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On the 2-rank of compact connected Lie groups

By

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§ 1. Introduction

Let G be a compact cnnected Lie qroup and p be a prime. Consider the following three conditions:

(1. 1) *$H^*(G;Z)$ is p-torsion free,*

(1. 2) *$H^*(G;Z_p)$ is generated by universally transgressive elements,*

(1. 3) *$H^*(G;Z_p)$ is generated by primitive elements.*

As is well known (1. 1) is equivalent to the following (1. 1').

(1. 1') $H^*(G;Z_p)=\Lambda(x_1, \cdots, x_n)$ where deg $x_i=$ odd.

By Borel's results in [7], (1. 1') implies (1. 2) and (1. 2) implies (1. 3). Browder [16] showed that if p is an odd prime then (1. 3) implies (1. 1). On the other hand if $p=2$ then (1. 3) does not imply (1. 2) and (1. 2) does not imply (1. 1). In fact $G=SO(n)$, $n\geq 3$, satisfies (1. 2) but does not satisyfy (1. 1). If $G=Spin(2^k+1)$, $k\geq 4$, then G satisfies (1. 3) but does not satisfy (1. 2).

In [11] Borel gave a characterization of (1. 1) by making use of elementary p-grous in G. The purpose of this paper is to give a characterization of (1. 2) by making use of elementary 2-group in G. Note that if x is universally transgressive (resp. primitive) then x^2 is also universally transgressive (resp. primitive). So (1. 2) (resp. (1. 3)) is equivalent to the following (1. 2') (resp. (1. 3')).

(1. 2') *$H^*(G;Z_2)$ has a simple system of universally transgressive generators,*

(1. 3') *$H^*(G;Z_2)$ has a simple system of primitive generators.*

Note that the number of simple system is an invariant of G (cf. § 2). We denote this number by $s(G)$ (cf. § 5). Then the main theorem of this paper is the following:

Theorem 6. 1. *Let G be a compact connected Lie group. Then the following three conditions are equivalent:*

(6. 1) *$s(G)\leq l_2(G)$ where $l_2(G)$ is the 2-rank of G (cf. § 5),*

(6. 2) *$s(G)=l_2(G)$,*

(6. 3) *G satisfies (1. 2).*

To prove this theorem we use May's spectral sequence [27] (cf. § 3) and the Eilenberg-Moore spectral sequence [33], [34] (cf. § 5). We also use the result of Quillen [32].

The paper is organized as follows:

In § 2 we introdue some algebraic definitions and tools used after. In § 3 May's spectral sequence which converges to $\mathrm{Cotor}^{A}(k, k)$, where A is a Hopf algebra over a field k is considered. In § 4 $\mathrm{Cotor}^{E_0(A)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$, the E_1-term of May's spectral sequence for some Hopf algebras over $\boldsymbol{Z}_2$, is computed by making use of the method of [36]. In § 5 some topological tools are given. In § 6 the main theorem of this paper is proved. In § 7 some properties of $\boldsymbol{G}$ satisfying (1. 2) are given. The final two sections, § 8 and § 9, are applications of § 7. In § 8 cohomology mod 2 of some homogeneous spaces are given and in § 9 cohomology operations of $H^*(\boldsymbol{BF}_4;\boldsymbol{Z}_2)$ are determined.

A compact connected Lie group $\boldsymbol{G}$ satisfing (1. 2) has various good properties (cf. § 7). For examples *$H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$ is a polynomial algebra and $H^*(\boldsymbol{BV};\boldsymbol{Z}_2)$ is a free $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$-module, where $\boldsymbol{V}$ is a maximal dimensional elementary 2-group of $\boldsymbol{G}$.*

§ 2. Definitions and algebraic tools.

