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Cosimplicial cohomology and two term extensions of coalgebras

By

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Introduction

In this paper we interpret the second cosimplicial cohomology group of a coassociative coalgebra A with coefficients in a two sided A-comodule M. In § 1, we refer to some wellknown facts. We shall use the notation

$$A \xrightarrow{(f^0, \cdots, f^n)} B \text{ instead of } A \begin{array}{c} \xrightarrow{f^0} \\ \xrightarrow{f^1} \\ \vdots \\ \xrightarrow[f^n]{} \end{array} B.$$

A cosimplicial object A_* in a category $\mathscr{A}$ is a diagram

$$A_0 \rightleftarrows A_1 \rightleftarrows A_2 \rightleftarrows \cdots \rightleftarrows A_{n-1} \rightleftarrows A_n \rightleftarrows \cdots$$

which satisfies certain commutation rules. If A_* is a cosimplicial module, then the A_n has the following representation, $A_n = \tilde{A}_n \oplus \sum_{r-1}^{n} \sum_{n \geq i_1 > i_2 > \cdots > i_r \geq 1} \varepsilon^{i_1} \varepsilon^{i_2} \cdots \varepsilon^{i_r} \tilde{A}_{n-r}$ (direct sum). Furthermore, there exists a cofree functor F from the category $\mathscr{M}$ of modules to the category $\mathscr{C}$ of coalgebras, which is the right adjoint functor of the underlying object functor $U: \mathscr{C} \rightarrow \mathscr{M}$ (see Lee [5]). The standard cosimplicial resolution of a coalgebra A is an augmented cosimplicial coalgebra

$$G_0A \rightleftarrows G_1A \rightleftarrows \cdots \rightleftarrows G_nA \rightleftarrows \cdots$$

where $G_n = (FU)^{n+1}A$. Using this complex and a functor Coder$(M, -)$, the n-th cohomology $H^n(M, A)$ is defined. In § 2, we discuss some properties of the cosimplicial cokernels. The cosimplicial cokernel of $A \Rightarrow B$ is written by Cosimp coker $(A \Rightarrow B)$. A cosimplicial object A_* in $\mathscr{A}$ is called acyclic if the canonical morphism Cosimp coker $(A_{n-2} \Rightarrow A_{n-1}) \rightarrow A_n$ is an monomorphism for every $n \geq 2$, and the acyclicity of an augmented cosimplicial object A_* over $A_{-1} = A$ is similarly defined. The main theorem is given in § 3. An augmented cosimplicial coalgebra

$$A \longrightarrow E_0 \rightleftarrows E_1 \rightleftarrows E_2 \cdots$$

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over A is called a two term extension of A by M, if it satisfies certain conditions (see § 3). For a given normal coderivation n-cocycle, the standard n-term extension is described in Proposition 14. With certain additional propositions, we get the main result, that the second cosimplicial cohomology $H^2(M, A)$ is in 1-1 correspondence with the set $Ex^2(M, A)$ of all equivalence classes of two term extensions.

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§ 1. Preliminaries

Throughout this paper K is a fixed field. We understand by a module a K-module, and by a tensor product $\otimes$ the one over K. A coalgebra A is a module with a coassociative comultiplication $\Delta_A : A \to A \otimes A$ and a counit $\varepsilon_A : A \to K$. A comodule M over a coalgebra A is a two sided A-comodule, i.e. a module with linear maps $\Delta_M{}^l : M \to A \otimes M$ and $\Delta_M{}^r : M \to M \otimes A$ which satisfy relations $(1 \otimes \Delta_M{}^l)\Delta_M{}^l = (\Delta_A \otimes 1)\Delta_M{}^l$, $(\Delta_M{}^r \otimes 1)\Delta_M{}^r = (1 \otimes \Delta_A)\Delta_M{}^r$, $(\Delta_M{}^l \otimes 1)\Delta_M{}^r = (1 \otimes \Delta_M{}^r)\Delta_M{}^l$ and $(\varepsilon_A \otimes 1)\Delta_M{}^l = (1 \otimes \varepsilon_A)\Delta_M{}^r = 1$. A coideal is a two sided coideal.

Let M be a two sided A-comodule. We define

$$\Delta_{A\oplus M} : A \oplus M \to (A \oplus M) \otimes (A \oplus M) \quad \text{by}$$
$$\Delta_{A\oplus M} = \Delta_A \text{ on } A \text{ and } \Delta_{A\oplus M} = \Delta_M{}^l + \Delta_M{}^r \text{ on } M,$$

and $\varepsilon_{A\oplus M}$ by the composition $A \oplus M \xrightarrow{\text{proj.}} A \xrightarrow{\varepsilon_A} K$. Then $A \oplus M$ is a coalgebra. It is called the coidealization of M, and we shall denote this by $A * M$.

We denote by $A \overset{(f^0, \cdots, f^n)}{\Longrightarrow} B$ a diagram $A \begin{array}{c} \xrightarrow{f^0} \\ \xrightarrow{f^1} \\ \cdots \\ \xrightarrow[f^n]{} \end{array} B$.

A cosimplicial object A_* in a category $\mathscr{A}$ is a diagram

$$A^0 \underset{\delta^0}{\overset{(\varepsilon^0, \varepsilon^1)}{\rightleftarrows}} A_1 \underset{(\delta^0, \delta^1)}{\overset{(\varepsilon^0, \varepsilon^1, \varepsilon^2)}{\rightleftarrows}} A_2 \rightleftarrows \cdots \rightleftarrows A_{n-1} \underset{(\delta^0, \cdots, \delta^{n-1})}{\overset{(\varepsilon^0, \cdots, \varepsilon^n)}{\rightleftarrows}} A_n \rightleftarrows \cdots$$

which satisfies the commutation rules (the cosimplicial relations)

(i) $\varepsilon^j \varepsilon^i = \varepsilon^i \varepsilon^{j-1}$ if $i < j$,

(ii) $\delta^j \delta^i = \delta^i \delta^{j+1}$ if $i \leq j$,

(iii) $\delta^j \varepsilon^i = \begin{cases} \varepsilon^i \delta^{j-1} & \text{if } i < j, \\ \text{identity} & \text{if } i = j \text{ or } i = j+1, \\ \varepsilon^{i-1} \delta^j & \text{if } i > j+1. \end{cases}$

Moreover, if a morphism $\varepsilon : A \to A_0$ satisfies the relation $\varepsilon^0 \varepsilon = \varepsilon^1 \varepsilon$, (A_*, ε) is said to be an augmented cosimplicial object over A.

