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The divisor class group of a certain Krull domain

By

Shiro GOTO

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1. Introduction.

In this paper let R be a commutative ring and let $r>t>0$ be integers. Let $S=R[\{X_{ij}\}_{1\leqq i\leqq j\leqq r}]$ be a polynomial ring and let $\mathfrak{a}$ denote the ideal of S generated by all the $(t+1)\times(t+1)$ minors of the symmetric $r\times r$ matrix $X=(X_{ij})_{1\leqq i,j\leqq r}$ where we put $X_{ij}=X_{ji}$ for $i>j$. We denote $S/\mathfrak{a}$ by A. The purpose of this paper is to give the following

Theorem. *If R is a Krull domain, then A is again a Krull domain and $C(A)=C(R)\oplus \boldsymbol{Z}/2\boldsymbol{Z}$.*

Here $C(A)$ (resp. $C(R)$) denotes the divisor class group of A (resp. R). Recall that $\mathfrak{a}$ is a prime ideal of S if R is an integral domain and that A is a Macaulay ring of $\dim A=rt-t(t-1)/2$ if R is a field (c.f. Theorem 1, [K]. Note that $\mathfrak{a}$ is a prime ideal of S even if R is not necessarily a Noetherian ring. In fact the problem can be reduced to the case where R is finitely generated over $\boldsymbol{Z}$.).

2. Proof of the theorem.

In the following we put $x_{ij}=X_{ij}$ mod $\mathfrak{a}$. For every $1\leqq i_1<i_2<\cdots<i_t\leqq r$ and $1\leqq j_1<j_2<\cdots<j_t\leqq r$, we define $x^{i_1 i_2 \cdots i_t}_{j_1 j_2 \cdots j_t}=\det(x_{i_\alpha j_\beta})_{1\leqq\alpha,\beta\leqq t}$ and let $\mathfrak{p}$ be the ideal of A generated by the set $\{x^{12\cdots t}_{j_1 j_2\cdots j_t}/1\leqq j_1<j_2<\cdots<j_t\leqq r\}$. In particular we put $d=x^{12\cdots t}_{12\cdots t}$.

Lemma 1. *Let K be a field and suppose that $Y=(y_{ij})_{1\leqq i,j\leqq r}$ is a symmetric $r\times r$ matrix with entries in K of* $\operatorname{rank} Y\leqq t$. *If* $\det(y_{ij})_{1\leqq i,j\leqq t}=0$, *then* $\operatorname{rank}(y_{ij})_{1\leqq i\leqq t,1\leqq j\leqq r}<t$.

Proof. We define $Z=(r_{ij})_{1\leqq i,j\leqq t}$ and put $s=\operatorname{rank} Z(<t)$. Since we may assume that K is an algebraically closed field, we have ${}^tPZP=\begin{pmatrix} E_s & 0 \\ 0 & 0 \end{pmatrix}$ for some invertible $t\times t$ matrix P with entries in K, where E_s denotes the $s\times s$ unit matrix. Therefore, after further suitable elementary transformations, we can assume without loss of generality that Y has the form

$$\begin{array}{c} \overbrace{\qquad\qquad\quad}^{t}\qquad \\ t\left\{\begin{pmatrix} E_s & 0 & 0 \\ 0 & 0 & F \\ 0 & {}^tF & * \end{pmatrix}\right. \end{array}.$$

If we put $u=\operatorname{rank} F$, then $s+u=\operatorname{rank}(y_{ij})_{1\leqq i\leqq t, 1\leqq j\leqq r}$ and $s+u\leqq t$ since $\operatorname{rank} Y \leqq t$. Now assume that $s+u=t$. Then $u=t-s>0$ and hence $F\neq 0$. Thus ${}^tF\neq 0$ and so $\operatorname{rank} Y>t$ — this is a contradiction. Therefore $s+u<t$.

Proposition. *Suppose that R is an integral domain. Then*
(1) *$\mathfrak{p}$ is a prime ideal of A and $\mathfrak{p}=\sqrt{dA}$.*
(2) *$A_{\mathfrak{p}}$ is a discrete valuation ring and $v_{\mathfrak{p}}(d)=2$. (Here $v_{\mathfrak{p}}$ denotes the discrete valuation corresponding to $A_{\mathfrak{p}}$.)*
(3) *$\mathfrak{p}$ is not a principal ideal.*

Proof. (1) By Theorem 1 of [**K**], it is known that $\mathfrak{p}$ is a prime ideal A. (Note that $\mathfrak{p}$ is a prime of A even if R is not necessarily a Noetherian ring. The problem can be reduced to the case where R is Noetherian.)
Let $\mathfrak{q}\in\operatorname{Spec} A$ and assume that $\mathfrak{q}\ni d$. If we denote by K the quotient field of $A/\mathfrak{q}$ and if we put $Y=(x_{ij}\bmod \mathfrak{q})_{1\leqq i,j\leqq r}$, applying Lemma 1 to this situation we have that $\mathfrak{q}$ contains $x^{1\,2\,\cdots\,t}_{j_1 j_2\cdots j_t}$ for every $1\leqq j_1<j_2<\cdots<j_t\leqq r$. Therefore $\mathfrak{q}\supset\mathfrak{p}$ and hence $\mathfrak{p}=\sqrt{dA}$.

