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SPLITTING ALGEBRAS AND GYSIN HOMOMORPHISMS

DAN LAKSOV

ABSTRACT. We give three algebraic constructions of splitting algebras of monic polynomials with coefficients in arbitrary commutative rings with a unit, and of the corresponding Gysin homomorphisms.

1. Introduction. We give three algebraic constructions of splitting algebras of monic polynomials with coefficients in arbitrary commutative rings with a unit, and of the corresponding Gysin homomorphisms. The first construction of splitting algebras is by induction, and is well known (see, e.g., [**2, 5, 12, 13, 15–17, 21, 22**]), the second is the most natural, via symmetric polynomials, and is a variant of the method used by Bourbaki [**2**], and the third, using the division algorithm over polynomial rings, was indicated to us by A. Thorup (University of Copenhagen), and has not appeared in the literature. We believe that all of the constructions of Gysin homomorphisms are new. Many of the ideas and methods used in our constructions evolved during our cooperation with Throup. We are also thankful to Bengt Ek (The Royal Institute of Technology, Stockholm) and Michael Shapiro (Michigan State University, East Lansing) for the elegant proofs of some auxiliary results. For other treatments of Gysin homomorphisms see [**1, 3, 4, 6–8, 10, 12, 13, 15–20**].

The importance of having several constructions of the same object and of the homomorphisms between them is that we in this way shed light on the area, but also that we by comparing the constructions can obtain interesting polynomial identities. We have not performed these comparisons here, but leave them to the reader.

Splitting algebras and Gysin homomorphisms appear in many different parts of mathematics. Best known is their usefulness to the cohomology theory of flag schemes, and in particular in Schubert calculus

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(see, e.g., [**1, 3, 4, 6– 16, 19, 20**]). Splitting algebras also provide a convenient tool for the study of symmetric polynomials (see, e.g., [**2, 5, 13, 19, 22**]). In [**5**] we indicated how splitting algebras can be used to develop Galois theory in the classical spirit. Splitting algebras can also be used to generate error correcting codes ([**12, 18**]), and appear as coordinate rings of certain Hilbert schemes of points [**9**].

To indicate the connection between splitting algebras and the cohomology of flag schemes we let X be a scheme that has a bivariant intersection ring $A(X)$, and we let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Denote by $F = \mathrm{Flag}^d_X(\mathcal{E})$ the flag scheme whose S-points are partial flags $\mathcal{E}_S \to \mathcal{F}_d \to \cdots \to \mathcal{F}_0$, where $\mathcal{F}_i$ is a locally free $\mathcal{O}_S$-module of rank i. Then the bivariant intersection ring $A(F)$ of F is the d'th splitting algebra over $A(X)$ of the Chern polynomial

$$c_T(\mathcal{E}) = T^n - c_1(\mathcal{E})T^{n-1} + \cdots + (-1)^n c_n(\mathcal{E})$$

of $\mathcal{E}$. Denote by $G = \mathrm{Grass}^d_X(\mathcal{E})$ the Grassmannian whose S-points are the surjections $\mathcal{E}_S \to \mathcal{F}$ to a locally free $\mathcal{O}_S$-module of rank d. Mapping $\mathcal{E}_S \to \mathcal{F}_d \to \cdots \to \mathcal{F}_0$ to $\mathcal{E}_S \to \mathcal{F}_d$ gives a morphism $F \to G$ and a corresponding Gysin homomorphism

$$\partial : A(F) \longrightarrow A(G).$$

Denote by ξ_i the Chern class of the kernel of $\mathcal{F}_i \to \mathcal{F}_{i-1}$ for $i = 1, \ldots, d$. Then, for all non-negative integers $h_1, \ldots, h_d$, we have

$$\partial(\xi_1^{h_1} \cdots \xi_d^{h_d}) = \det\left((s_{h_i-n+j}(\mathcal{Q}))_{ij}\right),$$

where the right hand side is the $d \times d$-matrix whose (i, j)'th coordinate is the $(h_i - n + j)$'th Segre class of the universal $\mathcal{O}_S$-module $\mathcal{Q}$ of G.

In general, for any commutative ring A with unit, and for every monic polynomial

$$p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$$

with coefficients in A, we have a splitting algebra $\mathrm{Split}^d_A(p) = A[\xi_1, \ldots, \xi_d]$ splitting $p(T)$ in d ordered linear factors

$$p(T) = (T - \xi_1) \cdots (T - \xi_d)q(T)$$

and a Gysin homomorphism

$$\partial^d(p) : \mathrm{Split}^d_A(p) \longrightarrow A$$

such that

$$\partial^d(p)(\xi_1^{h_1}\cdots\xi_d^{h_d}) = \operatorname{Res}\left(\frac{T^{h_1}}{p},\cdots,\frac{T^{h_d}}{p}\right),$$

where Res is the residue introduced in [**15, 16**]. We note that general splitting algebras are not only more general than those appearing in intersection theory in the sense that we have a theory over any ring A, and not only over intersection rings $A(X)$, but also that we split any monic polynomial $p(T)$, and not only Chern polynomials $c_T(\mathcal{E})$. The difference is most clearly visible when X is the spectrum of a field, when we always have that $c_T(\mathcal{E}) = T^n$.

1. Definition of splitting algebras. We first give a somewhat exotic definition of splitting algebras as algebras representing functors. Then we give the more usual definition via their universal properties.

1.1 Definition. Let A be a *commutative ring* with *unit*, and let

$$p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$$

be a polynomial in the variable T with coefficients in A. For $d = 1, 2, \ldots, n$ we have a *covariant* functor Split_p^d from A-*algebras* to *sets* that maps a homomorphism $\varphi : A \to B$ to

$$\begin{aligned}\operatorname{Split}_p^d(B) = \{&\text{splittings } (\varphi p)(T) = T^n - \varphi(c_1)T^{n-1} + \cdots + (-1)^n\varphi(c_n)\\ &= (T-b_1)\cdots(T-b_d)s(T) \text{ over } B \text{ where}\\ &\qquad b_1,\ldots,b_d \text{ is an ordered sequence of roots}\},\end{aligned}$$

and where the map

$$\operatorname{Split}_p^d(\psi) : \operatorname{Split}_p^d(B) \longrightarrow \operatorname{Split}_p^d(C)$$

corresponding to an A-algebra *homomorphism* $\psi : B \to C$ is defined by

$$\operatorname{Split}_p^d(\psi)((\varphi p)(T)) = (T - \psi(b_1))\cdots(T-\psi(b_n))(\psi s)(T).$$

An A-algebra $\operatorname{Split}_A^d(p)$ that *represents* the functor Split_p^d we call a d'th *splitting algebra* for $p(T)$ over A and if

$$p(T) = (T-\xi_1)\cdots(T-\xi_d)p_{d+1}(T)$$

is the *universal splitting* over $\mathrm{Split}_A^d(p)$ we call the *ordered set* of *roots* $\xi_1, \dots, \xi_d$, the *universal roots*.

It is often more convenient to define splitting algebras in the following more *concrete* way.

