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SEPARATORS OF ARITHMETICALLY COHEN-MACAULAY FAT POINTS IN $\mathbf{P}^1 \times \mathbf{P}^1$

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ABSTRACT. Let $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ be a set of fat points that is also arithmetically Cohen-Macaulay (ACM). We describe how to compute the degree of a separator of a fat point of multiplicity m for each point in the support of Z using only a numerical description of Z. Our formula extends the case of reduced points which was previously known.

1. Introduction. Fix an algebraically closed field k of characteristic zero. Let $R = k[x_0, x_1, y_0, y_1]$ be the $\mathbf{N}^2$-graded polynomial ring with $\deg x_i = (1,0)$ for $i = 0,1$ and $\deg y_i = (0,1)$ for $i = 0,1$. The ring R is the coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^1$. Consider now a set of points $X = \{P_1, \dots, P_s\} \subset \mathbf{P}^1 \times \mathbf{P}^1$, and fix positive integers $m_1, \dots, m_s$. The goal of this note is to study some of the properties of the scheme $Z = m_1P_1 + \cdots + m_sP_s$ of fat points (precise definitions are deferred until the next section). In particular, we are interested in describing the separator of P_i of multiplicity m_i.

Recall that for sets of points $X = \{P_1, \dots, P_s\} \subseteq \mathbf{P}^n$, a homogeneous form $F \in k[\mathbf{P}^n]$ is called a *separator* of $P \in X$ if $F(P) \neq 0$, but $F(Q) = 0$ for all $Q \in X \setminus \{P\}$. Over the years, a number of authors have shown how to exploit information about the separator of a point to describe properties of the set of reduced points $X \subseteq \mathbf{P}^n$ (e.g., see [**1–4, 11, 13, 14**]). In a series of papers, the authors, along with Marino, (see [**6, 10**]) generalized some of these results by studying separators of fat points, a family of non-reduced points. Roughly speaking, a *separator of a point P_i of multiplicity m_i* and the *degree of a point P_i of multiplicity m_i* are defined in terms of the generators of $I_{Z'}/I_Z$ in R/I_Z where $I_{Z'}$ is the defining ideal of $Z' = m_1P_1 + \cdots + (m_i - 1)P_i + \cdots + m_sP_s$.

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General properties of separators of both reduced points and fat points in a multiprojective space were studied in [**8–10, 12**]. We now specialize to the case of fat points in $\mathbf{P}^1 \times \mathbf{P}^1$. By restricting to this case, we can improve upon the results found in [**10**]. The main result of this paper (Theorem 3.4) is to show how to compute the degree of a point P_i of multiplicity m_i directly from the combinatorics of the scheme, i.e., the number of points on the various rulings of $\mathbf{P}^1 \times \mathbf{P}^1$ and the multiplicities, provided that the scheme is ACM. This result generalizes the reduced case as found in [**12**, Theorem 7.4] and [**9**, Theorem 4.4]. As an application, we can compute the Hilbert function of a scheme Z' where Z' is any fat point scheme (possibly not ACM) which has the property that if we increase the multiplicity of one of its points by one we get an ACM scheme Z.

2. Preliminaries and notation.

2.1. ACM fat points in $\mathbf{P}^1 \times \mathbf{P}^1$. Let X be a set of s distinct points in $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\pi_1 : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ denote the projection morphism defined by $P \times Q \mapsto P$. Let $\pi_2 : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ be the other projection morphism. The set $\pi_1(X) = \{P_1, \ldots, P_a\}$ is the set of $a \leq s$ distinct first coordinates that appear in X. Similarly, $\pi_2(X) = \{Q_1, \ldots, Q_b\}$ is the set of $b \leq s$ distinct second coordinates. For $i = 1, \ldots, a$, let L_{P_i} be the degree $(1,0)$ form that vanishes at all the points with first coordinate P_i. Similarly, for $j = 1, \ldots, b$, let L_{Q_j} denote the degree $(0,1)$ form that vanishes at points with second coordinate Q_j.

Let $D := \{(x,y) \mid 1 \leq x \leq a, 1 \leq y \leq b\}$. If $P \in X$, then $I_P = (L_{P_i}, L_{Q_j})$ for some $(i,j) \in D$. So, we can write each point $P \in X$ as $P_i \times Q_j$ for some $(i,j) \in D$. (Note that this does not mean that if $(i,j) \in D$, then $P_i \times Q_j \in X$; there may exist a tuple $(i,j) \in D$, but $P_i \times Q_j \notin X$.)