If A^* is a cosimplicial module, we define

$$\tilde{A}_0=A_0$$
$$\tilde{A}_n=\bigcap_{i=0}^{n-1}\operatorname{Ker}\delta^i,\quad (n>0).$$

The Moore complex of A_* is a cochain complex

$$0\to\tilde{A}_0\xrightarrow{\tilde{d}_0}\tilde{A}_1\to\cdots\to\tilde{A}_{n-1}\xrightarrow{\tilde{d}_n}\tilde{A}_n\to\cdots$$

where $\tilde{d}_n$ are induced by $d_n=\sum_{i=0}^n(-1)^i\varepsilon^i:A_{n-1}\to A_n$.

The module $\tilde{A}_n$ is a direct summand of A_n, and its projection is given by

$$t_n=(1-\varepsilon^n\delta^{n-1})\cdots(1-\varepsilon^1\delta^0),\quad (n\geq 1).$$

It satisfies the following relations:

$$t_n\varepsilon^0=d_nt_{n-1},\quad (n\geq 1),$$
$$t_n\varepsilon^i=0,\quad (1\leq i\leq n),$$
$$\delta^it_n=0,\quad (0\leq i\leq n-1).$$

Proposition 1. *If A_* is a cosimplicial module, then A_n has the following representation:*

$$A_n=\tilde{A}_n\oplus\sum_{r-1}^{n}\sum_{n\geq i_1>i_2>\cdots>i_r\geq 1}\varepsilon^{i_1}\varepsilon^{i_2}\cdots\varepsilon^{i_r}\tilde{A}_{n-r},\quad (\textit{direct sum}).$$

Proof. Since $1-t_n$ is written in the form $\sum_{i=1}^n\varepsilon^i\delta^{i-1}s_i$, we have

$$A_n=t_nA_n+(1-t_n)A_n=\tilde{A}_n+\sum_{i=1}^n\varepsilon^iA_{n-1}$$

Using an induction argument on n, we can easily get

$$A_n=\tilde{A}_n+\sum_{r-1}^{n}\sum_{n\geq i_1>i_2>\cdots>i_r\geq 1}\varepsilon^{i_1}\varepsilon^{i_2}\cdots\varepsilon^{i_r}\tilde{A}_{n-r}.$$

To prove the right hand side of this equation to be a direct sum, it suffices to show that if

$$\tilde{x}+\sum_{r-1}^{m}\sum_{n\geq i_1>\cdots>i_r\geq n-m+1}\varepsilon^{i_1}\cdots\varepsilon^{i_r}x_{i_1\cdots i_r}=0,\tag{1.1}$$
$$\tilde{x}\in A_n,\quad x_{i_1\cdots i_r}\in\tilde{A}_{n-r},\quad 1\leq m\leq n,$$

then $x_{i_1\cdots i_r}=0$ for $i_r=n-m+1$.

Operating δ^{n-m} on (1.1), we get

$$\sum_{r-1}^{m}\sum_{i_r-n-m+1}\varepsilon^{i_1-1}\varepsilon^{i_2-1}\cdots\varepsilon^{i_{r-1}-1}x_{i_1\cdots i_r}=0$$

Hence we can reduce to the case $n-1$. Q.E.D.

The cofree coalgebra associated with a given module V is a coalgebra FV with a linear map $\eta_V:FV\to V$ which satisfies the following universal property: For every coalgebra A and every linear map $\alpha:A\to V$, there exists one and only one morphism of coalgebras $\beta:A\to FV$ such that $\eta_V\beta=\alpha$. More precisely F is a right adjoint functor of the forgetful functor U from the category $\mathscr{C}$ of K-coalgebras to the category $\mathscr{M}$ of K-modules. For every coalgebra A there exists one and only one morphism $\varepsilon_A:A\to GA=FUA$ of coalgebras such that $\eta_{UA}\cdot U\varepsilon_A=1_{UA}$. η and ε are natural transformations. Denote by G_n the composite functor G^{n+1}. Put $\varepsilon^i = G^i\varepsilon G^{n-i}: G_{n-1}A \to G_nA$ and $\delta^i = G^iF\eta UG^{n-i}: G_{n+1}A \to G_nA$, $(0 \le i \le n)$. Then we get a (functorial) augmented cosimplicial coalgebra over A

$$G^*A : G_0A \rightleftarrows G_1A \rightleftarrows G_2A \rightleftarrows \cdots,$$

which is called the standard cosimplical resolution of A.

A coderivation from an A-comodule M to a coalgebra A is a linear map $f: M \to A$ such that $\Delta_A f = (1 \otimes f)\Delta_M{}^l + (f \otimes 1)\Delta_M{}^r$. Denote by $\mathrm{Der}_K(M, A)$ the module of all coderivations from M to A. If A_* is an augmented coalgebra over A, then for each $n \ge 0$ all the compositions $\varepsilon^{i_{n-1}} \cdots \varepsilon^{i_1}\varepsilon: A \to A_n$ are the same and independent of the choice of $(i_{n-1}, \cdots, i_1)$. Therefore A-comodule M can be understood as an A_n-comodule. We get a cochain complex

$$\mathrm{Der}_K(M, A_*): 0 \to \mathrm{Der}(M, A_0) \xrightarrow{\delta_0} \mathrm{Der}(M, A_1) \to \cdots \to \mathrm{Der}(M, A_n) \xrightarrow{\delta_n} \cdots$$

$$\delta_n(f) = d_{n+1}f = \sum_{i=0}^{n+1} (-1)^i \varepsilon^i f.$$

Denote by $H^n(M, A_*)$ the n-th cohomology of this complex. Put

$$\begin{aligned} \widetilde{\mathrm{Der}}_K(M, A_n) &= \{f \in \mathrm{Der}(M, A_n) \mid \mathrm{Im} f \subset \tilde{A}_n\} \\ &= \{f \in \mathrm{Der}(M, A_n) \mid t_n f = f\}. \end{aligned}$$

Then we have a cochain subcomplex $\widetilde{\mathrm{Der}}_K(M, A_*)$ of $\mathrm{Der}_K(M, A_*)$, and they are cochain homotopic (see Lee [5]). If $f \in \widetilde{\mathrm{Der}}_K(M, A_n)$ is a cocycle then it is called to be a normal coderivation n-cocycle. The n-th cohomology $H^n(M, A)$ of a coalgebra A with a coefficient comodule is M is defined to be $H^n(M, G_*A)$.

