. \end{cases}$$

Then $a'_{i_k, k} = 1$ for all $k = 1, 2, \dots, m + 1$. The next assertion can be verified easily.

Lemma 4.1. *Suppose $1 \leq i \leq m$ and $1 \leq j \leq m + 1$.*

- (1) *If $i + j \leq m + 2$, setting $k = i + j - 1$, we have $1 \leq k \leq m + 1$, $i \leq k$ and $a_{ij} = x_k^{\beta_k} \cdot a'_{ik}$.*
- (2) *If $i + j > m + 2$, setting $k = i + j - m - 2$, we have $1 \leq k < m$, $i > k$ and $a_{ij} = x_k^{\beta_k} \cdot a'_{ik}$.*

Let Q be the ideal of R generated by $x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_{m+1}^{\beta_{m+1}}$. Then $A \in \text{Mat}(m, m + 1; Q)$ by Lemma 4.1. Assertion (2) of Theorem 1.1 follows from the next

Proposition 4.2. $(I^m)^{\text{sat}} = I^m :_R Q$ and $(I^m)^{\text{sat}}/I^m \cong R/Q$.

Proof. Let S be the polynomial ring over R with variables T_1, T_2, \dots, T_{m+1} . We regard S as a graded ring by setting $\deg T_j = 1$ for any $j = 1, 2, \dots, m + 1$. Let

$$f_i = \sum_{j=1}^{m+1} a_{ij} T_j \in S_1$$

for any $i = 1, 2, \dots, m$, and let K_\bullet be the Koszul complex of f_1, f_2, \dots, f_m . Then K_\bullet is a graded complex. Let ∂_\bullet be the boundary map of K_\bullet , and let e_1, e_2, \dots, e_m be an S -basis of K_1 consisting of homogeneous elements of degree 1 such that $\partial_1(e_i) = f_i$ for all $i = 1, 2, \dots, m$. As is stated in the proof of Proposition 3.1,

$$0 \rightarrow [K_m]_m \xrightarrow{\partial_m} [K_{m-1}]_m \rightarrow \cdots \rightarrow [K_1]_m \xrightarrow{\partial_1} [K_0]_m \xrightarrow{\epsilon} R \rightarrow 0$$

\parallel
 S_m

is an acyclic complex, where ϵ is the R -linear map such that

$$\epsilon(T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{m+1}^{\alpha_{m+1}}) = d_1^{\alpha_1} d_2^{\alpha_2} \cdots d_{m+1}^{\alpha_{m+1}}$$

for any non-negative integers $\alpha_1, \alpha_2, \dots, \alpha_{m+1}$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_{m+1} = m$. We obviously have $\text{Im } \epsilon = I^m$. We set $e = e_1 \wedge e_2 \wedge \cdots \wedge e_m$ and $\check{e}_i = e_1 \wedge \cdots \wedge \widehat{e}_i \wedge \cdots \wedge e_m$ for any $i = 1, 2, \dots, m$. Let us take $\{e\}$ and $\{T_j \check{e}_i \mid 1 \leq i \leq m, 1 \leq j \leq m + 1\}$ as R -bases of $[K_m]_m$ and $[K_{m-1}]_m$, respectively. Because

$$(\#) \quad \partial_m(e) = \sum_{i=1}^m (-1)^{i-1} f_i \cdot \check{e}_i = \sum_{i=1}^m \sum_{j=1}^{m+1} (-1)^{i-1} a_{ij} \cdot T_j \check{e}_i,$$

we have $\partial_m([K_m]_m) \subseteq Q \cdot [K_{m-1}]_m$. Hence, by [14, 3.1] we get

$$(I^m :_R Q) / I^m \cong [K_m]_m / Q[K_m]_m \cong R / Q.$$

Here, for any $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m + 1$, we set

$$T_{ik} = \begin{cases} T_{k-i+1} & \text{if } i \leq k, \\ T_{k-i+m+2} & \text{if } i > k. \end{cases}$$

Then the following assertion holds:

Claim 1. *Suppose $1 \leq k, \ell \leq m + 1$ and $k \neq \ell$. Then $T_{ik} \neq T_{i\ell}$ for any $i = 1, 2, \dots, m$.*

In order to prove the above claim, we may assume $k < \ell$. Then the following three cases can happen: (i) $i \leq k < \ell$, (ii) $k < i \leq \ell$ or (iii) $k < \ell < i$. Because $k-i+1 < \ell-i+1$ and $k-i+m+2 < \ell-i+m+2$, we get $T_{ik} \neq T_{i\ell}$ in cases (i) and (iii). Furthermore, as $m+1 > \ell-k$, we get $k-i+m+2 > \ell-i+1$, and so $T_{ik} \neq T_{i\ell}$ holds also in the case of (ii). Thus, we have seen Claim 1.

Now, for any $k = 1, 2, \dots, m+1$, we set

$$v_{(k,e)} = \sum_{i=1}^m (-1)^{i-1} a'_{ik} \cdot T_{ik} \check{e}_i \in [K_{m-1}]_m.$$

Then the following equality holds:

Claim 2.
$$\partial_m(e) = \sum_{k=1}^{m+1} x_k^{\beta_k} \cdot v_{(k,e)}.$$

In fact, by (#) and Lemma 4.1, we have

$$\begin{aligned} \partial_m(e) &= \sum_{i=1}^m \left(\sum_{j=1}^{m-i+2} (-1)^{i-1} a_{ij} \cdot T_j \check{e}_i + \sum_{j=m-i+3}^{m+1} (-1)^{i-1} a_{ij} \cdot T_j \check{e}_i \right) \\ &= \sum_{i=1}^m \left(\sum_{k=i}^{m+1} (-1)^{i-1} x_k^{\beta_k} a'_{ik} \cdot T_{k-i+1} \check{e}_i + \sum_{k=1}^{i-1} (-1)^{i-1} x_k^{\beta_k} a'_{ik} \right. \\ &\qquad \qquad \qquad \left. \cdot T_{k-i+m+2} \check{e}_i \right) \\ &= \sum_{i=1}^m \sum_{k=1}^{m+1} (-1)^{i-1} x_k^{\beta_k} a'_{ik} \cdot T_{ik} \check{e}_i, \end{aligned}$$

and so the equality of Claim 2 follows.

Finally, we need the following:

Claim 3. $\{v_{(k,e)}\}_{1 \leq k \leq m}$ is a part of an R -basis of $[K_{m-1}]_m$.

If this is true, by [14, 3.4] (see [7, 3.4] for the case where $m = 2$) we get $\text{depth } R/(I^m :_R Q) > 0$, which means $(I^m)^{\text{sat}} = I^m :_R Q$. So, let us prove Claim 3. By Claim 1, we see that $T_{i_1,1} \check{e}_{i_1}, T_{i_2,2} \check{e}_{i_2}, \dots, T_{i_m,m} \check{e}_{i_m}$

are different to each other. We set

$$U = \{T_j \check{e}_i \mid 1 \leq i \leq m, 1 \leq j \leq m + 1\} \setminus \{T_{i_k, k} \check{e}_{i_k} \mid 1 \leq k \leq m\}$$

and aim to prove that $U \cup \{v_{(k, e)}\}_{1 \leq k \leq m}$ is an R -basis of $[K_{m-1}]_m$. By [14, 3.3], it is enough to show that the submodule of $[K_{m-1}]_m$ generated by $U \cup \{v_{(k, e)}\}_{1 \leq k \leq m}$ includes $T_{i_k, k} \check{e}_{i_k}$ for any $k = 1, 2, \dots, m$. This can be easily seen since

$$v_{(k, e)} = (-1)^{k-1} \cdot T_{i_k, k} \check{e}_{i_k} + \sum_{i \neq i_k} (-1)^{i-1} a'_{i_k} \cdot T_{i_k} \check{e}_i$$

and $T_{i_k} \check{e}_i \in U$ if $i \neq i_k$, which follows from Claim 1. Thus the assertion of Claim 3 follows, and the proof of Proposition 4.2 is complete. \square