Suppose that X is a set of distinct points in $\mathbf{P}^1 \times \mathbf{P}^1$, $|\pi_1(X)| = a$ and $|\pi_2(X)| = b$. Let $I_{P_i \times Q_j} = (L_{P_i}, L_{Q_j})$ denote the ideal associated to the point $P_i \times Q_j \in X$. For each $(i,j) \in D$, let m_{ij} be a positive integer if $P_i \times Q_j \in X$; otherwise, let $m_{ij} = 0$. Then we denote by Z the subscheme of $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the saturated bihomogeneous ideal

$$I_Z = \bigcap_{(i,j) \in D} I_{P_i \times Q_j}^{m_{ij}}$$

where $I^0_{P_i \times Q_j} := (1)$. We say Z is a *fat point scheme* or *a set of fat points* of $\mathbf{P}^1 \times \mathbf{P}^1$. The integer m_{ij} is called the *multiplicity* of the point $P_i \times Q_j$. The *support* of Z, written supp (Z), is the set of points X. We shall sometimes denote the scheme as $Z = \{(P_i \times Q_j; m_{ij}) \mid (i,j) \in D\}$, or as $Z = m_{11}(P_1 \times Q_1) + \cdots + m_{ab}(P_a \times Q_b)$, where if $P_i \times Q_j \notin \operatorname{supp}(Z)$ then we put $m_{ij} = 0$.

A fat point scheme is said to be *arithmetically Cohen-Macaulay* (ACM for short) if the associated coordinate ring is Cohen-Macaulay. We will need a classification of ACM fat point schemes of $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ due to the first author (see [**5**]).

We begin by recalling a construction of [**5**]. Let Z be a fat point scheme in $\mathbf{P}^1 \times \mathbf{P}^1$ where $Z = \{(P_i \times Q_j; m_{ij}) \mid 1 \leq i \leq a,\ 1 \leq j \leq b\}$ with $m_{ij} \geq 0$. For each $h \in \mathbf{N}$, and for each tuple (i,j) with $1 \leq i \leq a$ and $1 \leq j \leq b$, define

$$t_{ij}(h) := (m_{ij} - h)_+ = \max\{0, m_{ij} - h\}.$$

The set $\mathcal{S}_Z$ is then defined to be the set of b-tuples

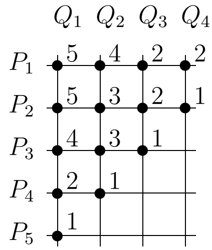
$$\mathcal{S}_Z = \{(t_{11}(h), \ldots, t_{1b}(h)), (t_{21}(h), \ldots, t_{2b}(h)), \ldots, \\ (t_{a1}(h), \ldots, t_{ab}(h)) \mid h \in \mathbf{N}\}.$$

The elements of $\mathcal{S}_Z$ belong to $\mathbf{N}^b$. Let $\succeq$ denote the partial order where $(i_1, \ldots, i_b) \succeq (j_1, \ldots, j_b)$ if and only if $i_\ell \geq j_\ell$ for all $\ell = 1, \ldots, b$. Then we have [**5**, Theorem 4.8]:

Theorem 2.1. *Let Z be a fat point scheme in $\mathbf{P}^1 \times \mathbf{P}^1$. Then Z is* ACM *if and only if the elements of $\mathcal{S}_Z$ can be totally ordered by $\succeq$, i.e., $\mathcal{S}_Z$ has no incomparable elements.*

Remark 2.2. Recall that the bigraded *Hilbert function* of Z is defined by $H_Z(i,j) = \dim_k R_{i,j} - \dim_k (I_Z)_{i,j}$. When $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ is an ACM fat point scheme, then $H_Z(i,j)$ can be computed for all (i,j) directly from the set $\mathcal{S}_Z$ (see [**5**] for details).

Example 2.3. We illustrate these ideas with the following set of fat points in $\mathbf{P}^1 \times \mathbf{P}^1$:

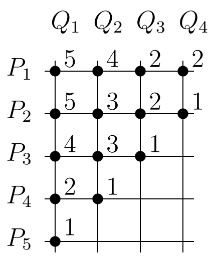
$Z =$ 

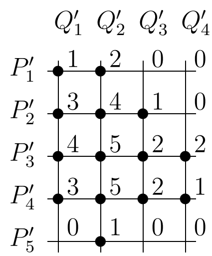
Consider the multiplicities 5, 4, 2 and 2 on the first ruling, i.e., the points whose first coordinate is P_1. From the construction of $\mathcal{S}_Z$, the tuples (5,4,2,2), (4,3,1,1), (3,2,0,0), (2,1,0,0), (1,0,0,0) all belong to $\mathcal{S}_Z$. Notice we successively subtract one from each entry, until we reach a zero. We carry out this procedure for each ruling to find:

$$\begin{aligned}\mathcal{S}_Z = \{&(5,4,2,2),(4,3,1,1),(3,2,0,0),(2,1,0,0),(1,0,0,0),(5,3,2,1),\\ &(4,2,1,0),(3,1,0,0),(2,0,0,0),(1,0,0,0),(4,3,1,0),(3,2,0,0),\\ &(2,1,0,0),(1,0,0,0),(2,1,0,0),(1,0,0,0),(1,0,0,0)\}.\end{aligned}$$

The partial order $\succeq$, when restricted to the set $\mathcal{S}_Z$, is a total ordering, i.e., there are no incomparable elements. Thus, the set Z is an ACM set of fat points.