1.2 Definition. For $d = 1, \dots, n$, a d'th *splitting algebra* for the polynomial

$$p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$$

in the variable T with coefficients in A is an A-algebra $\mathrm{Split}_A^d(p)$, over which $p(T)$, considered as a polynomial over $\mathrm{Split}_A^d(p)$, *splits* as

$$p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T),$$

and where $\mathrm{Split}_A^d(p)$ has the following *universal property*:

For every A-algebra $\varphi : A \to B$ over which we have a splitting

$$\begin{aligned}(\varphi p)(T) &= T^n - \varphi(c_1) T^{n-1} + \cdots + (-1)^n \varphi(c_n) \\ &= (T - b_1) \cdots (T - b_d) s(T),\end{aligned}$$

where $b_1, \dots, b_n$ is an ordered sequence of roots in B, there is a *unique* A-algebra homomorphism

$$\psi : \mathrm{Split}_A^d(p) \longrightarrow B$$

that satisfies $\psi(\xi_i) = b_i$ for $i = 1, \dots, d$.

We let $\mathrm{Split}_A^0(p) = A$ and $p(T) = p_1(T)$. The ordered set of roots $\xi_1, \dots, \xi_d$ of $p(T)$ in $\mathrm{Split}_A^d(p)$ we call the *universal roots*, and the splitting $p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T)$ we call the *universal splitting*.

1.3 *Remark.* It is clear from the Definitions 1.1 or 1.2 that the d'th splitting algebra is uniquely determined up to an A-algebra isomorphism.

2. The first construction of splitting algebras. The main properties of splitting algebras, including their existence, are contained in the following results.

2.1 Proposition. *Let* $\mathrm{Split}_A^d(p)$ *be a d'th splitting algebra for the polynomial* $p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$ *with coefficients in* A, *and with universal splitting*

$$p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T).$$

(1) *The* A*-algebra* $\mathrm{Split}_A^d(p)$ *is generated by* $\xi_1, \dots, \xi_d$. *That is,* $\mathrm{Split}_A^d(p) = A[\xi_1, \dots, \xi_d]$.

(2) *The* $\mathrm{Split}_A^d(p)$*-algebra* $\mathrm{Split}_A^d(p)[T]/(p_{d+1})$ *is a* $(d+1)$*'th splitting algebra for* $p(T)$ *over* A *with universal roots* $\xi_1, \dots, \xi_d, \xi_{d+1}$ *where* ξ_{d+1} *is the class of* T.

Proof. (1) It is clear that the A-algebra $A[\xi_1, \dots, \xi_d]$ has the universal property of a splitting algebra described in Definition 1.2. Since the universal property characterizes the d'th splitting algebra up to isomorphisms it follows that the homomorphism defined by the inclusion of $A[\xi_1, \dots, \xi_d]$ in $\mathrm{Split}_A^d(p)$ is an isomorphism. That is, we have $\mathrm{Split}_A^d(p) = A[\xi_1, \dots, \xi_d]$ as asserted.

(2) Let $\varphi : A \to B$ be an A-algebra over which we have a splitting

$$(\varphi p(T)) = (T - b_1) \cdots (T - b_{d+1}) s(T)$$

over B. We then have a unique A-algebra homomorphism $\chi : \mathrm{Split}_A^d(p) \to B$ such that $\chi(\xi_i) = b_i$ for $i = 1, \dots, d$. Since multiplication by monic polynomials is injective in $B[T]$ it follows from the equality

$$\begin{aligned}(\chi p)(T) &= (T - \chi(\xi_1)) \cdots (T - \chi(\xi_d))(\chi p_{d+1})(T) \\ &= (T - b_1) \cdots (T - b_d)(T - b_{d+1}) s(T)\end{aligned}$$

that $(\chi p_{d+1})(T) = (T - b_{d+1}) s(T)$ in the polynomial ring $B[T]$.

Let ξ_{d+1} be the class of T in $\mathrm{Split}_A^d(p)[T]/(p_{d+1})$. Since $(\chi p_{d+1})(T)$ has the root b_{d+1} in B it follows from the universal property of residue algebras that we can extend χ uniquely to a $\mathrm{Split}_A^d(p)$-algebra homomorphism $\psi : \mathrm{Split}_A^d(p)[\xi_{d+1}] \to B$ such that $\psi(\xi_{d+1}) = b_{d+1}$. Thus we have an A-algebra homomorphism

$$\psi : A[\xi_1, \dots, \xi_{d+1}] = A[\xi_1, \dots, \xi_d][\xi_{d+1}] \longrightarrow B$$

that takes the values $\psi(\xi_i) = b_i$ for $i = 1, \dots, d+1$. By the first part of the proposition ψ is uniquely determined by these values. Thus $A[\xi_1, \dots, \xi_{d+1}]$ is a $(d+1)$'st splitting algebra for $p(T)$ over A.

2.2 Corollary. *With the notation of the proposition we have*

(1) *For each $d = 0, 1, \dots, n$ there exists a d'th splitting algebra for $p(T)$ over A.*

(2) *A splitting algebra with universal roots $\xi_1, \dots, \xi_d$ is free as an A-module with a basis $\xi_1^{h_1} \cdots \xi_d^{h_d}$ for $0 \le h_j \le n - j$ and $j = 1, \dots, d$.*

Proof. (1) The existence of the d'th splitting algebra follows from assertion (2) of the proposition by induction on d, beginning at $\mathrm{Split}_A^0(p) = A$.

(2) This assertion also follows from assertion (2) of the proposition by induction on d since $\mathrm{Split}_A^0(p) = A$, and since $\mathrm{Split}_A^{d+1}(p) = \mathrm{Split}_A^d(p)[\xi_{d+1}]$ is a free $\mathrm{Split}_A^d(p)$-module with basis $1, \xi_{d+1}, \dots, \xi_{d+1}^{n-d-1}$.

3. The second construction of splitting algebras. We give a proof of the existence of splitting algebras depending on the algebraic independence of elementary symmetric polynomials. The proof is in the spirit of the proof for $n = d$ in [**2**].

3.1 Notation. Let $T_1, \dots, T_n$ be *independent variables* over the polynomial ring $A[T]$. For $i = 1, \dots, n$ denote by $C_i = c_i(T_1, \dots, T_n)$ the i'th *elementary symmetric polynomial* in $T_1, \dots, T_n$, and for $i = 1, \dots, n-d$ we denote by $D_i = c_i(T_{d+1}, \dots, T_n)$ the i'th *elementary symmetric polynomial* in the variables $T_{d+1}, \dots, T_n$.

Over the ring $A[T_1, \dots, T_n]$ we have a *splitting*

$$
\begin{aligned}
(3.1.1)\quad T^n - C_1 T^{n-1} + \cdots + (-1)^n C_n &= (T - T_1) \cdots (T - T_n) \\
&= (T - T_1) \cdots (T - T_d)(T^{n-d} - D_1 T^{n-d-1} + \cdots \\
&\quad + (-1)^{n-d} D_{n-d}).
\end{aligned}
$$

In particular, we have that the A-algebra $A[C_1, \dots, C_n, T_1, \dots, T_d]$ in $A[T_1, \dots, T_n]$ is equal to the A-algebra $A[D_1, \dots, D_{n-d}, T_1, \dots, T_d]$.