§ 2. Cosimplicial Cokernels

Definition 2. Let

$$A \xrightarrow{(\varepsilon^0, \cdots, \varepsilon^{n-1})} B \xrightarrow{(\varepsilon^0, \cdots, \varepsilon^n)} C,$$

$(n \ge 1)$ be a diagram in a category $\mathscr{A}$. We assume the following conditions:

(i) $\varepsilon^j\varepsilon^i = \varepsilon^i\varepsilon^{j-1}$ $(0 \le i < j \le n)$,

(ii) for every object X and morphism $(f^0, \cdots, f^n): B \Rightarrow X$ with $f^j\varepsilon^i = f^i\varepsilon^{j-1}$, $(0 \le i < j \le n)$, there exists one and only one morphism $f: C \to X$ with $f^i = f\varepsilon^i$, $(0 \le i \le n)$.

$$\begin{tikzcd} A \arrow[r, Rightarrow] & B \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & C \arrow[d, dashed, "f"] \\ & & X \end{tikzcd}$$

Then we say that $(\varepsilon^0, \cdots, \varepsilon^n): B \Rightarrow C$ (or roughly speaking C itself) is the cosimplicial cokernel of $(\varepsilon^0, \cdots, \varepsilon^{n-1}): A \Rightarrow B$, which is written by Cosimp coker $(A \Rightarrow B)$.

If $\mathscr{A}$ has a finite colimit, then any diagram $(f^0, \cdots, f^{n-1}): A \Rightarrow B$ in $\mathscr{A}$ has a cosimplical cokernel (Tierney-Vogel [7]). In the category of modules, the cosimplicial cokernel is represented as follows:

(2.1) Cosimp coker $((f^0, \cdots, f^{n-1}): A \Rightarrow B)$
$$= \oplus_{i=0}^{n} B / \sum_{0 \le i < j \le n} \mathrm{Im}(k^j f^i - k^i f^{j-1}),$$
where $k^i: B \to \oplus_{i=0}^{n} B$ is the canonical inclusion

Proposition 3. *Let $n \geq 2$. Suppose that a diagram*
$$D \underset{(\delta^0, \cdots, \delta^{n-3})}{\overset{(\varepsilon^0, \cdots, \varepsilon^{n-2})}{\rightleftarrows}} A \underset{(\delta^0, \cdots, \delta^{n-2})}{\overset{(\varepsilon^0, \cdots, \varepsilon^{n-1})}{\rightleftarrows}} B$$
in a category $\mathscr{A}$ satisfies the cosimplicial relations and C=Cosimp coker $(A \Rightarrow B)$. Then there exist unique morphism $(\delta^0, \cdots, \delta^{n-1}): C \Rightarrow B$ satisfying cosimplicial relations.

Proof. Put $X = B$ and $(f^0, \cdots, f^n) = (\varepsilon^0 \delta^{j-1}, \cdots, \varepsilon^{j-1}\delta^{j-1}, 1, 1, \varepsilon^{j+1}\delta^j, \cdots, \varepsilon^{n-1}\delta^j)$, $(0 \leq j \leq n-1)$. By Definition 2, we get the desired morphism δ^j. Q.E.D.

Proposition 4. *Let $(\varepsilon^0, \cdots, \varepsilon^{n-1}): A \Rightarrow B$ be coalgebra maps. If $(\varepsilon^0, \cdots, \varepsilon^n)$: $B \Rightarrow C$ is the cosimplicial cokernel of $(\varepsilon^0, \cdots, \varepsilon^{n-1}): A \Rightarrow B$ in the category $\mathscr{M}$ of modules, then C has a uniquely determined coalgebra structure such that $\varepsilon^i: B \to C$ are coalgebra maps.*

Proof. Consider the diagram
$$\begin{tikzcd} A \arrow[r, Rightarrow, "(\varepsilon^0, \cdots, \varepsilon^{n-1})"] \arrow[d, "\Delta_A"] & B \arrow[r, Rightarrow, "(\varepsilon^0, \cdots, \varepsilon^n)"] \arrow[d, "\Delta_B"] & C \arrow[d, dashed, "\Delta_C"] \\ A \otimes A \arrow[r, Rightarrow, "(\varepsilon^0 \otimes \varepsilon^0, \cdots, \varepsilon^{n-1} \otimes \varepsilon^{n-1})"] & B \otimes B \arrow[r, Rightarrow, "(\varepsilon^0 \otimes \varepsilon^0, \cdots, \varepsilon^n \otimes \varepsilon^n)"] & C \otimes C \end{tikzcd}$$
Since $\varepsilon^j \varepsilon^i = \varepsilon^i \varepsilon^{j-1}$, $(0 \leq i < j \leq n)$, $\Delta_B \varepsilon^i = (\varepsilon^i \otimes \varepsilon^i) \Delta_A$, we have
$$\begin{aligned}(\varepsilon^j \otimes \varepsilon^j) \Delta_B \varepsilon^i &= (\varepsilon^j \otimes \varepsilon^j)(\varepsilon^i \otimes \varepsilon^i) \Delta_A \\ &= (\varepsilon^i \otimes \varepsilon^i)(\varepsilon^{j-1} \otimes \varepsilon^{j-1}) \Delta_A \\ &= (\varepsilon^i \otimes \varepsilon^i) \Delta_B \varepsilon^{j-1}.\end{aligned}$$
Hence $X = C \otimes C$ and $f^i = (\varepsilon^i \otimes \varepsilon^i) \Delta_B$ satisfy the condition of Definition 2 and hence there exists a K-linear map Δ_C such that $\Delta_C \varepsilon^i = (\varepsilon^i \otimes \varepsilon^i) \Delta_B$, $(0 \leq i \leq n)$. Using these relations and $(\Delta_B \otimes 1) \Delta_B = (1 \otimes \Delta_B) \Delta_B$, we can easily check $(\Delta_C \otimes 1)$ $\Delta_C \varepsilon^i = (1 \otimes \Delta_C) \Delta_C \varepsilon^i$, $(0 \leq i \leq n)$. Hence $(\Delta_C \otimes 1) \Delta_C = (1 \otimes \Delta_C) \Delta_C$, i.e. Δ_C is coassociative. Similarly we can verify the existence of a counit $\varepsilon_C: C \to K$ with $(\varepsilon_C \otimes 1) \Delta_C = (1 \otimes \varepsilon_C) \Delta_C = 1$.
$$\begin{tikzcd} A \arrow[r, Rightarrow] \arrow[d, "\varepsilon_A"'] & B \arrow[r, Rightarrow] \arrow[d, "\varepsilon_B"] & C \arrow[d, dashed, "\varepsilon_C"] \\ K \arrow[r, Rightarrow, "(1, \cdots, 1)"] & K \arrow[r, Rightarrow, "(1, \cdots, 1)"] & K \end{tikzcd}$$
Q.E.D.

Definition 5. Let $\mathscr{A}$ be a categosy with finite colimits. A cosimplicial object A_* in $\mathscr{A}$ is said to be acyclic, if the canonical morphism

(2.2) Cosimp coker $(A_{n-2} \Rightarrow A_{n-1}) \to A_n$

is a monomorphism for every $n\geq 2$. An augmented cosimplicial object A_* over $A_{-1}=A$ is said to be acyclic if the morphism (2. 2) is a monomorphism for every $n\geq 1$ and $A\to A_0$ is a monomorphism.