Remark 2.4. We note that the following two schemes of fat points in $\mathbf{P}^1 \times \mathbf{P}^1$:

$Z =$ 

$Z' =$ 

are both ACM, and reordering in a suitable way the lines of type $(1,0)$ and $(0,1)$ become the same. Thus, if Z is ACM, we can always suppose that $(m_{i1},\ldots,m_{ib}) \succeq (m_{j1},\ldots,m_{jb})$ for $1 \le i < j \le a$. In addition, we can always suppose that $m_{i1} \ge m_{i2} \ge \cdots \ge m_{ib}$ for all $i = 1,\ldots,a$.

When Z is ACM, there are some relative bounds on the multiplicities:

Lemma 2.5. *Let $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ be an* ACM *set of fat points with lines ordered as in Remark* 2.4.

(i) *Suppose that there exist i, k, j, l such that $P_i \times Q_j$, $P_i \times Q_l$, $P_k \times Q_j$, and $P_k \times Q_l$ all belong to* $\mathrm{Supp}\,(Z)$, *and let m_{ij}, m_{il}, m_{kj}, and m_{kl} be the corresponding nonzero multiplicities. Then $m_{ij} \leq m_{il} + m_{kj} - m_{kl} + 1$.*

(ii) *Suppose that there exist i, k, j, l such that $P_i \times Q_j$, $P_i \times Q_l$, and $P_k \times Q_j$, all belong to* $\mathrm{Supp}\,(Z)$, *but $P_k \times Q_l \notin$* $\mathrm{Supp}\,(Z)$, *and let m_{ij}, m_{il} and m_{kj} be the corresponding nonzero multiplicities. Then $m_{il} \leq m_{ij} - m_{kj} + 1$.*

Proof. (i) Since Z is ACM, using Remark 2.4 we can always reorder the lines of type L_{P_i} and L_{Q_j} so that $m_{ij} \geq m_{il}$ and $m_{ij} \geq m_{kj}$. Suppose that $m_{ij} > m_{il} + m_{kj} - m_{kl} + 1$. From the construction of $\mathcal{S}_Z$, the tuples

$$(\star, m_{ij} - m_{il} + m_{kl} - 1, \star, m_{il} - m_{il} + m_{kl} - 1, \star) \text{ and } (\star, m_{kj}, \star, m_{kl}, \star)$$

are elements of $\mathcal{S}_Z$ where $\star$ denotes other elements of the tuple. But, since $m_{ij} - m_{il} + m_{kl} - 1 > m_{kj}$ but $0 \leq m_{il} - m_{il} + m_{kl} - 1 = m_{kl} - 1 < m_{kl}$, these tuples of $\mathcal{S}_Z$ will be incomparable, which contradicts the ACM property of Z.

(ii) The proof is similar to (i). Suppose that $m_{il} > m_{ij} - m_{kj} + 1$. In $\mathcal{S}_Z$, we will have tuples of the form $(\star, m_{ij} - (m_{ij} - m_{kj} + 1), \star, m_{il} - (m_{ij} - m_{kj} + 1), \star)$ and $(\star, m_{kj}, \star, 0, \star)$. But then $\mathcal{S}_Z$ will have incomparable elements since $m_{ij} - (m_{ij} - m_{kj} + 1) < m_{kj}$ but $m_{il} - (m_{ij} - m_{kj} + 1) > 0$. ◻

In the sequel we will require a bigraded version of Bezout's theorem:

Theorem 2.6. *Let $F, G \in k[x_0, x_1, y_1, y_0]$ be two bihomogeneous forms such that G is irreducible,* $\deg G = (a, b)$, *and* $\deg F = (c, d)$. *If the curves defined by F and G in $\mathbf{P}^1 \times \mathbf{P}^1$ meet at more than $ad + bc$ points (counting multiplicities), then $F = GF'$.*

2.2. Separators of fat points. In [**10**], the authors introduced the notion of a separator for a set of fat points in $\mathbf{P}^n \times \mathbf{P}^m$. We recall these results but specialize to the case of $\mathbf{P}^1 \times \mathbf{P}^1$. Note, we will denote a point of $\mathbf{P}^1 \times \mathbf{P}^1$ simply by P instead of $P_i \times Q_j$.