Let R be a field or the rational integer ring $\boldsymbol{Z}$. Let $A=\sum_{i\geq 0} A_i$ be a graded commutative R-algebra in the sence of Milnor-Moore [28]. If A is connected then A has a unique augmentation $\varepsilon: A\to R$ (see § 1 of [28]). Then we put as follows:

Definition 2. 1. $\bar{A}=\mathrm{Ker}\,\varepsilon$. $\bar{A}$ is called the augmentation ideal.

If A is of finite type and R is a field then we put as follow:

Definition 2. 2. $\mathbf{P.S.}(A;R)=\mathbf{P.S.}(A)=\sum_{i=0}^{\infty}(\mathrm{rank}_R A_i)t^i\in\boldsymbol{Z}[[t]]$.

If $\sum a_i t^i$ and $\sum b_i t^i\in\boldsymbol{Z}[[t]]$, $\sum a_i t^i\gg\sum b_i t^i$ means $a_i\geqq b_i$ for any $i\geq 0$.

If $\mathbf{P.S.}(A;R)$ is a rational function of t the following definition is due to Quillen [31] (cf. § 2 of [32] [26]).

Definition 2. 3. $\dim(A;R)=$ the order of a pole at $t=1$.

Note that if A is a finitely generated connected R algebra, $\mathbf{P.S.}(A;R)$ is a rational function of t.

Moreover $\mathbf{P.S.}(A)$ satisfies the following:

$$\mathbf{P.S.}(A)=P(t)/\sum_{i=1}^{n}(1-t^{a_i}), \tag{2. 1}$$

where $a_1, \cdots, a_n$ are positive integers and $P(t)\in\boldsymbol{Z}[t]$. (See Lemma 2. 7 of [32].)

If $\mathbf{P.S.}(A)$ and $\mathbf{P.S.}(B)$ satisfy (2. 1) and $\mathbf{P.S.}(A)\gg\mathbf{P.S.}(B)$, then $\dim(A;k)\geq\dim(B;k)$ (see P. 137 of [13]).

For the details of the following definition the reader is refered to [30] or Appendix 6 of vol. II of [38].

Definition 2. 4. A sequence of (homogeneous) elements $\{x_1, \cdots, x_n\in\bar{A}\}$

is called a regular sequence (or a prime sequence) if x_i is not a zero divisor in $A/(x_1, \cdots, x_{i-1})$ for any $i=1, 2, \cdots, n$.

Let k be a field. Let X_i, $1 \leq i \leq n+h$, and Y_j, $1 \leq j \leq n$, be indeterminants, where X_i and Y_j have positive degrees. Note that if the characteristic of k is not 2 then the degree of X_i and Y_j is even. Let $\{r_1, \cdots, r_h\}$ is a regular sequence in the graded polynomial algebra $k[X_1, \cdots, X_{n+h}]$. Put $A'=k[X_1, \cdots, X_{n+h}]/(r_1, \cdots, r_h)$ and $A=k[Y_1, \cdots, Y_n]$. Let $f: A \to A'$ be a homomorphism of graded algebras. Then we have the following:

Proposition 2. 5. *A' is a finite A-module under f if and only if $\{f(Y_1), \cdots, f(Y_n)\}$ is a regular sequence in A'. Moreover A' is a free A-module.* (Proof is given in p. 209 [30].)

Put $A''=k[X_1, \cdots, X_n]$. As a particular case of Proposition 2. 5:

Corollary 2. 6. *Let $f: A \to A''$ be a homomorphism of graded algebra then A'' is a finite A-module under f if and only if $\{f(Y_1), \cdots, f(Y_n)\}$ is a regular sequence in A. Moreover A'' is a free A-module.*
(Proof is given in p. 209 of [30] or [4].)

Let P be a commutative ring with unit.

Definition 2. 7. $\mathrm{Spec}(P) = \{\mathfrak{P}\,;$ a prime ideal of P and $\mathfrak{P} \neq P\}$.

(For details the reader is refered to chap. 2 of [15] or [19].)