Proposition 6. *If A_* is a cosimplicial module over A, then*

(2. 3) *Cosimp coker* $(A_{n-2}\Rightarrow A_{n-1})=(\sum_{i=1}^{n}\varepsilon^i A_{n-1})\oplus\mathrm{Coker}\,\tilde{d}_{n-1}$, $(n\geq 2)$.

Proof. Let $\underline{A}_n=\sum_{i=1}^{n}\varepsilon^i A_{n-1}$ and $C_n=\underline{A}_n\oplus\mathrm{Coker}\,\tilde{d}_{n-1}$. Since $1-t_n: A_n\to\underline{A}_n$ is a projection, we can define $\underline{\varepsilon}^i: A_{n-1}\to C_n$ by

$$\underline{\varepsilon}^i x=\begin{cases}(\varepsilon^i x, 0), & \text{for } i>0,\\ ((1-t_n)\varepsilon^0 x,\ pt_{n-1}x), & \text{for } i=0,\end{cases}$$

where $p:\tilde{A}_{n-1}\to\mathrm{Coker}\,\tilde{d}_{n-1}$ is the canonical projection. For $y\in\tilde{A}_{n-2}$, we have $\underline{\varepsilon}^0\varepsilon^0 y=((1-t^u)\varepsilon^0\varepsilon^0 y,\ pt_{n-1}\varepsilon^0 y)=((1-t_n)\varepsilon^1\varepsilon^0 y,\ pd_{n-1}t_{n-2}y)=(\varepsilon^1\varepsilon^0 y, 0)=\underline{\varepsilon}^1\varepsilon^0 y$. Hence $\underline{\varepsilon}^0\varepsilon^0 y=\underline{\varepsilon}^1\varepsilon^0$. The other relations $\underline{\varepsilon}^j\varepsilon^i=\underline{\varepsilon}^i\varepsilon^{j-1}$, $(i<j)$ are easily verified. Thus

$$A_{n-2}\xrightarrow{(\varepsilon^0,\cdots,\varepsilon^{n-1})}A_{n-1}\xrightarrow{(\underline{\varepsilon}^0,\cdots,\underline{\varepsilon}^n)}C_n$$

satisfies the cosimplicial relation. Suppose that

$$A_{n-2}\xrightarrow{(\varepsilon^0,\cdots,\varepsilon^{n-1})}A_{n-1}\xrightarrow{(f^0,\cdots,f^n)}X$$

satisfies the cosimplicial relations. Using the unique representation

$$y=\sum_{r-1}^{n}\ \sum_{n\geq i_1>i_2>\cdots>i_r\geq 1}\varepsilon^{i_1}\varepsilon^{i_2}\cdots\varepsilon^{i_r}y_{i_1\cdots i_r}$$

of an element of $\underline{A}_n$, we put

$$f(y)=\sum_{r-1}^{n}\ \sum_{n\geq i_1>\cdots>i_r\geq 1}f^{i_1}\varepsilon^{i_2}\cdots\varepsilon^{i_r}y_{i_1\cdots i_r}.$$

Since $(\sum_{i=0}^{n}(-1)^i f^i)d_{n-1}=0$, we can define $f(\tilde{x}_0)=\sum_{i=0}^{n}(-1)^i f^i(x_0)$ where $\tilde{x}_0$ is the canonical image of $x_0\in\tilde{A}_{n-1}$. Hence we have a linear map $f: C_n\to X$, which satisfies $f^i=f\underline{\varepsilon}^i$ $(0\leq i\leq n)$. The uniqueness of f is easily verified. Therefore $C_n=\mathrm{Cosimp\ coker}(A_{n-2}\Rightarrow A_{n-1})$. Q.E.D.

Corollary 7. *If* $C_n=\mathrm{Cosimp\ coker}(A_{n-2}\Rightarrow A_{n-1})$, *then*

$$\tilde{C}_n=\mathrm{Coker}(\tilde{d}_{n-1}:\tilde{A}_{n-2}\to\tilde{A}_{n-1}).$$

Proof. If we put

$$\underline{\delta}^i(x,y)=\delta^i x \quad\text{for } (x,y)\in\underline{A}_n\oplus\mathrm{Coker}\,d_{n-1}=C_n$$

then

$$A_{n-2}\ \underset{(\delta^0,\cdots,\delta^{n-2})}{\overset{(\varepsilon^0,\cdots,\varepsilon^{n-1})}{\rightleftarrows}}\ A_{n-1}\ \underset{(\underline{\delta}^0,\cdots,\underline{\delta}^{n-1})}{\overset{(\underline{\varepsilon}^0,\cdots,\underline{\varepsilon}^n)}{\rightleftarrows}}\ C_n$$

satisfy the cosimplicial relation (cf. Proposition 3). Put $\underline{t}_n=(1-\underline{\varepsilon}^n\underline{\delta}^{n-1})\cdots(1-\underline{\varepsilon}^1\underline{\delta}^0)$, $(n\geq 1)$, then $\underline{t}_n=0$ on $\underline{A}_n$ and $\underline{t}_n=1$ on $\mathrm{Coker}\,\tilde{d}_{n-1}$, hence $\tilde{C}_n=\underline{t}_n(\underline{A}_n\oplus\mathrm{Coker}\,\tilde{d}_{n-1})=\mathrm{Coker}\,\tilde{d}_{n-1}$. Q.E.D.

Note that the similar results also hold for augmented cosimplicial coalgebras.

Corollary 8. *If E_* is an acyclic augmented cosimplicial module, and if*

$$C_n=\text{Cosimp coker}(E_{n-2}\Rightarrow E_{n-1}),\ (n\geq 1),$$
$$\alpha: C_n\to E_n \text{ the canonical map,}$$

then there exists a K-linear map $\beta: E_n\to C_n$ such that $\underline{\delta}^i\beta=\delta^i$, $(0\leq i<n)$, and $\beta\alpha=1$.

Proposition 9. *A cosimplicial module A_* is acyclic if and only if the associated cochain complex $\tilde{A}_0\to\tilde{A}_1\to\cdots\to\tilde{A}_n\to\cdots$ is acyclic. An augmented cosimplicial module A_* over A is acyclic if and only if the associated cochain complex $0\to A\to A_0\to\tilde{A}_1\to\cdots\to\tilde{A}_n\to\cdots$ is acyclic.*

Proof. By Proposition 6 and its Corollary, $C_n=\underline{A}_n\oplus\tilde{C}_n$.
The canonical map $f_n: C_n\to A_n$ is of the form

$$1\oplus\tilde{f}_n: \underline{A}_n\oplus\tilde{C}_n\to\underline{A}_n\oplus\tilde{A}_n.$$

Hence f_n is a monomorphism if and only if so is $\tilde{f}_n$. Q.E.D.