Definition 2.7. Let $Z = m_1P_1 + \cdots + m_iP_i + \cdots + m_sP_s$ be a set of fat points in $\mathbf{P}^1 \times \mathbf{P}^1$. We say that F is a *separator of the point* P_i *of multiplicity* m_i if $F \in I_{P_i}^{m_i-1} \setminus I_{P_i}^{m_i}$ and $F \in I_{P_j}^{m_j}$ for all $j \neq i$.

If we let $Z' = m_1P_1 + \cdots + (m_i - 1)P_i + \cdots + m_sP_s$, then a separator of the point P_i of multiplicity m_i is also an element of $F \in I_{Z'} \setminus I_Z$. The set of minimal separators are defined in terms of the ideals $I_{Z'}$ and I_Z.

Definition 2.8. A set $\{F_1, \ldots, F_p\}$ is a set of *minimal separators of* P_i *of multiplicity* m_i if $I_{Z'}/I_Z = (\overline{F}_1, \ldots, \overline{F}_p)$, and there does not exist a set $\{G_1, \ldots, G_q\}$ with $q < p$ such that $I_{Z'}/I_Z = (\overline{G}_1, \ldots, \overline{G}_q)$.

Important for this paper is the following definition:

Definition 2.9. The *degree of the minimal separators of* P_i *of multiplicity* m_i, denoted $\deg_Z(P_i)$, is the tuple

$$\deg_Z(P_i) = (\deg F_1, \ldots, \deg F_p) \text{ where } \deg F_i \in \mathbf{N}^2$$

and $F_1, \ldots, F_p$ is any set of minimal separators of P_i of multiplicity m_i.

For a general fat point scheme $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$, there is no known formula for $p = |\deg_Z(P)|$. However, as a special case of Theorem 4.3 and Theorem 5.1 of [**10**], we can compute the exact value for p if we also assume that Z is ACM (as we shall assume throughout the next section):

Theorem 2.10. *Let* $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ *be an* ACM *set of fat points. If* P *is a fat point of multiplicity* m *of* Z*, then* $|\deg_Z(P)| = m$.

3. Main results.

3.1. Fat points on a ruling. We begin by looking at a special case, namely, $\mathrm{Supp}\,(Z) = \{P_1 \times Q_1, \ldots, P_1 \times Q_b\}$, i.e., all the points have the same first coordinate. The fact that these schemes are ACM follows directly from Theorem 2.1. We first require a lemma which depends upon the Hilbert functions of these schemes (see [**7**, Theorem 2.2]).

Lemma 3.1. *Let Z be a set of fat points in $\mathbf{P}^1 \times \mathbf{P}^1$ of the form*

$$Z = m_1(P \times Q_1) + m_2(P \times Q_2) + \cdots + m_b(P \times Q_b).$$

Let $m = \max\{m_j\}_{j=1}^b$. For $\ell = 0, \ldots, m-1$, set $c_\ell = \sum_{p=1}^b (m_p - \ell)_+$. If $(i,j) \not\succeq (\ell, c_\ell)$ for all $\ell \in \{0, \ldots, m-1\}$, then $\dim_k (I_Z)_{i,j} = 0$.

Theorem 3.2. *Let Z be a set of fat points in $\mathbf{P}^1 \times \mathbf{P}^1$ of the form*

$$\begin{aligned} Z = m_1(P \times Q_1) + m_2(P \times Q_2) + \cdots \\ + m_j(P \times Q_j) + \cdots + m_b(P \times Q_b), \end{aligned}$$

i.e., each point of $\mathrm{Supp}\,(Z)$ has the same first coordinate. Fix a $j \in \{1, \ldots, b\}$, and set

$$b_\ell = \sum_{p=1}^{b} (m_p - \ell)_+ \text{ for } \ell = 0, \ldots, m_j - 1.$$

Then $\deg_Z(P \times Q_j) = \{(\ell, b_\ell - 1) \mid \ell = 0, \ldots, m_j - 1\}$.

Proof. By Theorem 2.10, we have $|\deg_Z(P \times Q_j)| = m_j$. We first construct m_j separators $F_0, \ldots, F_{m_j - 1}$ of $P \times Q_j$ of multiplicity m_j where $\deg\,(F_\ell) = (\ell, b_\ell - 1)$ for $\ell = 0, \ldots, m_j - 1$. Our second step is to prove that these separators form a set of minimal separators.

To simplify our notation, let $R = L_P$ and $L_j = L_{Q_j}$ for $j = 1, \ldots, b$. Fix an $\ell \in \{0, \ldots, m_j - 1\}$, and let

$$A_\ell = R^\ell \quad \text{and} \quad B_\ell = L_1^{(m_1-\ell)_+} L_2^{(m_2-\ell)_+} \cdots L_j^{(m_j-\ell)_+ - 1} \cdots L_b^{(m_b-\ell)_+}.$$

We then set $F_\ell = A_\ell B_\ell$. By construction, $\deg\,(F_\ell) = (\ell, b_\ell - 1)$.