3.2 Proposition. *The residue algebra*

$$A[C_1, \dots, C_n, T_1, \dots, T_d]/(c_1 - C_1, \dots, c_n - C_n)$$

is a d'th splitting algebra for $p(T)$ over A. If $\xi_1, \dots, \xi_d$ denote the classes of $T_1, \dots, T_d$ and $q_1, \dots, q_{n-d}$ denote the classes of $D_1, \dots$, D_{n-d}, the universal splitting is

$$p(T) = (T - \xi_1) \cdots (T - \xi_d)(T^{n-d} - q_1 T^{n-d-1} + \cdots + (-1)^{n-d} q_{n-d}). \tag{3.2.1}$$

Proof. The existence of the splitting (3.2.1) of $p(T)$ follows from the equality (3.1.1), since c_i is the class of C_i for $i = 1, \dots, n$.

Let $\varphi : A \to B$ be an A-algebra over which there is a splitting

$$(\varphi p)(T) = (T - b_1) \cdots (T - b_d)(T^{n-d} - d_1 T^{n-d-1} + \cdots + (-1)^{n-d} d_{n-d}). \tag{3.2.2}$$

Since $T_1, \dots, T_d, D_1, \dots, D_{n-d}$ are algebraically independent over A we can define an A-algebra homomorphism

$$\chi : A[C_1, \dots, C_n, T_1, \dots, T_d] = A[D_1, \dots, D_{n-d}, T_1, \dots, T_d] \longrightarrow B$$

such that $\chi(T_i) = b_i$ for $i = 1, \dots, d$ and $\chi(D_i) = d_i$ for $i = 1, \dots, n-d$. It follows from (3.1.1) that

$$\begin{aligned} &T^n - \chi(C_1) T^{n-1} + \cdots + (-1)^n \chi(C_n) \\ &\quad = (T - b_1) \cdots (T - b_d)(T^{n-d} - d_1 T^{n-d-1} + \cdots + (-1)^{n-d} d_{n-d}). \end{aligned}$$

In particular it follows from (3.2.2) that we must have that $\chi(C_i) = \varphi(c_i)$ for $i = 1, \dots, n$. Consequently the homomorphism χ induces an A-algebra homomorphism

$$\psi : A[C_1, \dots, C_n, T_1, \dots, T_d]/(c_1 - C_1, \dots, c_n - C_n) \longrightarrow B$$

such that the condition $\psi(\xi_i) = b_i$ is fulfilled for $i = 1, \dots, d$. Since $\psi(C_i) = \varphi(c_i)$ it follows that ψ is uniquely determined by these conditions. Hence the residue algebra of the proposition is a d'th splitting algebra for $p(T)$ over A. □

4. The third construction of splitting algebras. In this section we give an elementary and natural construction of splitting algebras based upon the division algorithm over polynomial rings.

4.1 Notation. Let $T_1, \dots, T_d$ be *algebraically independent variables* over the polynomial ring $A[T]$. The *division algorithm* used over $A[T_1, \dots, T_d]$ to the polynomial $p(T)$ *modulo* the polynomial $P(T) = (T - T_1) \cdots (T - T_d)$ gives

$$\begin{aligned} p(T) = (T - T_1) \cdots (T - T_d)(T^{n-d} - q_1 T^{n-d-1} + \cdots \\ + (-1)^{n-d} q_{n-d}) \\ + r_{d-1} T^{d-1} + r_{d-2} T^{d-2} + \cdots + r_0. \end{aligned} \tag{4.1.1}$$

4.2 Proposition. *The residue algebra*

$$A[T_1, \dots, T_d]/(r_0, \dots, r_{d-1})$$

is a d'th splitting algebra for $p(T)$ over A. The classes $\xi_1, \dots, \xi_d$ of $T_1, \dots, T_d$ are the universal roots and the universal splitting is

$$p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T) \tag{4.2.1}$$

where the coefficient of T^i in $p_{d+1}(T)$ is the class of $(-1)^{n-d-i} q_{n-d-i}$ for $i = 0, \dots, n - d - 1$.

Proof. The existence of the splitting (4.2.1) follows from (4.1.1).

Let $\varphi : A \to B$ be an A-algebra over which we have a splitting

$$(\varphi p)(T) = (T - b_1) \cdots (T - b_d)(T^{n-d} - d_1 T^{n-d-1} + \cdots + (-1)^{n-d} d_{n-d}). \tag{4.2.2}$$

We define a homomorphism of A-algebras

$$\chi : A[T_1, \dots, T_d] \longrightarrow B$$

by $\chi(T_i) = b_i$ for $i = 1, \dots, d$. It follows from (4.1.1) that

$$\begin{aligned} (\chi p)(T) = (T - b_1) \cdots (T - b_d)(T^{n-d} - \chi(q_1) T^{n-d-1} + \cdots \\ + (-1)^{n-d} \chi(q_{n-d})) \\ + \chi(r_{d-1}) T^{d-1} + \cdots + \chi(r_0) \end{aligned}$$

over B. Hence it follows from (4.2.2) that we have equalities

$$0=\chi(r_{d-1})=\cdots=\chi(r_0)\quad\text{and}\quad\chi(q_i)=d_i\quad\text{for } i=1,\ldots,n-d,$$

in B. Thus χ factors via a unique A-algebra homomorphism

$$\psi : A[T_1,\ldots,T_d]/(r_0,\ldots,r_{d-1})\longrightarrow B$$

such that $\psi(\xi_i)=b_i$ for $i=1,\ldots,d$, and

$$(\varphi p)(T)=(T-b_1)\cdots(T-b_d)(\psi p_{d+1})(T)$$

over B. We have thus that the residue algebra of the proposition is a d'th splitting algebra for $p(T)$ over A. □

5. Residues. We shall in this section introduce *residues*. Residues are among the principal tools for studying splitting algebras.

5.1 Notation. Let

$$p(T)=T^n-c_1T^{n-1}+\cdots+(-1)^nc_n$$

be a polynomial in the variable T with coefficients in the ring A. We define elements $s_1,s_2,\ldots$ in the ring A by the relation

$$1=(1-c_1T+\cdots+(-1)^nc_nT^n)(1+s_1T+s_2T^2+\cdots)$$

in the algebra of formal power series in T over A, and we let $s_0=1$, and $0=s_{-1}=s_{-2}=\cdots$.

5.2 Definition. Let $g_i=\cdots+a_{i\,-1}T+a_{i0}+(a_{i1}/T)+(a_{i2}/T^2)+\cdots$ for $i=1,\ldots,d$ be *formal Laurent series* in the variable $1/T$. We let

$$\operatorname{Res}(g_1,\ldots,g_d)=\det\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1d}\\ \vdots&\vdots&\ddots&\vdots\\ a_{d1}&a_{d2}&\cdots&a_{dd}\end{pmatrix}.$$

The main properties of residues are summarized in the following results.