Note that $(0) \in \mathrm{Spec}(P)$ if and only if P is an integral domain. In particular a polynomial algebra is an integral domain. So we have the following:

Proposition 2. 8. *If P is a polynomial algebra then* (0) *is a unique minimal prime ideal (cf. §7).*

Let $R = \boldsymbol{Z}_2$ then we use the following definition.

Definition 2. 10. A sequence of elements $\{x_1, \cdots, x_n \in \bar{A}\}$ is called a simple system of generators if $\{x_1^{\varepsilon_1} \cdots x^{\varepsilon_n}; \varepsilon_i = 0$ or $1\}$ is a module base of A.

Notation 2. 11. $A = \Delta(x_1, \cdots, x_n)$ if $\{x_1, \cdots, x_n\}$ is a simple system of generators of A.

Note that if $A = \Delta(x_1, \cdots, x_n)$, $\mathbf{P.\,S.}(A; \boldsymbol{Z}_2) = (1+t^{\deg x_1}) \cdots (1+t^{\deg x_n})$ and so $\mathbf{P.\,S.}(A; \boldsymbol{Z}_2) \in \boldsymbol{Z}[t]$.

Proposition 2. 12. *If $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ are both simple systems of generators of A, then $n=m$ and there exists $\sigma \in \Sigma_n$ such that* $\deg y_j = \deg x_{\sigma(j)}$, *$j=1, 2, \cdots, n$, where Σ_n is the symmetric group of order n.*

Proof. $\mathbf{P.\,S.}(A; \boldsymbol{Z}_2) \in \boldsymbol{Z}[t]$ and $\boldsymbol{Z}[t]$ is a unique factorization domain [38] and so the result follows. Q.E.D.

Definition 2. 13. If $A = \Delta(x_1, \cdots, x_n)$ then we put $s(A) = n$.

Note that if $A = \Lambda(x_1, \cdots, x_n)$ then $\{x_1, \cdots, x_n\}$ is a simple system of generators of A.

§ 3. May's spectral sequence.

Let (A, ϕ) be a Hopf algebra over a field k with an augmentation $\varepsilon: A \to k$. Let $I(A) = \operatorname{Ker} \varepsilon$. Then we put as follows:

Definition 3.1. $\begin{cases} F_p(A) = A & \text{if } p \geq 0 \\ F_p(A) = I(A)^{-p} & \text{if } p < 0. \end{cases}$

Then $E_0(A) = \sum F_p(A)/F_{p-1}(A)$ is also a Hopf algebra (cf. Milnor-Moore [28]). Moreover the following is known:

Proposition 3.2. *$(E_0(A), \phi_0)$ is primitively generated where ϕ_0 is the induced diagonal map $\phi_0: E_0(A) \to E_0(A \otimes A) \cong E_0(A) \otimes E_0(A)$.* (See 7.4 of Milnor-Moore [28].)

Let $C(A)$ be the cobar construction of A (cf. § 4). Then by making use of a suitable filtration of $C(A)$, May constructed a spectral sequence as follows [27]:

Theorem 3.3. *There exists a spectral sequence of algebra E_r, $r \geq 1$, satisfying the following conditions:*

i) $E_1 = \operatorname{Cotor}^{E_0(A)}(k, k)$,

ii) $E_\infty = gr\ (\operatorname{Cotor}^A(k, k))$.

For the proof see [27].

Remark 3.4. $C(A) = A \otimes T(I(A))$, *where $T(I(A))$ is the free tensor algebra over $I(A)$ (cf. § 4).*

May used a filtration induced by F_p in Definition 3.1.

In the next section we compute E_1-term of this spectral sequence.

§ 4. Primitively generated Hopf algebras over Z_2.

Let A be Hopf algebra over Z_2 with a commutative multiplication. We also assume that **P. S.** (A, Z_2) is a polynomial. Then the following is due to Milnor-Moore [28]:

Lemma 4.1. i) $A = Z_2[x_1, \cdots, x_n]/(x_1^{2^{i_1+1}}, \cdots, x_n^{2^{i_n+1}})$ *as algebra,*

ii) $E_0(A) \cong A$ *as algebra.*

To compute the E_1-term of May's spectral sequence we use the twised tensor product construction due to Brown ([17], [21] or [36]).