We now show that $F_\ell \in I_{Z'} \setminus I_Z$ where Z' denotes the set of fat points with the multiplicity of $P \times Q_j$ reduced by one. Note that

$F_\ell = R^\ell L_j^{(m_j-\ell)_+-1}F'_\ell$, and since $\ell + (m_j-\ell)_+ - 1 = m_j - 1$, we have $F_\ell \notin I_Z$ since $F_\ell \in I_{P\times Q_j}^{m_j-1} \setminus I_{P\times Q_j}^{m_j}$. This follows from the fact that F'_ℓ does not pass through $P \times Q_j$. Now let $P \times Q_f$ be any other point in Supp(Z) distinct from $P \times Q_j$. Since $I_{P\times Q_f} = (R, L_f)^{m_f}$, and since the exponents of R and L_f in F_ℓ sum up to at least m_f, we have that $F_\ell \in I_{P\times Q_f}^{m_f}$. So $F_\ell \in I_{Z'} \setminus I_Z$.

Let us now show that the F_ℓ are minimal separators. Let G be any separator of $P \times Q_j$ of multiplicity m_j with $\deg(G) = (c, d)$. We want to show that $\deg(G) \succeq (\ell, b_\ell - 1)$ for some $\ell \in \{0, \ldots, m_j - 1\}$. If we can verify this fact, then the explicit separators described above would form a minimal set of minimal separators.

Suppose that, for every ℓ, $(c, d) \not\succeq (\ell, b_\ell - 1)$. Since Z' is also a fat point scheme of points on a line, by Lemma 3.1, $\deg_k(I_{Z'})_{c,d} = 0$ ($b_\ell - 1$ appears as some c_ℓ because we have reduced the multiplicity of m_j by 1). So, $0 \neq G \in (I_{Z'})_{c,d} = (0)$, a contradiction. $\square$

Remark 3.3. By swapping the roles of the grading, we can prove a similar result for sets of points whose second coordinates are the same. We leave it to the reader to write out the corresponding statement of Theorem 3.2.

3.2. Separators of ACM fat points. The main result of this paper is a formula to compute the degree of a minimal separator for each fat point in an ACM fat point scheme in $\mathbf{P}^1 \times \mathbf{P}^1$.

Theorem 3.4. *Let $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ be an* ACM *set of fat points with $a = |\pi_1(\mathrm{Supp}(Z))|$ and $b = |\pi_2(\mathrm{Supp}(Z))|$. Suppose $P_i \times Q_j \in \mathrm{Supp}(Z)$ is a point with multiplicity m_{ij}. Set*

$$a_\ell = \sum_{s=1}^{a} (m_{sj} - \ell)_+$$

and

$$b_\ell = \sum_{p=1}^{b} (m_{ip} - \ell)_+ \textit{ for } \ell = 0, \ldots, m_{ij} - 1.$$

Then

$$\deg_Z(P_i \times Q_j) = \{(a_{m_{ij}-1-\ell} - 1, b_\ell - 1) \mid \ell = 0, \ldots, m_{ij} - 1\}.$$

Before proving this result, let us illustrate how to use it.

Example 3.5. We continue to use the example of Example 2.3. For convenience, we recall that

$$Z = \begin{array}{c|cccc} & Q_1 & Q_2 & Q_3 & Q_4 \\ \hline P_1 & 5 & 4 & 2 & 2 \\ P_2 & 5 & 3 & 2 & 1 \\ P_3 & 4 & 3 & 1 & \\ P_4 & 2 & 1 & & \\ P_5 & 1 & & & \end{array}$$

So $a = |\pi_1(\mathrm{Supp}\,(Z))| = 5$ and $b = |\pi_2(\mathrm{Supp}\,(Z))| = 4$. We will compute $\deg_Z(P_3 \times Q_2)$. The multiplicity of $P_3 \times Q_2$ is $m_{32} = 3$, so we will have $|\deg_Z(P_3 \times Q_2)| = 3$. In the notation of Theorem 3.4, we now calculate a_0, a_1, a_2 and b_0, b_1, b_2:

$$\begin{aligned} a_0 &= 4+3+3+1+0 & b_0 &= 4+3+1+0 \\ a_1 &= 3+2+2+0+0 & b_1 &= 3+2+0+0 \\ a_2 &= 2+1+1+0+0 & b_2 &= 2+1+0+0. \end{aligned}$$