5.3 Proposition. *We have that* $Res(g_1, \dots, g_d)$ *is multilinear and alternating in* $g_1, \dots, g_d$*, and it is zero if at least one of the* g_i *is a polynomial in* T*.*

Proof. All the assertions follow immediately from Definition 5.2. □

5.4 Proposition. *For all natural numbers* $h_1, \dots, h_d$ *we have*
(5.4.1)

$$\operatorname{Res}\left(\frac{T^{h_1}}{p}, \cdots, \frac{T^{h_d}}{p}\right) = \det\begin{pmatrix} s_{h_1-n+1} & s_{h_1-n+2} & \cdots & s_{h_1-n+d} \\ \vdots & \vdots & \ddots & \vdots \\ s_{h_d-n+1} & s_{h_d-n+2} & \cdots & s_{h_d-n+d} \end{pmatrix}.$$

In particular, when $0 \le h_j \le n-j$ *for* $j = 1, \dots, d$ *we have*

$$Res\left(\frac{T^{h_1}}{p}, \cdots, \frac{T^{h_d}}{p}\right) = \begin{cases} 1 & \text{when } h_j = n-j \text{ for } j = 1, \dots, d \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first part of the proposition follows from Definition 5.2 and the equality

$$\frac{T^h}{p} = T^{h-n}\left(1 - \frac{c_1}{T} + \cdots + (-1)^d \frac{c_d}{T^d}\right) = T^{h-n}\left(1 + \frac{s_1}{T} + \frac{s_2}{T^2} + \cdots\right)$$

of formal Laurent series in $1/T$.

The second part follows from the first since, when $0 \le h_j \le n-j$ for $j = 1, \dots, d$, the $d \times d$-matrix (s_{h_i-n+j}) is upper triangular. When at least one of the inequalities $h_j \le n-j$ is strict there is a zero on the diagonal. Otherwise all the diagonal elements are one. □

6. Two auxiliary results. To prove the result on Gysin homomorphisms we need the following auxiliary results on matrices. We are thankful to Bengt Ek and Michael Shapiro for the elegant proofs of the first parts of the results.

6.1 Lemma. *Let* (a_{ij}) *and* (b_{ij}) *be two* $d \times d$*-matrices with coordinates in the ring* A*.*

(1) *For $j = 1, \dots, d$ we have*

$$\sum \det \begin{pmatrix} a_{11} & \cdots & b_{1i_1} & \cdots & b_{1i_j} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{d1} & \cdots & b_{di_1} & \cdots & b_{di_j} & \cdots & a_{dd} \end{pmatrix} = \sum \det \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ & \cdots & \\ b_{i_1 1} & \cdots & b_{i_1 d} \\ & \cdots & \\ b_{i_j 1} & \cdots & b_{i_j d} \\ & \cdots & \\ a_{d1} & \cdots & a_{dd} \end{pmatrix} \tag{6.1.1}$$

where both sums are over all indices $1 \leq i_1 < \cdots < i_j \leq d$.

(2) *When*

$$\begin{pmatrix} b_{11} & \cdots & b_{1d} \\ \vdots & \ddots & \vdots \\ b_{d1} & \cdots & b_{dd} \end{pmatrix} = \begin{pmatrix} a_{1\,2} & \cdots & a_{1\,d+1} \\ \vdots & \ddots & \vdots \\ a_{d\,2} & \cdots & a_{d\,d+1} \end{pmatrix} \tag{6.1.2}$$

for some elements $a_{1d+1}, \dots, a_{dd+1}$ of A we obtain for $j = 0, 1, \dots, d$

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1d-j} & a_{1d-j+2} & \cdots & a_{1d+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd-j} & a_{dd-j+2} & \cdots & a_{dd+1} \end{pmatrix} = \sum_{1 \leq i_1 < \cdots < i_j \leq d} \det \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ & \cdots & \\ a_{i_1 2} & \cdots & a_{i_1 d+1} \\ & \cdots & \\ a_{i_j 2} & \cdots & a_{i_j d+1} \\ & \cdots & \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}. \tag{6.1.3}$$

(3) *When*

$$a_{i\,d+1} - c_1 a_{i\,d} + \cdots + (-1)^d c_d a_{i\,1} = 0 \quad \textit{for} \quad i = 1, \dots, d \tag{6.1.4}$$

we have for $j = 1, \dots, d$ the equality

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1d-j} & a_{1d-j+2} & \cdots & a_{1d+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd-j} & a_{dd-j+2} & \cdots & a_{dd+1} \end{pmatrix} = c_j \det \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix} \tag{6.1.5}$$

Proof. We obtain the left and right sides of (6.1.1) by expanding the determinant of the $d \times d$-matrix $(a_{ij}) + T(b_{ij}) = (a_{ij} + Tb_{ij})$ along "columns," respectively "rows," and comparing the coefficient of T^j.

When the equality (6.1.2) holds the equality (6.1.3) follows from (6.1.1) because, on the left hand side of (6.1.1), the matrix of which we take the determinant has columns i_k and i_k+1 equal when $i_k+1 < i_{k+1}$, and columns i_j and $i_j + 1$ are equal if $i_j < d$.

To obtain (6.1.5) under the condition (6.1.4) we substitute $c_1 a_{i_k d} - c_2 a_{i_k d-1} + \cdots + (-1)^{d+1} c_d a_{i_k 1}$ for $a_{i_k d+1}$ on the left hand side of (6.1.5).

6.2 Lemma. *Let d be a positive integer, and let*

$$(x_{ij}(k)) = \begin{pmatrix} x_{11}(k) & \cdots & x_{1d}(k) \\ \vdots & \ddots & \vdots \\ x_{d1}(k) & \cdots & x_{dd}(k) \end{pmatrix} \quad \text{for } k = 0, 1, \dots,$$

be $d \times d$-matrices with coefficients in the ring A. For $k = 0, 1, \dots,$ we let $\mathcal{I}_k$ be all multi-indices $(k_1, \dots, k_d)$ of natural numbers such that $k_1 + \cdots + k_d = k$.

(1) *For $k = 0, 1, \dots,$ we have*

$$\sum \det \begin{pmatrix} x_{11}(k_1) & \cdots & x_{1d}(k_d) \\ \vdots & \ddots & \vdots \\ x_{d1}(k_1) & \cdots & x_{dd}(k_d) \end{pmatrix} = \sum \det \begin{pmatrix} x_{11}(k_1) & \cdots & x_{1d}(k_1) \\ \vdots & \ddots & \vdots \\ x_{d1}(k_d) & \cdots & x_{dd}(k_d) \end{pmatrix}, \tag{6.2.1}$$

where both sums are over all $(k_1, \dots, k_d) \in \mathcal{I}_k$.

(2) *Assume that for* $k = 0, 1, \dots,$ *we have*

$$\begin{pmatrix} x_{11}(k) & \cdots & x_{1d}(k) \\ \vdots & \ddots & \vdots \\ x_{d1}(k) & \cdots & x_{dd}(k) \end{pmatrix} = \begin{pmatrix} x_{11+k} & \cdots & x_{1d+k} \\ \vdots & \ddots & \vdots \\ x_{d1+k} & \cdots & x_{dd+k} \end{pmatrix} \tag{6.2.2}$$

for elements x_{ij} *in* A. *Then*

$$\det \begin{pmatrix} x_{11} & \cdots & x_{1d-1} & x_{1d+k} \\ \vdots & \ddots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd-1} & x_{dd+k} \end{pmatrix} = \sum_{(k_1,\dots,k_d)\in\mathcal{I}_k} \det \begin{pmatrix} x_{11+k_1} & \cdots & x_{1d+k_1} \\ \vdots & \ddots & \vdots \\ x_{d1+k_d} & \cdots & x_{dd+k_d} \end{pmatrix}, \tag{6.2.3}$$

for $k = 0, 1, \dots$.