Let (A, ϕ) be a graded coalgebra over a field k with augmentation $\eta: k \to A$. We may consider $A = k \otimes J(A)$ where $J(A) = \operatorname{Coker} \eta$. Let L be a graded submodule of $J(A)$. Let $\iota: L \to A$ be the inclusion and $\theta: A \to L$ be a map satisfying $\theta \circ \iota = 1_L$. Let $s: L \to sL$ be the suspension. Let $\bar{\theta} = s \circ \theta$ and $\bar{\iota} = \iota \circ s^{-1}$. Let $T(sL)$ be a free tensor algebra and I be the ideal of $T(sL)$ generated by $\operatorname{Im} \psi\ (\bar{\theta} \otimes \bar{\theta})(\operatorname{Ker} \bar{\theta})$, where ψ is the multiplication of $T(sL)$. Let $\bar{X} = T(sL)/I$. Then the map $\bar{d} = -\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi \circ \bar{\iota}$ on sL define a map $\bar{d}: \bar{X} \to \bar{X}$ satisfying $\bar{d} \circ \bar{d} = 0$

(cf. § 1 of [36]). Since $\bar{d}\circ\theta+\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi=0$ holds, we now can construct the twisted tensor product construction $X=A\otimes\bar{X}$ with respect to $\bar{\theta}$. That is $X=A\otimes\bar{X}$ is an A-comodule with the differential operator

$d=1\otimes\bar{d}+(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)\circ(\phi\otimes 1)$.

If θ is the projection $A=k\oplus J(A)\to J(A)$ and $L=J(A)$ then X is $C(A)$; the cobar construction of A.

So if θ is the projection (X, d) is a quotient of the cobar construction. So if (X, d) is acyclic $H(\bar{X}, \bar{d})=\mathrm{Cotor}^A(k, k)$ as algebra (cf. [21] or [36]).

Now consider the following Hopf algebra $(A_0, \phi_0): A_0=\boldsymbol{Z}_2[x_1, \cdots, x_n]/(x_1^{2^{i_1+1}}, \cdots, x_n^{2^{i_n+1}})$ as algebra and $\bar{\phi}_0(x_i)=0$ for $i=1, 2, \cdots, n$, where $\bar{\phi}_0(x)=\phi_0(x)+x\otimes 1+1\otimes x$.

Let $L=\{x_1, x_1^2, x_1^4, \cdots, x_1^{2^{i_1}}, \cdots. x_n, x_n^2, \cdots, x_n^{2^{i_n}}\}$. $\theta: A_0\to L$ be the projection and $y_{ij}=s(x_i^{2^j})$ for $x_i^{2^j}\in L$. Then we can easily prove the following:

Lemma 4.2. *L is a simple system of primitive generators of A_0.*

Let $I=(\varepsilon_{1,0}, \varepsilon_{1,1}, \cdots, \varepsilon_{n,i_n})$ for $\varepsilon_{i,j}=0$ or 1.

Let

$x^I=x_1^{\varepsilon_{1,0}+2\varepsilon_{1,1}+\cdots+2^{i_1}\varepsilon_{1,i_1}}\cdots x_n^{\varepsilon_{n,0}+\cdots+2^{i_n}\varepsilon_{n,i_n}}$.

Let $J=(\varepsilon_1, \cdots, \varepsilon_s)$ and $J'=(\varepsilon_1', \cdots, \varepsilon_s')$ for $\varepsilon_i+\varepsilon_i'\leqq 1$ and $s=i_1+\cdots+i_n+n$. Put $J+J'=(\varepsilon_1+\varepsilon_1', \cdots, \varepsilon_s+\varepsilon_s')$. Let $|J|=\varepsilon_1+\cdots+\varepsilon_s$ then we can easily prove the following:

Lemma 4.3. $\phi_0(x^I)=\sum_{J+J'=I} x^J\otimes x^{J'}$.