Note that $m_{52} = 0$ since the point $P_5 \times Q_2$ is not in the support of Z. Similarly for m_{34}. We thus get

$$\begin{aligned} \deg_Z(P_3 \times Q_2) &= \{(a_0 - 1, b_2 - 1), (a_1 - 1, b_1 - 1), (a_2 - 1, b_0 - 1)\} \\ &= \{(10, 2), (6, 4), (3, 7)\}. \end{aligned}$$

Furthermore, as we will describe in the proof of Theorem 3.4, we can explicitly determine these minimal separators. If L_{P_i} denotes the degree $(1, 0)$ form that passes through P_i and L_{Q_j} denotes the degree $(0, 1)$ form that passes through Q_j, then the forms $F_1 = L_{P_1}^4 L_{P_2}^3 L_{P_3}^2 L_{P_4} L_{Q_1}^2$, $F_2 = L_{P_1}^3 L_{P_2}^2 L_{P_3} L_{Q_1}^3 L_{Q_2}$ and $F_3 = L_{P_1}^2 L_{P_2} L_{Q_1}^4 L_{Q_2}^2 L_{Q_3}$ are the minimal separators of $P_3 \times Q_2$ of multiplicity $m_{32} = 3$ with the required degrees.

Proof of Theorem 3.4. Fix a point $P_i \times Q_j$ of multiplicity m_{ij} in Z. There are two main steps. First, we construct m_{ij} separators $F_0, \ldots, F_{m_{ij}-1}$ of $P_i \times Q_j$ of multiplicity m_{ij} where $\deg\,(F_\ell) =$

$(a_{m_{ij}-\ell-1}-1, b_\ell-1)$ for $\ell = 0,\dots,m_{ij}-1$. Second, we prove that these separators form a set of minimal separators of $P_i \times Q_j$ of multiplicity m_{ij}.

To simplify our notation slightly, let L_i denote the degree $(1,0)$ form that passes through P_i for $i = 1,\dots,a$, and let R_j denote the degree $(0,1)$ form that passes through Q_j for $j = 1,\dots,b$. Fix an $\ell \in \{0,\dots,m_{ij}-1\}$, and let

$$A_\ell = L_1^{(m_{1j}-(m_{ij}-\ell-1))_+} L_2^{(m_{2j}-(m_{ij}-\ell-1))_+} \cdots L_i^{(m_{ij}-(m_{ij}-\ell-1))_+-1} \cdots L_a^{(m_{aj}-(m_{ij}-\ell-1))_+}$$

and

$$B_\ell = R_1^{(m_{i1}-\ell)_+} R_2^{(m_{i2}-\ell)_+} \cdots R_j^{(m_{ij}-\ell)_+-1} \cdots R_b^{(m_{ib}-\ell)_+}.$$

We then set $F = F_\ell = A_\ell B_\ell$. By construction, $\deg(F) = (a_{m_{ij}-\ell-1}-1, b_\ell-1)$.

We now need to show that $F \in I_{Z'} \setminus I_Z$ where Z' denotes the set of fat points with the multiplicity of $P_i \times Q_j$ reduced by one. Note that $F = L_i^{(m_{ij}-(m_{ij}-\ell-1))_+-1} R_j^{(m_{ij}-\ell)_+-1} F'$, and since

$$[(m_{ij}-(m_{ij}-\ell-1))_+ - 1 + (m_{ij}-\ell)_+ - 1 = m_{ij}-1,$$

we have $F \notin I_Z$ since $F \in I_{P_i\times Q_j}^{m_{ij}-1} \setminus I_{P_i\times Q_j}^{m_{ij}}$. This is because F' does not pass through $P_i \times Q_j$.

Now take any other point $P_e \times Q_f$ in the support of Z distinct from $P_i \times Q_j$. We need to show that $F \in I_{P_e\times Q_f}^{m_{ef}}$. Since $I_{P_e\times Q_f} = (L_e, R_f)^{m_{ef}} = (L_e^u R_f^v \mid u+v = m_{ef})$, it will suffice to show that the exponents of L_e and R_f in F sum up to at least m_{ef}. We break this problem into a number of cases and we use the relation

$$(\star) \qquad x_+ + y_+ \geq (x+y)_+ \geq x+y.$$

Recall that, since we are assuming that Z is ACM, we can assume that $m_{uv} \geq m_{cv}$ if $u < c$ and $m_{uv} \geq m_{ud}$ if $v < d$.