(3) *Let*

$$R_k = \sum_{(k_1,\dots,k_d)\in\mathcal{I}_k} \det \begin{pmatrix} x_{11+k_1} & \cdots & x_{1d+k_1} \\ \vdots & \ddots & \vdots \\ x_{d1+k_d} & \cdots & x_{dd+k_d} \end{pmatrix},$$

and let $c_1, \dots, c_n$ *be elements in* A. *Assume that*

$$x_{id+k} - c_1 x_{id+k-1} + \cdots + (-1)^n c_n x_{id+k-n} = 0 \tag{6.2.4}$$

for $i = 1, \dots, d$, *and for* $k = n-d+1, n-d+2, \dots$. *Then we have*

$$R_k - c_1 R_{k-1} + \cdots + (-1)^k c_k R_0 = 0 \quad \textit{for } k = n-d+1, \dots, n. \tag{6.2.5}$$

Proof. (1) For all permutations σ of $\{1, \dots, d\}$ the monomial $x_{\sigma(1)1}(k_1) \cdots x_{\sigma(d)d}(k_d)$ on the left hand side of (6.2.1) is equal to the monomial $x_{\sigma(1)1}(k'_{\sigma(1)}) \cdots x_{\sigma(d)d}(k'_{\sigma(d)})$ on the right hand side in (6.2.1) when $k_i = k'_{\sigma(i)}$ for $i = 1, \dots, d$. Hence (6.2.1) holds.

(2) Under the assumption (6.2.2) the left hand side of (6.2.1) becomes the coefficient of T^k in the determinant of the matrix
(6.2.6)

$$\begin{pmatrix} x_{11}+x_{12}T+x_{13}T^2+\cdots & \cdots & x_{1d}+x_{1d+1}T+x_{1d+2}T^2+\cdots \\ \vdots & \ddots & \vdots \\ x_{d1}+x_{d2}T+x_{d3}T^2+\cdots & \cdots & x_{dd}+x_{dd+1}T+x_{dd+2}T^2+\cdots \end{pmatrix}.$$

For $1,2,\dots,d-1$ we subtract T times the $(i+1)$'st column in this matrix from the i'th column. The result is that the i'th column has coordinates $x_{1i},\dots,x_{di}$. Thus the determinant of the matrix (6.2.6) is the determinant of

$$\begin{pmatrix} x_{11} & \cdots & x_{1d-1} & x_{1d}+x_{1d+1}T+x_{1d+2}T^2+\cdots \\ \vdots & \ddots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd-1} & x_{dd}+x_{dd+1}T+x_{dd+2}T^2+\cdots \end{pmatrix}. \tag{6.2.7}$$

The coefficient of T^k in the determinant of the matrix (6.2.7) is clearly the determinant to the left of (6.2.3). Finally the right hand side of (6.2.3) is, under the assumption (6.2.2), equal to the right hand side of (6.2.1). Thus equation (6.2.3) follows from (6.2.1).

(3) From (6.2.3) it follows that the left hand side of (6.2.5) is

$$\begin{aligned} &\det\begin{pmatrix} x_{11} & \cdots & x_{1d-1} & x_{1d+k} \\ \vdots & \ddots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd-1} & x_{dd+k} \end{pmatrix} \\ &\qquad - c_1\det\begin{pmatrix} x_{11} & \cdots & x_{1d-1} & x_{1d+k-1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd-1} & x_{dd+k-1} \end{pmatrix} + \cdots \\ &\qquad\qquad + (-1)^k c_k\det\begin{pmatrix} x_{11} & \cdots & x_{1d-1} & x_{1d} \\ \vdots & \ddots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd-1} & x_{dd} \end{pmatrix} \\ &= \det\begin{pmatrix} x_{11} & \cdots & x_{1d-1} & x_{1d+k}-c_1x_{1d+k-1}+\cdots+(-1)^kc_kx_{1d} \\ \vdots & \ddots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd-1} & x_{dd+k}-c_1x_{dd+k-1}+\cdots+(-1)^kc_kx_{dd} \end{pmatrix}. \end{aligned} \tag{6.2.8}$$

The right hand side of (6.2.8) is equal to

$$\det\begin{pmatrix} x_{11} & \cdots & x_{1d-1} & x_{1d+k} - c_1 x_{1d+k-1} + \cdots + (-1)^n c_n x_{1d+k-n} \\ \vdots & \ddots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd-1} & x_{dd+k} - c_1 x_{dd+k-1} + \cdots + (-1)^n c_n x_{dd+k-n} \end{pmatrix}$$

when $k = n-d+1, n-d+2, \dots, n$. Thus it follows from the recursion formula (6.2.4) that the right hand side of (6.2.8) is zero for $k = n-d+1, \dots, n$. The equations (6.2.5) thus follow from (6.2.8). □

7. Some results on residues. As a consequence of the auxiliary results of Section 6 we obtain some useful results on residues.

7.1 Notation. Let $A[T_1, \dots, T_d]$ be the polynomial ring in the independent variables $T_1, \dots, T_d$ over A. Moreover, let d be an integer with $1 \le d \le n$ and write

$$P(T) = (T - T_1) \cdots (T - T_d)$$

in $A[T_1, \dots, T_d]$, and

$$p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$$

in $A[T]$. We denote by $U_i = s_i(T_1, \dots, T_d)$ the i'th *complete symmetric polynomial* in $T_1, \dots, T_d$ for $i = 0, 1, \dots,$ and we let $U_i = 0$ for $i < 0$. For $d = 1, \dots, n$ we write

$$\begin{aligned} V_j &= U_{n-d+j} - c_1 U_{n-d+j-1} + \cdots + (-1)^n c_n U_{n-d+j-n} \\ &= U_{n-d+j} - c_1 U_{n-d+j-1} + \cdots + (-1)^{n-d+j} c_{n-d+j} U_0, \end{aligned}$$

for $j = 1, \dots, d$. Then

$$\begin{aligned} \frac{p}{P} &= T^{n-d}\left(1 - \frac{c_1}{T} + \cdots + (-1)^n \frac{c_n}{T^n}\right)\left(1 + \frac{U_1}{T} + \frac{U_2}{T^2} + \cdots\right) \\ &= \cdots + \frac{V_1}{T} + \frac{V_2}{T^2} + \cdots + \frac{V_d}{T^d} + \cdots . \end{aligned} \tag{7.1.1}$$

7.2 Proposition. *Let $g_i = \cdots + (a_{i1}/T) + (a_{i2}/T^2) + \cdots + (a_{id}/T^d) + \cdots$ for $i = 1, \dots, d$ be formal Laurent series in $1/T$, and let*