Lemma 10. (Cosimplicial Five Lemma)
*Given an integer $n\geq 2$ and a morphism $\theta_*: E_*\to B_*$ of cosimplicial modules. Suppose that the sequence $B_{n-2}\Rightarrow B_{n-1}\Rightarrow B_n$ is acyclic, $E_n=$*Cosimp coker $(E_{n-2}\Rightarrow E_{n-1})$, *$\theta_{n-2}$ is an epimorphism and θ_{n-1} is a monomorphism, then θ_n is a monomorphism.*

$$\begin{tikzcd}
E_{n-2} \arrow[r, Rightarrow] \arrow[d, "\theta_{n-2}"] & E_{n-1} \arrow[r, Rightarrow] \arrow[d, tail, "\theta_{n-1}"] & E_n \arrow[d, tail, dashed, "\theta_n"] \\
B_{n-2} \arrow[r, Rightarrow] & B_{n-1} \arrow[r, Rightarrow] & B_n
\end{tikzcd}$$

Proof. If $x\in\operatorname{Ker}\theta_n$, then $\theta_{n-1}(\delta^i x)=\delta^i\theta_n(x)=0$ $(0\leq i<n)$. Since θ_{n-1} is a monomorphism, $\delta^i x=0$ therefore $x\in\tilde{E}_n$. It suffices to show that $\operatorname{Ker}\tilde{\theta}_n=(\operatorname{Ker}\theta_n)\cap\tilde{E}_n=0$. Since

$$\tilde{B}_{n-2}=t_{n-2}B_{n-2}=t_{n-2}\theta_{n-2}(E_{n-2})=\theta_{n-2}t_{n-2}(E_{n-2})=\theta_{n-2}(\tilde{E}_{n-2}),$$

we get a commutative diagram with exact rows.

$$\begin{tikzcd}
\tilde{E}_{n-2} \arrow[r, "\tilde{d}_{n-1}"] \arrow[d, "\tilde{\theta}_{n-2}"] & \tilde{E}_{n-1} \arrow[r, "\tilde{d}_n"] \arrow[d, tail, "\tilde{\theta}_{n-1}"] & \tilde{E}_n \arrow[r] \arrow[d, "\tilde{\theta}_n"] & 0 \\
\tilde{B}_{n-2} \arrow[r, "\tilde{d}_{n-1}"] & \tilde{B}_{n-1} \arrow[r, "\tilde{d}_n"] & \tilde{B}_n &
\end{tikzcd}$$

By the usual Five Lemma, $\tilde{\theta}_n$ is a monomorphism. Q.E.D.

Given a module M and a positive integer n, we define a cosimplicial module M_* as follows.

$$M_k=0,\ (0\leq k<n-1),$$
$$M_{n-1}=M,$$

$$M_k = \text{Cosimp coker}(M_{k-2} \Rightarrow M_{k-1}), \ (k \geq n).$$

Especially, M_n, $\tilde{M}_n$, $\varepsilon^i : M_{n-1} \Rightarrow M_n$ and $\delta^i : M_n \Rightarrow M_{n-1}$ are represented as follows.

$$M_n = M \oplus M \oplus \cdots \oplus M, \ (n+1 \text{ times}),$$
$$\varepsilon^i \text{ is the } i\text{-th injection}, \ (0 \leq i \leq n),$$
$$\delta^i(m_0, m_1, \cdots, m_n) = m_i + m_{i+1}, \ (0 \leq i \leq n-1),$$
$$M_n = \{(m, -m, m, \cdots, (-1)^n m) \mid m \in M\},$$

we have the following diagram of isomorphisms

$$\begin{tikzcd} & M \arrow[dl, "\cong"'] \arrow[dr, "\tau'"] & \\ M_{n-1} \arrow[rr, "d_n"'] & & M_n \end{tikzcd}$$

Morevoer, assume that A_* is an augmented coalgebra over a coalgebra A and M is a two sided A-comodule. Then M_k is a two-sided A_k-comodule *via* the morphism $(\varepsilon^1)^k \varepsilon : A \to A_k$. Hence the direct sum $B_* = A_* \oplus M_*$ is an augmented coalgebra over A such that $B_k = A_k * M_k$ for each k. We denote $B_* = A_* * M_*$. Since the inclusion $M = M_{n-1} \to A_{n-1} * M_{n-1} = B_{n-1}$ is a coderivation, the composite map $M \xrightarrow{\tau'} M_n \to B_n$ is also a coderivation. $\tau' : M \to B_n$ is called the canonical coderivation. If A_* is acyclic then B_* is also acylic.

Proposition 11. *If E_* is an acyclic augmented cosimplicial coalgebra over a coalgebra A, then there exists a cosimplicial coalgebra map $\theta_* : E_* \to G_* A$ over the identity of A.*

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, equal] & E_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, dashed, "\theta_0"] & E_1 \arrow[l] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[d, dashed, "\theta_1"] & E_2 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, dashed, "\theta_2"] \cdots \\ A \arrow[r, "\varepsilon"] & G_0 A \arrow[r, shift left] \arrow[r, shift right] & G_1 A \arrow[l] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & G_2 A \arrow[l, shift left] \arrow[l, shift right] \cdots \end{tikzcd}$$

Proof. We shall construct θ_n by induction on n. If $n=0$, then the assertion is obvious.

Let C_n, $\underline{\varepsilon}^i$, $\underline{\delta}^j$, α, and β be the same as in Corollary 8 and let $\underline{\theta}_n : C_n \to G_n A$ be the map with $\underline{\theta}_n \underline{\varepsilon}^i = \varepsilon^i \theta_{n-1}$, $(0 \leq i \leq n)$.

$$\begin{tikzcd} E_{n-2} \arrow[r, shift left] \arrow[r] \arrow[d, "\theta_{n-2}"] & E_{n-1} \arrow[l] \arrow[rr, shift left] \arrow[rr] \arrow[rr, shift right] \arrow[d, "\theta_{n-1}"] \arrow[dr, shift left] \arrow[dr] & & E_n \arrow[ll, shift left] \arrow[ll, shift right] \arrow[dl, "\beta"'] \arrow[dd, dashed, "\theta_n"] \\ & & C_n \arrow[ul] \arrow[ur, "\alpha"'] \arrow[dr, "\underline{\theta}_n"] & \\ G_{n-2} A \arrow[r, shift left] \arrow[r] & G_{n-1} A \arrow[l] \arrow[rr, shift left] \arrow[rr] \arrow[rr, shift right] & & G_n A \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