Case 1. $e < i$ and $f < j$. We have $F = L_e^{(m_{ej}-(m_{ij}-\ell-1))_+} R_f^{(m_{if}-\ell)_+} F'$. By $(\star)$ and Lemma 2.5, we have

$$(m_{ej}-(m_{ij}-\ell-1))_+ + (m_{if}-\ell)_+ \geq m_{ej}+m_{if}-m_{ij}+1 \geq m_{ef}.$$

Case 2. $e < i$ and $f = j$. We observe that $F = L_e^{(m_{ej}-(m_{ij}-\ell-1))_+} R_j^{(m_{ij}-\ell)_+-1}F'$. So, we have

$$(m_{ej}-(m_{ij}-\ell-1))_+ + (m_{ij}-\ell)_+ - 1 = m_{ej}-m_{ij}+\ell+1+m_{ij}-\ell-1 = m_{ej}$$

since $m_{ej} \geq m_{ij}$ and $m_{ij} - \ell \geq 1$.

Case 3. $e < i$ and $j < f \leq b$. In this case, $F = L_e^{(m_{ej}-(m_{ij}-\ell-1))_+} R_f^{(m_{if}-\ell)_+}F'$. Using $(\star)$ gives us

$$(m_{ej} - (m_{ij} - \ell - 1))_+ + (m_{if} - \ell)_+ \geq m_{ej} + m_{if} - m_{ij} + 1.$$

The point $P_i \times Q_f$ may or may not be in the support of Z. If it is, then $m_{ej} + m_{if} - m_{ij} + 1 \geq m_{ef}$ by Lemma 2.5 (a). If the point is not in the support, the conclusion follows by Lemma 2.5 (b).

Case 4. $e = i$. If $e = i$, then $f \in \{1, \ldots, \widehat{j}, \ldots, b\}$, and furthermore, $F = L_i^{(m_{ij}-(m_{ij}-\ell-1))_+-1} R_f^{(m_{if}-\ell)_+}F'$. We have $(m_{ij}-(m_{ij}-\ell-1))_+ - 1 + (m_{if} - \ell)_+ \geq m_{if}$ by $(\star)$.

Case 5. $i < e \leq a$ and $f < j$. This case is similar to Case 3.

Case 6. $i < e \leq a$ and $f = j$. In this situation, $F = L_e^{(m_{ej}-(m_{ij}-\ell-1))_+} R_j^{(m_{ij}-\ell)_+-1}F'$. Then using $(\star)$ gives $(m_{ej} - (m_{ij} - \ell - 1))_+ + (m_{ij} - \ell)_+ - 1 \geq m_{ej}$.

Case 7. $i < e \leq a$ and $j < f \leq b$. We have $F = L_e^{(m_{ej}-(m_{ij}-\ell-1))_+} R_f^{(m_{if}-\ell)_+}F'$. So $(m_{ej}-(m_{ij}-\ell-1))_+ + (m_{if}-\ell)_+ \geq m_{ej} - m_{ij} + m_{if} + 1 \geq m_{ef}$.

These seven cases show that each F_ℓ with $\ell \in \{0, \ldots, m_{ij} - 1\}$ is a separator of $P_i \times Q_j$ of multiplicity m_{ij}. We now demonstrate that these are the minimal separators.

Let F be any separator of $P_i \times Q_j$ of multiplicity m_{ij} with $\deg(F) = (c, d)$. To simplify our notation, set $m = m_{ij}$. We want to show that $\deg(F) \succeq (a_{m-1-\ell} - 1, b_\ell - 1)$ for some $\ell \in \{0, \ldots, m - 1\}$. If we can verify this fact, then the explicit separators described above would form a set of minimal separators of $P_i \times Q_j$ of multiplicity m.

For any $P_i \times Q_j \in \operatorname{Supp}(Z)$, let

$$Y = m_{i1}(P_i \times Q_1) + \cdots + m_{ij}(P_i \times Q_j) + \cdots + m_{ib}(P_i \times Q_b)$$

be all the fat points of Z whose support has P_i as its first coordinate, and let

$$W = m_{1j}(P_1 \times Q_j) + \cdots + m_{ij}(P_i \times Q_j) + \cdots + m_{aj}(P_a \times Q_j)$$

be all the fat points of Z whose support has Q_j as its second coordinate.

So, suppose that, for every ℓ, $(c,d) \not\succeq (a_{m-1-\ell} - 1, b_\ell - 1)$. Note, however, that F is a separator of $P_i \times Q_j$ of multiplicity of m of W. So, there exists a t such that $(c,d) \succeq (a_{m-1-t} - 1, m-1-t)$ by Remark 3.3 and Theorem 3.2. However, since $(c,d) \not\succeq (a_{m-1-t} - 1, b_t - 1)$, this implies that $d < b_t - 1$. On the other hand, by Theorem 3.2, there exists a k such that $(c,d) \succeq (k, b_k - 1)$ since F is also a separator of $P_i \times Q_j$ of multiplicity of m of Y. We thus have that $b_k - 1 \leq d < b_t - 1$, whence $t < k$.