$g_{d+1} = \cdots + (a_1/T) + (a_2/T^2) + \cdots + (a_{d+1}/T^{d+1}) + \cdots$. *Then*

$$\operatorname{Res}(g_1, \dots, g_{d+1}) = \sum_{j=0}^{d} (-1)^j a_{d-j+1} \sum_{1 \le i_1 < \cdots < i_j \le d} \operatorname{Res}(g_1, \dots, Tg_{i_1}, \dots, Tg_{i_j}, \dots, g_d).$$

Proof. Expand the determinant $\operatorname{Res}(g_1, \dots, g_{d+1})$ along the last row. We obtain

$$\operatorname{Res}(g_1, \dots, g_{d+1}) = \sum_{j=0}^{d} (-1)^j a_{d-j+1} A_j \tag{7.2.1}$$

where A_j is the determinant to the left of (6.1.3). Since $Tg_k = \cdots + (a_{k2}/T) + (a_{k3}/T^2) + \cdots + (a_{kd+1}/T^d) + \cdots$ we obtain from (6.1.3) that

$$A_j = \sum_{1 \le i_1 < \cdots < i_j \le d} \operatorname{Res}(g_1, \dots, Tg_{i_1}, \dots, Tg_{i_j}, \dots, g_d).$$

Thus the proposition follows from (7.2.1). □

7.3 Proposition. *Let*

$$\rho^d(p) : A[T_1, \dots, T_d] \to A$$

be the A-linear homomorphism determined by

$$\rho^d(p)(f_1(T_1) \cdots f_d(T_d)) = \operatorname{Res}\left(\frac{f_1}{p}, \dots, \frac{f_d}{p}\right)$$

for all polynomials $f_1(T), \dots, f_d(T)$ in $A[T]$. Then we have, for all natural numbers $h_1, \dots, h_d$, that

$$\begin{aligned} \rho^d(p)(U_{n-d+j} T_1^{h_1} \cdots T_d^{h_d}) & \\ - c_1 \rho^d(p)(U_{n-d+j-1} T_1^{h_1} \cdots T_d^{h_d}) + \cdots & \\ + (-1)^{n-d+j} c_{n-d+j} \rho^d(p)(U_0 T_1^{h_1} \cdots T_d^{h_d}) &= 0 \end{aligned}$$

for $j = 1, \dots, d$.

Proof. Let $\mathcal{I}_k$ consist of all multiindices $(k_1, \dots, k_d)$ of natural numbers such that $k_1 + \cdots + k_d = k$. Then, for all natural numbers $h_1, \dots, h_d$ we have

(7.3.1)
$$\rho^d(p)(U_{n-d+j}T_1^{h_1}\cdots T_d^{h_d}) = \sum_{(k_1,\dots,k_d)\in\mathcal{I}_{n-d+j}} \operatorname{Res}\left(\frac{T^{h_1+k_1}}{p}, \dots, \frac{T^{h_d+k_d}}{p}\right).$$

Let
$$x_{1j} = s_{h_1-n+j}, \dots, x_{dj} = s_{h_j-n+j} \quad \text{for all integers } j.$$

Then
$$\operatorname{Res}\left(\frac{T^{h_1+k_1}}{p}, \dots, \frac{T^{h_d+k_d}}{p}\right) = \begin{pmatrix} x_{1k_1+1} & \cdots & x_{k_1+d} \\ \vdots & \ddots & \vdots \\ x_{dk_d+1} & \cdots & x_{dk_d+d} \end{pmatrix}.$$

Thus, in the notation of Lemma 6.2 (3) and with $k = n-d+j$ we have

$$R_{n-d+j} = \sum_{(k_1,\dots,k_d)\in\mathcal{I}_{n-d+j}} \operatorname{Res}\left(\frac{T^{h_1+k_1}}{p}, \dots, \frac{T^{h_d+k_d}}{p}\right). \tag{7.3.2}$$

Moreover,

$$\begin{aligned} &x_{in+j} - c_1 x_{in+j-1} + \cdots + (-1)^n c_n x_{ij} \\ &\qquad = s_{h_i+j} - c_1 s_{h_i+j-1} + \cdots + (-1)^n c_n s_{h_i-n+j} = 0 \end{aligned}$$

for $j = 1, 2, \dots$. Thus it follows from Lemma 6.2 (3) with $k = n-d+j$ that

(7.3.3)
$$R_{n-d+j} - c_1 R_{n-d+j-1} + \cdots + (-1)^{n-d+j} c_{n-d+j} R_0 \quad \text{for } j = 1, \dots, d.$$

We obtain from (7.3.1) and (7.3.2) that

$$\rho^d(p)(U_{n-d+j}T_1^{h_1}\cdots T_d^{h_d}) = R_{n-d+j}$$

and the proposition follows from (7.3.3). □

7.4 Proposition. *Let*

$$p(T) = P(T)q(T) + r_{d-1}T^{d-1} + \cdots + r_0 T^0$$

be the result of using the division algorithm in $A[T_1, \dots, T_d]$ to the polynomial $p(T)$ modulo $P(T)$. Then

$$r_{d-i} = \operatorname{Res}\left(\frac{T^{d-1}}{P}, \dots, \frac{T^{d-i+1}}{P}, \frac{p}{P}, \frac{T^{d-i-1}}{P}, \dots, \frac{T^0}{P}\right) \tag{7.4.1}$$

for $i = 1, \dots, d$.

In particular, the ideal in $A[T_1, \dots, T_d]$ generated by the elements $r_0, r_1, \dots, r_{d-1}$ is generated by the elements

$$V_j = U_{n-d+j} - c_1 U_{n-d+j-1} + \cdots + (-1)^{n-d+j} c_{n-d+j} U_0$$

for $j = 1, \dots, d$.

Proof. We have by Proposition 5.4 that $\operatorname{Res}((T^{d-1}/P), \dots, (T^0/P)) = 1$. The first part of the proposition follows since $(p/P) = q(T) + r_{d-1}(T^{d-1}/P) + \cdots + r_0(T^0/P)$ by (7.1.1), and since Res is $A[T_1, \dots, T_d]$-linear, alternating, and zero if one of the factors is a polynomial.

Since $(T^{d-i}/P) = (U_0/T^i) + (U_1/T^{i+1}) + \cdots$ for $i = 1, \dots, d$ it follows from (7.1.1) and (7.4.1) that

$$r_{d-i} = \begin{pmatrix} U_0 & \cdots & U_{i-2} & U_{i-1} & U_i & \dots & U_{d-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & U_0 & U_1 & U_2 & \cdots & U_{d-i+1} \\ V_1 & \cdots & V_{i-1} & V_i & V_{i+1} & \cdots & V_d \\ 0 & \cdots & 0 & 0 & U_0 & \cdots & U_{d-i-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & U_0 \end{pmatrix}.$$

Hence we have $r_{d-1} = V_1, r_{d-2} = V_2 + W_2, \dots, r_0 = V_d + W_d$, where W_i lies in the ideal in $A[T_1, \dots, T_d]$ generated by $V_1, V_2, \dots, V_{i-1}$. □

8. First construction of Gysin homomorphisms. We give a proof based upon the first construction of splitting algebras.

8.1 Theorem. *Let $p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$ be a polynomial in the variable T with coefficients in the ring A, and let*

$\mathrm{Split}^d_A(p) = A[\xi_1, \dots, \xi_d]$ be a d'th splitting algebra for $p(T)$ over A with universal splitting

$$p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T).$$

Then there is an A-module homomorphism

$$\partial^d(p) : A[\xi_1, \dots, \xi_d] \longrightarrow A$$

determined by $\partial^d(p)(f_1(\xi_1) \cdots f_d(\xi_d)) = \mathrm{Res}\,((f_1/p), \dots, (f_d/p))$ for all collections of polynomials $f_1(T), \dots, f_d(T)$ in $A[T]$.