By the cofreeness of $G_n A$, there exists a unique coalgebra map $\theta_n : E_n \to G_n A$ such that $\eta_{n-1} \theta_n = \eta_{n-1} \underline{\theta}_n \beta$, where $\eta_{n-1} = \eta_{G_{n-1}A}$ (see p. 57). Using relations $\eta_{n-1} \varepsilon^0 = 1$, $\eta_{n-1} \varepsilon^i = \varepsilon^{i-1} \eta_{n-2}$, $(0 < i \leq n)$, $\delta^{i-1} \eta_{n-1} = \eta_{n-2} \delta^i$ $(0 < i \leq n-1)$, we can check

$\eta_{n-1}\theta_n\varepsilon^i=\eta_{n-1}\varepsilon^i\theta_{n-1}$, $(0\leq i\leq n)$. We get $\theta_n\varepsilon^i=\varepsilon^i\theta_{n-1}$, since the both side are coalgebra maps. Similarly, we have $\eta_{n-2}\delta^i\theta_n=\eta_{n-2}\theta_{n-1}\delta^i$ and hence $\delta^i\theta_n=\theta_{n-1}\delta^i$, $(0\leq i\leq n-1)$. Q.E.D.

§ 3. Interpretation of $H^2(M, A)$

In this section, A is a coalgebra and M is a two sided A-comodule.

Definition 12. An augmented cosimplicial coalgebra E_* over A with a K-isomorphism $\tau: M\to\tilde{E}_2$ is called a two term extension of A by M, if it satisfies the following conditions:

(1) E_* is acyclic,
(2) E_r is the cosimplicial cokernel of $E_{r-2}\Rightarrow E_{r-1}$, for $r\geq 2$,
(3) τ makes the following diagram commutative

$$\begin{array}{ccc}
M & \xrightarrow{\Delta_M{}^l+\Delta_M{}^l} & (A\otimes M)\oplus(M\otimes A) \\
\tau\downarrow & & \downarrow\text{cano.} \\
\tilde{E}_2 & & ((\varepsilon^1)^2A_0\otimes E_2)\oplus(E_2\otimes(\varepsilon^1)^2A_0) \\
\cap & & \cap \\
E_2 & \xrightarrow[\Delta_{E_2}]{\qquad\qquad} & E_2\otimes E_2
\end{array}$$

Definition 13. Let (E_*, τ) and (E_*', τ') be two term extensions of A by M. A morphism $\phi_*: E_*\to E_*'$ is defined to be a morphism of augmented cosimpliclal coalgebras such that $\tilde{\phi}_2\tau=\tau'$.

If there exists a sequence of morphisms of extensions

$$E_*{}^0\leftarrow E_*{}^1\rightarrow E_*{}^2\leftarrow E_*{}^3\rightarrow\cdots\leftarrow E_*{}^{2r-1}\rightarrow E_*{}^{2r} \tag{3.1}$$

then $E_*{}^0$ and $E_*{}^{2r}$ are called to be equivalent, denoted by $E_*{}^0\sim E_*{}^{2r}$.

Let A_* be an augmented cosimplicial coalgebra over A and $f: M\to A_n$ a normal coderivation n-cocycle. Denote by $B_*=A_* * M_*$ the coidealization.

Put
$$\begin{aligned} E_{n-1} &= \mathrm{Ker}(t_n\varepsilon^0p-fq) \\ &= \{a+m \mid a\in A_{n-1},\ m\in M_{n-1},\ t_n\varepsilon^0(a)=f(m)\}, \end{aligned}$$

where $p: B_{n-1}\to A_{n-1}$, $q: B_{n-1}\to M_{n-1}=M$ are the canonical projections. Since $\varepsilon^iA_{n-2}\subset E_{n-1}$, $(0\leq i\leq n-1)$, we can define

$$E_n=\text{Cosimp coker}(A_{n-2}\Rightarrow E_{n-1}).$$

We shall show E_{n-1} is a subcoalgebra of B_{n-1}. If $a\in A_{n-1}$, $m\in M_{n-1}$ and $t_n\varepsilon^0(a)=f(m)$, then

$$\Delta_{A_n}t_n\varepsilon^0(a)=\Delta_{A_n}f(m),$$
$$\Delta_{A_n}t_n\varepsilon^0(a)=((\varepsilon^1)^n\varepsilon\otimes f)\Delta^l_{M_{n-1}}(m)+(f\otimes(\varepsilon^1)^n\varepsilon)\Delta^r_{M_{n-1}}(m).$$

Operating $t_n\otimes\delta^0$ on the both side of the above equation, we get

$$(t_n\varepsilon^0\otimes 1)\Delta_{A_{n-1}}(a)=(f\otimes(\varepsilon^1)^{n-1}\varepsilon)\Delta^r_{M_{n-1}}(m),$$
$$((t_n\varepsilon^0p-fq)\otimes 1)\Delta_{B_{n-1}}(a+m)=0.$$

Hence $\Delta_{B_{n-1}}(a+m)\in E_{n-1}\otimes B_{n-1}$.

Symmetrically we get $\Delta_{B_{n-1}}(a+m)\in B_{n-1}\otimes E_{n-1}$, and hence

$$\Delta_{B_{n-1}}(a+m)\in(E_{n-1}\otimes B_{n-1})\cap(B_{n-1}\otimes E_{n-1})=E_{n-1}\otimes E_{n-1}.$$

By Proposition 4, E_n has the canonical (and unique) structure of a coalgebra. By Lemma 10, E_n is a submodule of B_n, therefore E_n is a subcoalgebra of B_n.

Define a map $\tau: M\to B_n=A_n\oplus M_n$ by

$$\tau(m)=f(m)+\tau'(m)$$

where $\tau': M\to B_n$ is the canonical coderivation. Since $d_{n+1}f(m)=0, f(m)=d_n(a)$ for some $a\in A_{n-1}$, and therefore $\tau(m)=d_n(a+m)$. Hence $\tau(M)=\operatorname{Im}\tilde{d}_n=\tilde{E}_n$.

Since $\tau': M\xrightarrow{\tau} B_n\xrightarrow{\text{proj.}} M_n$ is a monomorphism, $\tau: M\to E_n$ is a monomorphism, and therefore $\tau: M\to E$ is an isomorphism. Denoting by $E(A_*, f)$ the cosimplicial coalgebra E_* defined as above and by $\theta_*: B_*\to A_*$ the canonical projection, we obtain the following proposition.