So if $|I|\geq 3$, $x^J\notin L$ or $x^{J'}\notin L$. If $|I|=2$, that is $x^I=x\cdot x'$ for $x, x'\in L$, $\phi_0(x\cdot x')=x\cdot x'\otimes 1+1\otimes x\cdot x'+x'\otimes x+x\otimes x'$. Clearly $\mathrm{Ker}\,\theta$ is generated by $\{x^I; |I|\geq 2\}$. If $|I|\geq 3$, $\psi(\bar{\theta}\otimes\bar{\theta})\phi_0=0$. If $|I|=2$, $\psi(\bar{\theta}\otimes\bar{\theta})\phi_0(x\cdot x')=\psi(sx, sx')+\psi(sx', sx)=[sx, sx']$.

Clearly $\bar{d}(x)=\psi(\bar{\theta}\otimes\bar{\theta})(x\otimes 1+1\otimes x)=0$ for $x\in L$. So $X=A_0\otimes\bar{X}=A_0\otimes\boldsymbol{Z}_2[sL]$. $d(x\otimes 1)=(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)\circ(\phi_0\otimes 1)(x\otimes 1)=(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)(x\otimes 1\otimes 1+1\otimes x\otimes 1)=1\otimes s(x)$ for $x\in L$ and so $d(1\otimes s(x))=0$. Now we compare this with Koszul resolution of the exterior algebra $\Lambda(x_{1,0}, x_{1,1}, \cdots, x_{1,i_1}, \cdots, x_{n,0}, \cdots, x_{n,i_n})$. Clearly these are chain equivalent and so (X, d) is acyclic. So we have the following:

Theorem 4.4. $\mathrm{Cotor}^{A_0}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)\cong\boldsymbol{Z}_2[sL]$ *as algebra.*

So by Lemma 4.1 and Theorem 4.4 we can compute the E_1-term of May's spectral sequence for some Hopf algebra over $\boldsymbol{Z}_2$.

§ 5. Topological tools.

In this section various topological tools used after are introduced. Let $\boldsymbol{G}$ be a compact connected Lie group and p be a prime, then $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$ is a Hopf algebra over $\boldsymbol{Z}_p$. So $H^*(\boldsymbol{G}; \boldsymbol{Z}_2)$ has a simple system of generators. So we define as follows:

Definition 5.1. $s(G)=s(H^*(G;Z_2))$ (cf. §2).

Let G be a compact Lie group then the following definition is due to [13]:

Definition 5.2. $l_p(G)$ is the dimension of a maximal dimensional elementary p-group in G.

The following theorem is due to Venkov [37].

Theorem 5.3. *$H^*(BG;Z_p)$ is finitely generated.*

So $\mathrm{P.S.}(BG;Z_p)=\mathrm{P.S.}(H^*(BG;Z_p);Z_p)$ is a rational function of t. So $\dim(H^*(BG;Z_p);Z_p)<\infty$.

Theorem 5.4. $l_p(G)=\dim(H^*(BG;Z_p);Z_p)$.

The above theorem is Corollary 7.8 of Quillen [32]. (See also [31].) The part $l_p(G)\leq\dim(H^*(BG;Z_p);Z_p)$ is due to [13].

The following two theorems are also due to Quillen:

Theorem 5.5. *Let G and G' be compact Lie groups and $f:G\to G'$ be a homomorphism of Lie groups. Then $H^*(BG;Z_p)$ is a finite $H^*(BG';Z_p)$ module under f^* if and only if* $\mathrm{Ker} f$ *is a finite group and* (ord ($\mathrm{Ker} f$), p) $=1$. (See Corollary 2.4 of [32].)