In particular, on the basis of Corollary 2.2 for the A-module $A[\xi_1, \dots, \xi_d]$, it takes the value

$$\partial^d(p)(\xi_1^{h_1} \cdots \xi_d^{h_d}) = \begin{cases} 1 \text{ when } h_j = n - j \text{ for } j = 1, \dots, d \\ 0 \text{ otherwise when } 0 \le h_j \le n - j \text{ for } j = 1, \dots, d. \end{cases}$$

Proof. We prove the assertion by induction on d. For $d = 0$ it is clear. Assume that we have an A-module homomorphism

$$\partial^d(p) : A[\xi_1, \dots, \xi_d] \longrightarrow A$$

satisfying the properties of the theorem. We shall show how to obtain an A-linear homomorphism

$$\partial^{d+1}(p) : A[\xi_1, \dots, \xi_{d+1}] \longrightarrow A$$

satisfying the properties of the theorem for $d + 1$.

Since $\mathrm{Res}\,(g) = 0$ when $g(T)$ is a polynomial, we have, with the notation of Proposition 2.1 an $A[\xi_1, \dots, \xi_d]$-module homomorphism

$$\partial' : A[\xi_1, \dots, \xi_{d+1}] = A[\xi_1, \dots, \xi_d][T]/(p_{d+1}) \longrightarrow A[\xi_1, \dots, \xi_d]$$

such that $\partial'(f_{d+1}(\xi_{d+1})) = \mathrm{Res}\,(f_{d+1}/p_{d+1})$ for all $f_{d+1}(T)$ in $A[T]$.

We now show that the composite A-module homomorphism

$$\partial^{d+1}(p) = \partial^d(p)\partial' : A[\xi_1, \dots, \xi_{d+1}] \longrightarrow A$$

satisfies the properties of the homomorphism of the theorem with $d+1$ instead of d. Since $\partial^{d+1}(p) = \partial^d(p)\partial'$ is linear in $f_{d+1}(T)$ we can assume that $f_{d+1}(T) = T^i$ for some non-negative integer i.

Let $t_0, \dots, t_d$ be the elementary symmetric polynomials in variables $T_1, \dots, T_d$ evaluated at the elements $\xi_1, \dots, \xi_d$. We have

$$\begin{aligned}\frac{T^i}{p_{d+1}} &= \frac{T^i(T-\xi_1)\cdots(T-\xi_d)}{p}\\ &= T^{i+d-n}\left(1-\frac{t_1}{T}+\cdots+(-1)^d\frac{t_d}{T^d}\right)\left(1+\frac{s_1}{T}+\frac{s_2}{T^2}+\cdots\right),\end{aligned}$$

and thus

$$\operatorname{Res}\left(\frac{T^i}{p_{d+1}}\right) = s_{i+d-n+1} - t_1 s_{i+d-n} + \cdots + (-1)^d t_d s_{i+d-n-d+1}.$$

Consequently,

(8.1.1)

$$f_1(\xi_1)\cdots f_d(\xi_d)\operatorname{Res}\left(\frac{f_{d+1}}{p_{d+1}}\right) = \sum_{j=0}^{d}(-1)^j s_{i+d-n+1-j} t_j f_1(\xi_1)\cdots f_d(\xi_d).$$

Since $t_j = \sum_{1\le i_1<\cdots<i_j\le d}\xi_{i_1}\xi_{i_1}\cdots\xi_{i_j}$ we obtain, by the induction assumption,

$$\begin{aligned}&\partial^d(p)(t_j f_1(\xi_1)\cdots f_d(\xi_d))\\ &\qquad = \sum_{1\le i_1<\cdots<i_j\le d}\operatorname{Res}\left(\frac{f_1}{p},\dots,\frac{Tf_{i_1}}{p},\dots,\frac{Tf_{i_j}}{p},\dots,\frac{f_d}{p}\right).\end{aligned}$$

Consequently we obtain from (8.1.1) that

$$\begin{aligned}(8.1.2)\quad &\partial^d(p)(f_1(\xi_1)\cdots f_d(\xi_d)\partial'(f_{d+1}(\xi_{d+1})))\\ &= \partial^d(p)\left(f_1(\xi_1)\cdots f_d(\xi_d)\operatorname{Res}\left(\frac{f_{d+1}}{T^{d+1}}\right)\right)\\ &= \sum_{j=0}^{d}(-1)^j s_{i+d-n+1-j}\\ &\qquad\sum_{1\le i_1<\cdots<i_j\le d}\operatorname{Res}\left(\frac{f_1}{p},\dots,\frac{Tf_{i_1}}{p},\dots,\frac{Tf_{i_j}}{p},\dots,\frac{f_d}{p}\right).\end{aligned}$$

By Proposition 7.2 the right hand side of (8.1.2) is $\operatorname{Res}((f_1/p),\ldots,(f_{d+1}/p))$ with $f_{d+1}=T^i$. Thus $\partial^{d+1}(p)=\partial^d(p)\partial'$ gives an A-module homomorphism as asserted in the theorem.

The asserted values on the A-module basis of $A[\xi_1,\ldots,\xi_{d+1}]$ follow from Proposition 5.4. □

9. Second construction of Gysin homomorphisms.

We prove a result that with the second construction of splitting algebras immediately implies the existence of Gysin homomorphisms.

As in Theorem 10.1 we shall use that if we let

$$\rho^n(p)(T_1^{h_1}\cdots T_n^{h_m})=\operatorname{Res}\left(\frac{T^{h_1}}{p},\cdots,\frac{T^{h_n}}{p}\right)$$

for all monomials in $A[t_1,\ldots,T_n]$ we obtain, extending by A-linearity, an A-linear homomorphism $\rho^n(p):A[T_1,\ldots,T_n]\to A$ such that $\rho^n(p)(f_1(T_1)\cdots f_n(T_n))=\operatorname{Res}((f_1/p),\ldots,(f_n/p))$.

9.1 Theorem. *Let*

$$\rho^n(p):A[T_1,\ldots,T_n]\longrightarrow A$$

be the A-module homomorphism determined by

$$\rho^n(p)(f_1(T_1)\cdots f_n(T_n))=\operatorname{Res}\left(\frac{f_1}{p},\ldots,\frac{f_n}{p}\right)$$

for all polynomials $f_1(T),\ldots,f_n(T)$ in $A[T]$. Then $\rho^n(p)$ vanishes on the ideal in $A[T_1,\ldots,T_n]$ generated by the polynomials $c_1-C_1,\ldots,c_n-C_n$.