Proposition 14. *Let A_* be an acyclic augmented cosimplicial coalgebra over A, and $f: M\to A_n$ a normal coderivation n-cocycle. Then there exists an acyclic cosimplicial coalgebra $E(A_*, f)$ over A with a normal coderivation $\tau: M\to E(A_*, f)_n$ and a morphism $\sigma_*: E(A_*, f)\to A_*$ of cosimplicial coalgebra such that σ_k is an isomorphism for $k\leq n-2, f=\sigma_n\tau$ and $\tau: M\cong E(A_*, \tilde{f})_n$.*

$$\begin{tikzcd}
E(A_*, f)_0 \arrow[r, Rightarrow] \arrow[d, equal, "\sigma_0"] & \cdots \arrow[r, Rightarrow] & E(A_*, f)_{n-2} \arrow[r, Rightarrow] \arrow[d, equal, "\sigma_{n-2}"] & E(A_*, f)_{n-1} \arrow[r, Rightarrow] \arrow[d, "\sigma_{n-1}"] & E(A_*, f)_n \arrow[d, "\sigma_n"] & \\
A_0 \arrow[r, Rightarrow] & \cdots \arrow[r, Rightarrow] & A_{n-2} \arrow[r, Rightarrow] & A_{n-1} \arrow[r, Rightarrow] & A_n & M \arrow[ul, "\tau"'] \arrow[l, "f"]
\end{tikzcd}$$

In particular, if $A_*=G_*A$ we write $E(A_*, f)=E(f)$, which we shall refer to as the standard n-term extension of A by M.

Proposition 15. *If two normal coderivation 2-cocycles f and f' are cohomologous, then $E(A_*, f)\sim E(A_*, f')$.*

$$\begin{tikzcd}
A \arrow[r, "\varepsilon"] \arrow[d, equal] & A_0 \arrow[r, shift left] \arrow[d, equal] & E(A_*, f)_1 \arrow[l, shift left] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, dashed, "\phi_1"] & E(A_*, f)_2 \arrow[l] \arrow[d, dashed, "\phi_2"] \\
A \arrow[r, "\varepsilon"] & A_0 \arrow[r, shift left] & E(A_*, f')_1 \arrow[l, shift left] \arrow[r, shift left] \arrow[r, shift right] & E(A_*, f')_2 \arrow[l]
\end{tikzcd}$$

Proof. If f and f' are cohomologous, there exists a normal coderivation $h: M\to A_1$ such that $f-f'=d_2h$, and therefore $d_2t_1(x-hm)=f(m)-d_2h(m)=f'(m)$ for $(x, m)\in E(A_*, f)_1$. We can define a linear map $\phi_1: E(A_*, f)_1\to E(A_*, f')_1$ by $\phi_1(x, m)=(x-hm, m)$. It is easy to see $\phi_1\varepsilon^i=\varepsilon^i$, $(i=0, 1)$ and $\delta^0=\delta^0\phi_1$. Next, we should prove that ϕ_1 is a coalgebra map. Let $p: A_1*M\to A_1$, $q: A_1*M\to M$ be projections, and let $i: A_1\to A_1*M$, $j: M\to A_1*M$ be inclusions. Thus $\phi_1=ip-ihq+iq$. Note that $\Delta_{A_1}h=(\varepsilon^1\varepsilon\otimes h)\Delta_M{}^l+(h\otimes\varepsilon^1\varepsilon)\Delta_M{}^r$,

$$\Delta_{E(A_*,-)_1}=\Delta_{A_1*M}=(i\otimes i)\Delta_{A_1}p+(i\varepsilon^1\varepsilon\otimes j)\Delta_M{}^l q+(j\otimes i\varepsilon^1\varepsilon)\Delta_M{}^r q.$$

By straightforward calculations, we obtain $(\psi_1\otimes\psi_1)\Delta_{E(A_*,f)_1}=\Delta_{E(A_*,f')_1}\psi_1$.

The existence of a required morphism ψ_2 follows from the universal property of cosimplicial cokernel (see proposition 4).

ψ_1 is an isomorphism with the inverse $\psi_1{}^{-1}(x, m)=(x+hm, m)$ and hence ψ_2 is also isomorphism. Especially ψ_2 induces an isomorphism of $\widetilde{E(A_*, f)}_2$ and $\widetilde{E(A_*, f')}_2$. Q.E.D.

Proposition 16. *If E_* is a two term extension of A by M, then there exists a normal coderivation 2-cocycle $f:M\to G_2A$ and a morphism of extension $\phi_*:E_*\to E(f)$.*

Proof. By Proposition 11 there is a morphism $\theta_*:E_*\to G_*A$ of augmented cosimplicial coalgebras. Using a normal coderivation 2-cocycle $\tau:M\to E_2$ in Definition 12 we put $f=\theta_2\tau$, which is also a normal coderivation 2-cocycle. Let $\sigma=\tau^{-1}t_2\varepsilon^0:E_1\to M$, then $t_2\varepsilon^0\theta_1=\theta_2 t_2\varepsilon^0=f\sigma$ and we get a linear map $\phi_1=\theta_1+\sigma:E_1\to E(f)_1\subset G_1A*M$.

It follows from Definition 12 that $(1\otimes t_2)\Delta_{E_2}\tau=(\varepsilon^0\varepsilon\otimes\tau)\Delta_M{}^l$.

$$(\varepsilon^0\otimes t_2\varepsilon^0)\Delta_{E_1}=(1\otimes t_2)\Delta_{E_2}t_2\varepsilon^0=(\varepsilon^0\varepsilon\otimes\tau)\Delta_M{}^l\sigma.$$

Operating $\delta^0\otimes\tau^{-1}$, we get $(1\otimes\sigma)\Delta_{E_1}=(\varepsilon\otimes 1)\Delta_M{}^l\sigma$. Similarly we get $(\sigma\otimes 1)\Delta_{E_1}=(1\otimes\varepsilon)\Delta_M{}^r\sigma$. Therefore

$$\begin{aligned}(\phi_1\otimes\phi_1)\Delta_{E_1}&=(\theta_1\otimes\theta_1+\theta_1\otimes\sigma+\sigma\otimes\theta_1+\sigma\otimes\sigma)\Delta_{E_1}\\&=\Delta_{G_1A}\theta_1+(\theta_1\varepsilon\otimes 1)\Delta_M{}^l\sigma+(1\otimes\theta_1\varepsilon)\Delta_M{}^r\sigma\\&=\Delta_{G_1A*M}(\theta_1+\sigma)\\&=\Delta_{E(f)_1}\phi_1.\end{aligned}$$

Namely ϕ_1 is a coalgebra map. Let $\phi_0=\theta_0$, then it is easy to see $\phi_1\varepsilon^i=\varepsilon^i\phi_0$ for $i=0, 1$ and $\delta^0\phi_1=\phi_0\delta^0$. Therefore we obtain a coalgebra map ϕ_2 such that $\phi_2\varepsilon^i=\varepsilon^i\phi_1$, $(0\leq i\leq 2)$ and $\delta^i\phi_2=\phi_1\delta^i$, $(0\leq i\leq 1)$. ϕ_2 induces an isomorphism of $\tilde{E}_2$ onto $\widetilde{E(f)}_2$ by the following commutative diagram

$$\begin{tikzcd}
& \tilde{E}_1 \arrow[r, "\phi_1"] \arrow[d, "\tilde{d}_2"] \arrow[rr, bend left, "\sigma"] & \tilde{E}(f)_1 \arrow[r, "\text{proj.}"] \arrow[d, "\tilde{d}_2"] & M \arrow[d, equal] \\
M \arrow[ur, "\sigma"] \arrow[r, "\cong", "\tau"'] & E_2 \arrow[r, "\phi_2"] & E(f)_2 \arrow[r, "\cong"] & M.
\end{tikzcd}$$

Q.E.D.