If we assume a suitable condition on $\{\alpha_{ij}\}$, we can describe a generator of $(I^m)^{\text{sat}}/I^m$. For any $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, m+1$, we set

$$b_{ik} = \begin{cases} a'_{i_k} d_{k-i+1} & \text{if } i \leq k \\ a'_{i_k} d_{k-i+m+2} & \text{if } i > k, \end{cases}$$

and $B = (b_{ik}) \in \text{Mat}(m, m + 1; I)$. Then the next equality holds:

Lemma 4.3.
$$B \begin{pmatrix} x_1^{\beta_1} \\ x_2^{\beta_2} \\ \vdots \\ x_{m+1}^{\beta_{m+1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Proof. For all $i = 1, 2, \dots, m$, we have

$$\sum_{j=1}^{m+1} a_{ij} d_j = 0.$$

Let us divide the left side of this equality as follows:

$$\sum_{j=1}^{m-i+2} a_{ij} d_j + \sum_{j=m-i+3}^{m+1} a_{ij} d_j = 0.$$

If $1 \leq j \leq m - i + 2$, setting $k = i + j - 1$, we have $i \leq k \leq m + 1$ and

$$a_{ij} d_j = x_k^{\beta_k} a'_{i_k} \cdot d_{k-i+1} = x_k^{\beta_k} \cdot b_{ik}.$$

On the other hand, if $m - i + 3 \leq j \leq m + 1$, setting $k = i + j - m - 2$, we have $1 \leq k < i$ and

$$a_{ij}d_j = x_k^{\beta_k} a'_{ik} \cdot d_{k-i+m+2} = x_k^{\beta_k} \cdot b_{ik}.$$

Thus, we get

$$\sum_{k=1}^{m+1} b_{ik} \cdot x_k^{\beta_k} = 0$$

for all $i = 1, 2, \dots, m$, which means the required equality. □

For any $k = 1, 2, \dots, m + 1$, we denote by B_k the submatrix of B determined by removing the k -th column. We set $b_k = (-1)^{k-1} \det B_k \in I^m$.

Proposition 4.4. *Suppose $\beta_k = \alpha_{k,1}$ for any $k = 1, 2, \dots, m$ (For example, this holds if $\alpha_{k,1} = 1$ for any $k = 1, 2, \dots, m$). Then, there exists $\delta \in R$ such that $x_k^{\beta_k} \cdot \delta = b_k$ for any $k = 1, 2, \dots, m + 1$ and $(I^m)^{\text{sat}} = I^m + (\delta)$.*

Proof. The existence of δ such that $x_k^{\beta_k} \cdot \delta = b_k$ for any $k = 1, 2, \dots, m + 1$ follows from Lemmas 2.3 and 4.3. Then $\delta \in I^m :_R Q \subseteq (I^m)^{\text{sat}}$. We put $Q' = (x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_m^{\beta_m})R$. Then

$$A_1 \equiv \begin{pmatrix} 0 & & x_{m+1}^{\alpha_{1,m+1}} \\ & x_{m+1}^{\alpha_{2,m}} & \\ & \ddots & 0 \\ x_{m+1}^{\alpha_{m,2}} & & \end{pmatrix} \pmod{Q'},$$

and so $d_1 \equiv \pm x_{m+1}^\alpha \pmod{Q'}$, where $\alpha := \alpha_{1,m+1} + \alpha_{2,m} + \dots + \alpha_{m,2}$. Furthermore, if $2 \leq k \leq m + 1$, we have $d_k \in Q'$ since the entries of the first column of M_k are $x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_m^{\beta_m}$. Hence $Q' + I = Q' + (x_{m+1}^\alpha)$. On the other hand, the assumption of Proposition 4.4 implies that, for all $k = 1, 2, \dots, m$, we can take k itself as i_k , and then $a'_{kk} = 1$. Hence,