(2) and (3). We will prove by induction on t.

($t=1$) First we will show that $A_{\mathfrak{p}}$ is a discrete valuation ring. Since $\mathfrak{p}\cap R=(0)$, $A_{\mathfrak{p}}$ contains the quotient field of R and so it suffices to prove in case R is a field. Let $T=R[X_1, X_2, \cdots, X_r]$ be a polynomial ring and let $f: S\to T$ be the R-algebra map such that $f(X_{ij})=X_iX_j$ for every $1\leqq i\leqq j\leqq r$. Then $\operatorname{Ker} f=\mathfrak{a}$ and it is well-known that $\operatorname{Im} f=R[\{X_iX_j\}_{1\leqq i\leqq j\leqq r}]$ is a Noetherian normal domain. Thus A is a Noetherian normal domain in this case. On the other hand, by [**K**], we know that $\operatorname{ht}_A\mathfrak{p}=1$ and therefore $A_{\mathfrak{p}}$ is a discrete valuation ring. (Here $\operatorname{ht}_A\mathfrak{p}$ denotes the height of $\mathfrak{p}$.)

Next we will prove that $v_{\mathfrak{p}}(d)=2$ and that $\mathfrak{p}$ is not a principal ideal. Since $d=x_{11}$ and since $x_{11}x_{ii}=x_{1i}^2$ for every $2\leqq i\leqq r$, we conclude that $v_{\mathfrak{p}}(d)=2$. (Note that $x_{ii}\notin\mathfrak{p}$ for every $2\leqq i\leqq r$.) Of course $\mathfrak{p}=(x_{11}, x_{12}, \cdots, x_{1r})$ is not a principal ideal.

($t\geqq 2$) We put $\tilde{A}=A[x_{11}^{-1}]$. Then $\mathfrak{p}\tilde{A}$ is generated by all the $(t-1)\times(t-1)$ minors of the matrix $(x_{ij}-x_{i1}x_{1j}/x_{11})_{2\leqq i\leqq t, 2\leqq j\leqq r}$. If we put $\tilde{S}=S[X_{11}^{-1}]$ and $\tilde{R}=R[\{X_{1j}\}_{1\leqq j\leqq r}, X_{11}^{-1}]$, then $\{X_{ij}-X_{i1}X_{1j}/X_{11}\}_{2\leqq i\leqq j\leqq r}$ are algebraically independent over $\tilde{R}$ and $\tilde{S}=\tilde{R}[\{X_{ij}-X_{i1}X_{1j}/X_{11}\}_{2\leqq i\leqq j\leqq r}]$. Moreover if we put $\tilde{X}=(X_{ij}-X_{i1}X_{1j}/X_{11})_{2\leqq i,j\leqq r}$, $\mathfrak{a}\tilde{S}$ coincides with the ideal of $\tilde{S}$ generated by all the $t\times t$ minors of the matrix $\tilde{X}$ and $\tilde{A}=\tilde{S}/\mathfrak{a}\tilde{S}$. Therefore, by the hypothesis of induction, we have that $\tilde{A}_{\mathfrak{p}\tilde{A}}$ is a discrete valuation ring with $v_{\mathfrak{p}\tilde{A}}(d/x_{11})=2$ and that $\mathfrak{p}\tilde{A}$ is not a principal ideal. Hence $A_{\mathfrak{p}}=\tilde{A}_{\mathfrak{p}\tilde{A}}$ is a discrete valuation ring with $v_{\mathfrak{p}}(d)=2$ since $x_{11}\notin\mathfrak{p}$. Of course $\mathfrak{p}$ is not a principal ideal.

Corollary. *d is not a zero-divisor of A. (Here R is not assumed to be an integral domain.)*

Proof. We denote A by A_0 if $R=\boldsymbol{Z}$. Since A_0 is a Macaulay ring by Theorem 1 of [**K**], we have that dA_0 is a $\mathfrak{p}$-primary ideal. Hence A_0/dA_0 is $\boldsymbol{Z}$-flat as $\mathfrak{p}\cap\boldsymbol{Z}=(0)$. For an arbitrary R, applying $R\underset{\boldsymbol{Z}}{\otimes}$ to the exact sequence $0\to A_0\xrightarrow{d}A_0\to A_0/dA_0\to 0$ we see that the sequence $0\to A\xrightarrow{d}A\to A/dA\to 0$ is also exact.