Theorem 5.6. i) *Let $\mathscr{A}(G;p)$ be all conjugacy classes of elementary p-groups in G. Then the correspondence $\Phi:\mathscr{A}(G)\to\mathrm{Spec}(H^*(BG;Z_p))$ given by $\Phi(V)=\mathrm{Ker}\, i^*;H^*(BG;Z_p)\to H^*(BV;Z_p)/\sqrt{0}$ is an injection.*

ii) *$\Phi(V)\subset\Phi(V')$ if and only if there exists $g\in G$ such that $gV'g^{-1}\subset V$.*

iii) *The correspondence Φ gives a one-one correspondence between conjugacy classes of maximal elementary p-groups and minimal prime ideals of $H^*(BG;Z_p)$.*

iv) *$\mathfrak{P}\in\mathrm{Spec}(H^*(BG;Z_p))$ is contained in* $\mathrm{Im}\,\Phi$ *if and only if $\mathfrak{P}$ is homogeneous and invariant under $\mathscr{P}^i, i\geq 0$, where $\mathscr{P}^i$ is the Steenrod reduced power operation.*

Proof. Put $X=\{\text{one point}\}$ in Proposition 11.2 and Theorem 12.1 of [32]. See else [31]. Note that $H^*(BZ_2;Z_2)$ is a polynomial algebra.

Remark 5.7. i) If G is a closed subgroup of G' then by Theorem 5.5 $H_*(BG;Z_p)$ is a finite $H^*(BG;Z_p)$-module for any p.

ii) We can prove Theorem 5.3 by Theorem 5.5 as follows: Since G is a closed subgroup of $U(N)$ for sufficiently large N. So $H^*(BG;Z_p)$ is a finite $H^*(BU(N);Z_p)=Z_p[c_1,\cdots,c_N]$ module. But as is well known $H^*(BU(N);Z_p)$ is Noetherian and so is $H^*(BG;Z_p)$ (cf. [37]).

iii) The part $l_p(G)\leq\dim(BG;Z_p);Z_p)$ is also proved by Theorem 5.5.

Let G be a compactly generated topological monoid (e.g. a compact Lie group). In 1959 Eilenberg-Moore constructed a new type of spectral sequence as follows:

Theorem 5. 8. *There exists a spectral sequence of algebra* $\{E_r(\boldsymbol{G}), d_r(\boldsymbol{G})\}$ *such that*

(1) $E_2(\boldsymbol{G}) = \mathrm{Cotor}^{H^*(G;Z_p)}(\boldsymbol{Z}_p, \boldsymbol{Z}_p)$,

(2) $E_\infty(\boldsymbol{G}) = gr(H^*(\boldsymbol{BG};\boldsymbol{Z}_p))$,

(3) *Furthermore, this spectral sequence satisfies naturality for a homomorphism* $f:\boldsymbol{G}\to\boldsymbol{G}'$. (See [33] and [34].)

Remark 5. 9. The above spectral sequence is very useful for computing $H^*(\boldsymbol{BG};\boldsymbol{Z}_p)$. In fact it collapses if $\boldsymbol{G}$ satisfies (1. 2). This is an easy proof of Borel's theorem (Proposition 16. 1 of [7]) (cf. § 6).

ii) The above spectral sequence is usually called the Eilenberg-Moore spectral sequence.

§ 6. Main theorm

The purpose of this section is to prove the following theorem.

Theorem 6. 1. *Let* $\boldsymbol{G}$ *be a compact connected Lie group and* $s(\boldsymbol{G}) = n$. *Then the following three conditions are equivalent:*

(6. 1) $\quad l_2(\boldsymbol{G}) \geq s(\boldsymbol{G})$,

(6. 2) $\quad l_2(\boldsymbol{G}) = s(\boldsymbol{G})$,

(6. 3) $\quad \boldsymbol{G}$ *satisfies (1. 2).*

For the proof of this theorem we need the following Lemma 6. 2.

Lemma 6. 2. *(6. 2) is equivalent to the follwing (6. 4):*

(6. 4) May's spectral in Theorem 3. 3 collapses for $A = H^*(\boldsymbol{G};\boldsymbol{Z}_2)$ *and the Eilenberg-Moore spectral sequence collapses for* $\boldsymbol{G}$.