$$B = \begin{pmatrix} d_1 & a'_{12}d_2 & a'_{13}d_3 & \cdots & a'_{1m}d_m & a'_{1,m+1}d_{m+1} \\ a'_{21}d_{m+1} & d_1 & a'_{23}d_2 & \cdots & a'_{2m}d_{m-1} & a'_{2,m+1}d_m \\ a'_{31}d_m & a'_{32}d_{m+1} & d_1 & \cdots & a'_{3m}d_{m-2} & a'_{3,m+1}d_{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{m1}d_3 & a'_{m2}d_4 & a'_{m3}d_5 & \cdots & d_1 & a'_{m,m+1}d_2 \end{pmatrix},$$

and so

$$B_{m+1} \equiv \begin{pmatrix} \pm x_{m+1}^\alpha & & & 0 \\ & \pm x_{m+1}^\alpha & & \\ & 0 & \ddots & \\ & & & \pm x_{m+1}^\alpha \end{pmatrix} \pmod{Q'},$$

which means $b_{m+1} \equiv \pm x_{m+1}^{m\alpha} \pmod{Q'}$. Thus, we get

$$x_{m+1}^{\beta_{m+1}} \cdot \delta \equiv x_{m+1}^{m\alpha} \pmod{Q'}.$$

Here we notice $\beta_{m+1} \leq \alpha_{1,m+1} < \alpha$. Because $x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_{m+1}^{\beta_{m+1}}$ is an R -regular sequence, it follows that

$$\delta \equiv \pm x_{m+1}^{m\alpha - \beta_{m+1}} \pmod{Q'},$$

and so

$$Q' + (\delta) = Q' + (x_{m+1}^{m\alpha - \beta_{m+1}}) \supseteq Q' + (x_{m+1}^{m\alpha}) = Q' + I^m.$$

Now we consider the R -linear map

$$f : R \rightarrow \frac{Q' + (x_{m+1}^{m\alpha - \beta_{m+1}})}{Q' + (x_{m+1}^{m\alpha})} = \frac{Q' + (\delta)}{Q' + I^m}$$

such that $f(1)$ is the class of $x_{m+1}^{m\alpha - \beta_{m+1}}$. Then we have the following:

Claim. $\text{Ker } f = Q$.

If this is true, then $R/Q \cong (Q' + (\delta))/(Q' + I^m)$, and so

$$\ell_R(R/Q) = \ell_R\left(\frac{Q' + (\delta)}{Q' + I^m}\right).$$

Because $(Q' + (\delta))/(Q' + I^m)$ is a homomorphic image of $(I^m + (\delta))/I^m$ and $I^m + (\delta) \subseteq (I^m)^{\text{sat}}$, we have

$$\ell_R\left(\frac{Q' + (\delta)}{Q' + I^m}\right) \leq \ell_R\left(\frac{I^m + (\delta)}{I^m}\right) \leq \ell_R((I^m)^{\text{sat}}/I^m) = \ell_R(R/Q),$$

where the last equality follows from Proposition 4.2. Thus, we see

$$\ell_R\left(\frac{I^m + (\delta)}{I^m}\right) = \ell_R((I^m)^{\text{sat}}/I^m),$$

and so $I^m + (\delta) = (I^m)^{\text{sat}}$ holds. \square

Proof of Claim. Let us take any $z \in \text{Ker } f$. Then, there exists $w \in R$ such that

$$z \cdot x_{m+1}^{m\alpha - \beta_{m+1}} \equiv w \cdot x_{m+1}^{m\alpha} \pmod{Q'}.$$

This congruence implies

$$x_{m+1}^{m\alpha - \beta_{m+1}}(z - w \cdot x_{m+1}^{\beta_{m+1}}) \in Q'.$$

Because $x_1^{\beta_1}, \dots, x_m^{\beta_m}, x_{m+1}^{m\alpha - \beta_{m+1}}$ is an R -regular sequence, we have $z - w \cdot x_{m+1}^{\beta_{m+1}} \in Q'$, which means $z \in Q$. Hence, $\text{Ker } f \subseteq Q$. As the converse inclusion is obvious, we get the equality of the claim, and the proof of Proposition 4.4 is complete. \square

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