Since $b_k < b_{k-1} < \cdots < b_t$, there exists a p such that $b_{p+1} - 1 \leq d < b_p - 1$ with $t \leq p < k$. Now $(c,d) \succeq (k, b_k - 1)$, and $k \geq p+1$ and $d \geq b_{p+1} - 1$, so we also have $(c,d) \succeq (p+1, b_{p+1} - 1)$. However, because $(c,d) \not\succeq (a_{m-1-(p+1)} - 1, b_{p+1} - 1)$, we have $c < a_{m-p-2} - 1$.

Consider the line L_i, i.e., the degree $(1,0)$ form that contains Y. Then the curves defined by F and L_i meet at $m_{i1} + m_{i2} + \cdots + (m_{ij} - 1) + \cdots + m_{ib} = b_0 - 1$ points. By Bezout's theorem (see Theorem 2.6), since L_i is irreducible and $c \cdot 0 + d \cdot 1 = d < b_p - 1 < b_0 - 1$, we have that $F = F_0 L_i$. But now consider F_0. This is a form of degree $(c-1, d)$, and the curve it defines meets L_i at

$$(m_{i1} - 1)_+ + (m_{i2} - 1)_+ + \cdots + (m_{ij} - 1 - 1)_+ + \cdots + (m_{ib} - 1)_+ \geq b_1 - 1$$

points (the inequality comes from the fact that $(m_{ij} - 1 - 1)_+ \geq [(m_{ij} - 1)_+ - 1]$). By Bezout's theorem $F_0 = F_1 L_i$ because $(c-1) \cdot 0 + d \cdot 1 = d < b_p - 1 < b_1 - 1$. We can continue this argument until we arrive at $F = F_p L_i^{p+1}$, where $\deg(F_p) = (c - p - 1, d)$.

The form F_p and the degree $(0,1)$ form R_j which contains W meet at

$$m_{1j} + m_{2j} + \cdots + (m_{ij} - p - 2)_+ + \cdots + m_{aj} \geq a_0 - p - 2$$

points, counting multiplicities. To see this, note that L_i^{p+1} already passes through the point $P_i \times Q_j$ $(p+1)$ times. Since F passes through $P_i \times Q_j$ with multiplicity $m-1$, F_p must pass through $P_i \times Q_j$ $(m-p-2)_+$ times. The inequality comes from the fact that $(m_{ij}-p-2)_+ \geq m_{ij}-p-2$.

Now, since $c < a_{m-p-2}-1$, we have $c-p-1 < a_{m-p-2}-p-2 < \cdots < a_0-p-2$. So, by Bezout's theorem (Theorem 2.6), we get $F_p = G_0R_j$. But then G_0 has degree $(c-p-1, d-1)$, and we can repeat the above argument to show that $G_0 = G_1R_j$. Continuing in this fashion, we arrive at $F_p = G_{m-p-2}R_j^{m-p-1}$.

We therefore have $F = L_i^{p+1}R_j^{m-p-1}G_{m-p-2}$ where G_{m-p-2} has degree $(c-p-1, d-m+p+1)$. The exponents of L_i and R_j sum to m, which means that $F \in (L_i, R_j)^m$, which contradicts the fact that a separator of $P_i \times Q_j$ of multiplicity $m = m_{ij}$ belongs to $(L_i, R_j)^{m-1} \setminus (L_i, R_j)^m$. Thus, there cannot be a separator F with degree $(c, d) \not\succeq (a_{m-1-\ell}-1, b_\ell-1)$ for all ℓ. $\square$

Application 3.6. We sketch out how one might use Theorem 3.4 to compute some Hilbert functions. When Z is an ACM fat point scheme in $\mathbf{P}^1 \times \mathbf{P}^1$, then $H_Z(i,j)$ can be computed for all (i,j) directly from the set $\mathcal{S}_Z$ introduced in Section 2 (see [**5**] for complete details). If we pick any fat point $P_i \times Q_j$ in Z of multiplicity m_{ij}, then by Theorem 3.4, we can compute

$$\deg_Z(P_i \times Q_j) = ((c_1, d_1), \ldots, (c_{m_{ij}}, d_{m_{ij}})).$$

Let Z' be the scheme formed by reducing the multiplicity of $P_i \times Q_j$ by one. As shown in [**10**, Corollary 4.4], we can compute the Hilbert function of Z' as follows:

$$H_{Z'}(r,s) = H_Z(r,s) - |\{(c,d) \in \deg(P_i \times Q_j) | (c,d) \preceq (r,s)\}|.$$

In other words, if Z' is any fat point scheme (possibly not ACM) which has the property that if we increase the multiplicity of one of its points by one to get an ACM scheme, then the Hilbert function of Z' can be computed directly from numerical information describing Z'.

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