Proposition 17. *If $\phi_*:E_*\to E(f)$ and $\phi_*':E_*\to E(f')$ are morphisms from a two term extension of A by M to the standard two term extensions of A by M, then f and f' are cohomologous.*

Proof. By Proposition 16, we get the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & E_0 \arrow[r, shift left=2] \arrow[r] \arrow[d, "\phi_0"] & E_1 \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r] \arrow[d, "\phi_1"] & E_2 \arrow[l, shift left=2] \arrow[d, "\phi_2"] & \\
A \arrow[r] \arrow[d, equal] & G_0A \arrow[r, shift left=2] \arrow[r] \arrow[d, equal] & E(f)_1 \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r] \arrow[d, "\sigma_1"] & E(f)_2 \arrow[l, shift left=2] \arrow[d, "\sigma_2"] & M \arrow[ul, "\tau"'] \arrow[dl, "f"] \\
A \arrow[r] & G_0A \arrow[r, shift left=2] \arrow[r] & G_1A \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r] & G_2A \arrow[l, shift left=2] & \\
A \arrow[r] \arrow[u, equal] & G_0A \arrow[r, shift left=2] \arrow[r] \arrow[u, equal] & E(f')_1 \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r] \arrow[u, "\sigma_1'"] & E(f')_2 \arrow[l, shift left=2] \arrow[u, "\sigma_2'"] & M \arrow[ul, "f'"'] \arrow[dl, "\tau"] \\
A \arrow[r] \arrow[u, equal] & E_0 \arrow[r, shift left=2] \arrow[r] \arrow[u, "\phi_0"] & E_1 \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r] \arrow[u, "\phi_1'"] & E_2 \arrow[l, shift left=2] \arrow[u, "\phi_2'"] &
\end{tikzcd}$$

Put $\theta_i=\sigma_i\phi_i$ and $\theta_i'=\sigma_i'\phi_i'$, $(1\leq i\leq 2)$, since $E_2=$Cosimp coker $(E_0\Rightarrow E_1)$ and the morphisms

$$E_0 \xRightarrow{(\varepsilon^0,\, \varepsilon^1)} E_1 \xRightarrow{(\varepsilon^0\theta_0\delta^0,\, \theta_1',\, \theta_1)} G_1A$$

satisfies the cosimplicial relations, we have a coalgebra map $\omega: E_2\rightarrow G_1A$ such that $(\omega\varepsilon^0, \omega\varepsilon^1, \omega\varepsilon^2)=(\varepsilon^0\theta_0\delta^0, \theta_1', \theta_1)$. Put $\gamma=t_1(\theta_1'\delta^0-\delta^0\theta_2-\omega+\delta^1\theta_2)$, then $d_2\gamma\varepsilon^i=t_2(\theta_2'-\theta_2)\varepsilon^i$, $(0\leq i\leq 2)$, hence $d_2\gamma=t_2(\theta_2'-\theta_2)$ and $d_2\gamma\tau=t_2f'-t_2f=f'-f$. Since γ is a linear combination of coalgebra maps and τ is a coderivation, so $h=\gamma\tau$ is a coderivation, and since $\delta^0h=\delta^0\gamma\tau=0$, h is normal. Therefore we proved $f-f'=d_2h$. Q.E.D.

Proposition 18. *If $E(f)\sim E(f')$, then f and f' are cohomologous.*

Proof. If $E(f)\sim E(f')$, there exists a sequence (3. 1) of morphisms of extensions with $E_*{}^0=E(f)$, $E_*{}^{2r}=E(f')$. By Proposition 16 there are normal coderivation 2-cocycles f_i and morphisms of extensions $E_*{}^{2i}\rightarrow E(f_i)$, $(1\leq i\leq r)$.

$$\begin{tikzcd}[column sep=small]
& E_*{}^1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & E_*{}^{2r-1} \arrow[dl] \arrow[dr] & \\
E(f) & & E_*{}^2 \arrow[d] & & E_*{}^{2r-2} \arrow[d] & & E(f') \\
& & E(f_1) & & E(f_{r-1}) & &
\end{tikzcd}$$

Therefore f and f' are cohomologous by Proposition 17. Q.E.D.

Theorem. *Let $H^2(M, A)$ be the second cosimplicial cohomology of a coalgebra A with a coefficient comodule M and $Ex^2(M, A)$ the set of all equivalence classes of two term extensions of A by M. Then there is a bijection between $Ex^2(M, A)$ and $H^2(M, A)$.*

Proof. Let $[f]$ denote the cohomology class containing f, and $[E(f)]$ the equivalence class of $E(f)$. Then from Proposition 15 we can define a map

$$\Phi : [f]\mapsto[E(f)] : H^2(M, A)\rightarrow Ex^2(M, A).$$

By Proposition 16, we see that Φ is a surjection, and Φ is an injection by Proposition 18. Hence Φ is a bijection. Q.E.D.

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References

[1] J. Beck: Triples, Algebras and cohomology, Doctoral dissertation, Columbia University (1967).
[2] S. Eilenberg and J. C. Moore: Adjoint functors and Triples, Ill. J. Math., **9** (1965), 381-399.
[3] A. Iwai: Simplicial cohomology and n-term extensions of algebras, J. Math. Kyoto Univ., **9-3** (1969), 449-470.
[4] D. W. Jonah: Cohomology of coalgebras, Memoirs of A. M. S., **82** (1968).
[5] K. Lee: Cosimplicial cohomology of coalgebras, Nagoya Math. J., **47** (1972), 199-226.
[6] N. Shimada, H. Uehara, F. Brenneman and A. Iwai: Triple cohomology of algebras and two term extensions, Publ. RIMS Kyoto Univ., **5** (1969), 267-285.
[7] M. Tierney and W. Vogel: Simplicial derived functors, Category theory, Homology theory and their applications I, Lecture Notes in Math., **86** (1969), 167-180.