Proof. We must prove that for all natural numbers $h_1,\ldots,h_n$ we have

$$\rho^n(p)(T_1^{h_1}\cdots T_n^{h_n}C_j)=c_j\rho^n(p)(T_1^{h_1}\cdots T_n^{h_n})\quad\text{for } j=1,\ldots,n.$$

However,

$$\text{(9.1.1)}\quad \rho^n(p)(T_1^{h_1}\cdots T_n^{h_n}c_j)$$
$$=\sum_{1\le i_1<\cdots<i_j\le d}\operatorname{Res}\left(\frac{T^{h_1}}{p},\ldots,\frac{T^{h_{i_1}+1}}{p},\ldots,\frac{T^{h_{i_j}+1}}{p},\ldots,\frac{T^{h_n}}{p}\right).$$

Let

$$a_{1i} = s_{h_1-n+i}, \dots , a_{ni} = s_{h_n-n+i} \quad \text{for all integers } i.$$

Then

$$\operatorname{Res}\left(\frac{T^{h_1}}{p}, \dots , \frac{T^{h_{i_1}+1}}{p}, \dots , \frac{T^{h_{i_j}+1}}{p}, \dots , \frac{T^{h_n}}{p}\right) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{i_1 2} & \cdots & a_{i_1 n+1} \\ & \cdots & \\ a_{i_j 2} & \cdots & a_{i_j n+1} \\ & \cdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

From Lemma 6.1 (2) we obtain

$$(9.1.2) \quad \det \begin{pmatrix} a_{11} & \cdots & a_{1n-j} & a_{1n-j+2} & \cdots & a_{1n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn-j} & a_{nn-j+2} & \cdots & a_{nn+1} \end{pmatrix} = \sum_{1 \le i_1 < \dots < i_j \le d} \operatorname{Res}\left(\frac{T^{h_1}}{p}, \dots , \frac{T^{h_1+1}}{p}, \dots , \frac{T^{h_j+1}}{p}, \dots , \frac{T^{h_d}}{p}\right).$$

Since

$$a_{in+1} - c_1 a_{in} + \dots + (-1)^n c_n a_{i1} = s_{h_i+1} - c_1 s_{h_i} + \dots + (-1)^n c_n s_{h_i-n+1} = 0,$$

for $i = 1, \dots , n$ we obtain from Lemma 6.1 (3) that the left hand side of (9.1.2) is equal to

$$c_j \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n_1} & \cdots & a_{nn} \end{pmatrix} = c_j \det \begin{pmatrix} s_{h_1-n+1} & \cdots & s_{h_1-n+n} \\ \vdots & \ddots & \vdots \\ s_{h_n-n+1} & \cdots & s_{h_n-n+n} \end{pmatrix} = c_j \operatorname{Res}\left(\frac{T^{h_1}}{p}, \cdots , \frac{T^{h_n}}{p}\right).$$

Thus the theorem follows from (9.1.1) and (9.1.2). $\square$

9.2 *Remark.* The theorem together with the second construction for $n = d$ gives the Gysin homomorphism

$$\partial^n(p) : A[\xi_1, \dots, \xi_n] \longrightarrow A$$

when $n = d$. To obtain the Gysin homomorphism

$$\partial^d(p) : A[\xi_1, \dots, \xi_d] \longrightarrow A$$

we note that by the construction of Proposition 2.1 (2), or by Corollary 2.2 (2) we have an injective homomorphism of A-modules

$$u : A[\xi_1, \dots, \xi_d] \longrightarrow A[\xi_1, \dots, \xi_n]$$

given by $u(f_1(\xi_1) \cdots f_d(\xi_d)) = f_1(\xi_1) \cdots f_d(\xi_d)\xi_{d+1}^{n-d-1} \cdots \xi_n^0$ for all collections of polynomials $f_1(T), \dots, f_d(T)$ in $A[T]$. Then

$$\begin{aligned}\partial^n(p)u(f_1(\xi_1) \cdots f_d(\xi_d)) &= \partial^n(p)(f_1(\xi_1) \cdots f_d(\xi_d)\xi_{d+1}^{n-d-1} \cdots \xi_n^0) \\ &= \operatorname{Res}\left(\frac{f_1}{p}, \dots, \frac{f_d}{p}, \frac{T^{n-d-1}}{p}, \dots, \frac{T^0}{p}\right).\end{aligned}$$

However, it is clear that

$$\operatorname{Res}\left(\frac{f_1}{p}, \dots, \frac{f_d}{p}\right) = \operatorname{Res}\left(\frac{f_1}{p}, \dots, \frac{f_d}{p}, \frac{T^{n-d-1}}{p}, \dots, \frac{T^0}{p}\right).$$

Hence, the A-module homomorphism

$$\partial^n(p)u : A[\xi_1, \dots, \xi_d] \longrightarrow A$$

satisfies

$$\partial^n(p)u(f_1(\xi_1) \cdots f_d(\xi_d)) = \operatorname{Res}\left(\frac{f_1}{p}, \dots, \frac{f_d}{p}\right).$$

Hence $\partial^d(p) = \partial^n(p)u$ is the Gysin homomorphism we wanted to construct.

10. Third construction of Gysin homomorphisms. We prove a result that with the third construction of splitting algebras immediately implies the existence of Gysin homomorphisms.

10.1 Theorem. *Let*

$$\rho^d(p) : A[T_1, \dots, T_d] \longrightarrow A$$

be the A-module homomorphism that is determined by

$$\rho^d(p)(f_1(T_1)\cdots f_d(T_d)) = \operatorname{Res}\left(\frac{f_1}{p}, \dots, \frac{f_d}{p}\right)$$

for all polynomials $f_1(T), \dots, f_d(T)$ in $A[T]$. Then $\rho^d(p)$ is zero on the ideal generated by the elements $r_{d-i} = \operatorname{Res}((T^{d-1}/P), \dots, (T^{d-i+1}/P), (p/P), (T^{d-i-1}/P), \dots, (T^0/P))$ for $i = 1, \dots, d$.

Proof. We use the notation of Section 7. It follows from Proposition 7.4 that it suffices to show that $\rho^d(p)$ is zero on the ideal generated by V_j for $j = 1, \dots, d$. That is, it is zero on the elements $V_j T_1^{h_1} \cdots T_d^{h_d}$ for all natural numbers $h_1, \dots, h_d$, and $j = 1, \dots, d$. In other words, with the notation of 7.1, it suffices to show that

$$\begin{aligned}\rho^d(p)(U_{n-d+j}T_1^{h_1}\cdots T_d^{h_d}) - c_1\rho^d(p)(U_{n-d+j-1}T_1^{h_1}\cdots T_d^{h_d}) + \cdots \\ + (-1)^{n-d+j}c_{n-d+j}\rho^d(p)(U_0T_1^{h_1}\cdots T_d^{h_d}) = 0 \quad \text{for } j = 1, \dots, d.\end{aligned}$$

However, this follows from Proposition 7.3. □

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Department of Mathematics, KTH, S-100 44 Stockholm, Sweden
Email address: laksov@math.kth.se