In the following we assume that R is an integral domain. We put $P=R[\{x_{ij}\}_{1\leqq i\leqq j\leqq r, 1\leqq i\leqq t}]$ in A and $B=P[d^{-1}]$.

Lemma 2. $A=A_{\mathfrak{p}}\cap B$ and $B=A[d^{-1}]$.

Proof. Let $t<i$, $j\leqq r$ be integers. As $x_{ij}=\sum_{k=1}^{t}(-1)^{k+t}x_{ik}\cdot x_{1\cdots\hat{k}\cdots tj}^{1\,2\,\cdots\,t}/d$ we have $x_{ij}\in B$. Thus $A\subset B$ and so $B=A[d^{-1}]$. Next we will prove that $A\supset A_{\mathfrak{p}}\cap B$. First we assume that R is a field. Then we know $A=\bigcap_{\mathrm{ht}_A\mathfrak{q}=1}A_{\mathfrak{q}}$ since A is a Macaulay domain by Theorem 1 of [**K**]. Let $\mathfrak{q}\in\operatorname{Spec}A$ of $\mathrm{ht}_A\mathfrak{q}=1$ and suppose that $\mathfrak{q}\neq\mathfrak{p}$. Then $\mathfrak{q}\not\ni d$ as $\mathfrak{p}=\sqrt{dA}$ and so $A_{\mathfrak{q}}\ni d^{-1}$. Thus $A_{\mathfrak{q}}\supset B=A[d^{-1}]$ and hence $A=\bigcap_{\mathrm{ht}_A\mathfrak{q}=1}A_{\mathfrak{q}}\supset A_{\mathfrak{p}}\cap B$. Now suppose that R is not necessarily a field and let $f\in A_{\mathfrak{p}}\cap B$. Then $rf\in A$ for some $r\in R-\{0\}$ by virtue of the result in case R is a field. On the other hand, since $f\in B=A[d^{-1}]$, we can express $f=g/d^s$ for some $g\in A$ and some integer $s>0$. Therefore $d^s a=rg$ in A where $a=rf$. Since $\{r, d\}$ is an A-sequence by the corollary of the above proposition, we conclude that $a\in rA$. This implies that $f\in A$.

Corollary. *P is a polynomial ring with $\{x_{ij}\}_{1\leqq i\leqq j\leqq r, 1\leqq i\leqq t}$ as indeterminates over R and d is a prime element of P.*

Proof. To prove the first assertion, we may assume that R is a field. Since P and A have the same quotient field, the transcedence degree of P over R is equal to $\dim A=rt-t(t-1)/2$. This shows the first assertion. The second one follows from the first (c.f. Theorem 1, [**K**]).

Proof of the the theorem. Because $A_{\mathfrak{p}}$ is a discrete valuation ring and $B=P[d^{-1}]$ is a Krull domain, $A=A_{\mathfrak{p}}\cap B$ is also a Krull domain. Recalling that $B=A[d^{-1}]$, we have an exact sequence $0\to\boldsymbol{Z}\mathfrak{p}\to C(A)\xrightarrow{j}C(B)\to 0$. Since $v_{\mathfrak{p}}(d)=2$ and since $\mathfrak{p}$ is not a principal ideal, $\mathfrak{p}$ has order 2 in $C(A)$. Moreover, as $B=P[d^{-1}]$ and as d is a prime element of P, we have that $C(B)=C(P)=C(R)$. Of course j is a split epimorphism.

Remark. Let R be a Noetherian normal domain and let r, s, t be integers such that $0<t<\min\{r, s\}$. Let $S=R[\{X_{ij}\}_{1\leqq j\leqq r, 1\leqq j\leqq s}]$ be a polynomial ring and let $\mathfrak{a}$ denote the ideal of S generated by all the $(t+1)\times(t+1)$ minors of the $r\times s$ matrix $X=(X_{ij})_{1\leqq i\leqq r, 1\leqq j\leqq s}$. We put $A=S/\mathfrak{a}$. Then it is known that A is again a Noetherian normal domain (c.f. M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory and the generic perfection of determinantal loci*, Amer. J. Math., **93** (1972), 1020–1058). Moreover W. Bruns gave the following remark: $C(A)=C(R)\oplus \boldsymbol{Z}$ (c.f. W. Bruns, *Die Divisorenklassengruppe der Restklassenringe von Polynomringen nach Determinantenidealen*, Rev. Roum. Math. Pures et Appl., **20** (1975), 1109–1111) and our theorem has been inspired by his work.

DEPARTMENT OF MATHEMATICS
NIHON UNIVERSITY

Reference

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