Proof. The part (6. 4) implies (6. 2):

Since the two spectral sequences collapse, by easy arguments $H^*(\boldsymbol{BG};\boldsymbol{Z}_2) = \boldsymbol{Z}_2[y_1, \cdots, y_n]$. Thus $l_2(\boldsymbol{G}) = \dim(H^*(\boldsymbol{BG};\boldsymbol{Z}_2);\boldsymbol{Z}_2) = s(\boldsymbol{G})$ by Theorem 5. 4.

The part (6. 2) implies (6. 4):

If the spectral sequence in Corollary 3. 4 collapses, $\mathrm{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2) = \boldsymbol{Z}_2[\bar{y}_1, \cdots, \bar{y}_n]$ where $n = s(\boldsymbol{G})$. So we only need the following Lemma 6. 3.

Lemma 6. 3. *Let* k *be a field and* $R = k[y_1, \cdots, y_n]$. *Let* $d:R\to R$ *be a derivation,* $d^2 = 0$ *and* $d \neq 0$ *then* $\mathbf{P.S.}(H(R;d)) \ll \mathbf{P.S.}(R)\cdot(1-t^a)$ *for some* $a > 0$.

Proof of Lemma 6. 2 (continued). If the spectral sequence of May does not collapse, by Lemma 6. 3 $\mathbf{P.S.}(\mathrm{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2)) \ll \mathbf{P.S.}(\boldsymbol{Z}_2[y_1, \cdots, y_n])\cdot(1-t^a)$ for some $a > 0$. So $\mathbf{P.S.}(H^*(\boldsymbol{BG};\boldsymbol{Z}_2)) \ll \mathbf{P.S.}(\boldsymbol{Z}_2[y_1, \cdots, y_n])\cdot(1-t^a)$ and so $\dim(H^*(\boldsymbol{BG};\boldsymbol{Z}_2)) < n$. If May's spectral sequence collapses and the Eilenberg-Moore spectral sequence does not collapse then also by Lemma 6. 3 $\dim(E_\infty(\boldsymbol{G});\boldsymbol{Z}_2) < n = s(\boldsymbol{G})$. Q.E.D.

Proof of Lemma 6. 3.

Since d is a derivation and $d \neq 0$ we may assume that $d(y_1) = f \neq 0$. Let

I be the ideal of R generated by f. Consider the following diagram:

$$\begin{array}{ccc} \operatorname{Ker} d & \xrightarrow{\psi} & R/I \\ {\scriptstyle p}\downarrow & & \\ \operatorname{Ker} d/\operatorname{Im} d = H(R;d), & & \end{array}$$

where ψ and p are natural projections. Note that $\operatorname{Ker}\psi = \operatorname{Ker} d \cap I$ and $\operatorname{Ker} p = \operatorname{Im} d$.

If $g \in \operatorname{Ker}\psi$ then $g = g_1 \cdot f$ for $g_1 \in R$. Then $d(g) = d(g_1)\cdot f$, since $d(f) = d^2(Y_1) = 0$. But $g \in \operatorname{Ker} d$ and R is an intergral domain, $d(g_1) = 0$. So $d(g_1 \cdot y_1) = \pm g_1 f = \pm g$ and so $\operatorname{Ker} p \supset \operatorname{Ker}\psi$. Thus $\mathbf{P.S.}(R/I) \gg \mathbf{P.S.}(H(R;d))$. But $\mathbf{P.S.}(R/I) = \mathbf{P.S.}(R)\cdot(1-t^{\deg f})$ and so the result follows. Q.E.D.

Proof of Theorem 6.1. Put $A = H^*(\boldsymbol{G};\boldsymbol{Z}_2)$.

(6.1) implies (6.2): Consider the following two spectral sequences in (6.4):

$E_1 = \boldsymbol{Z}_2[y_1, \cdots, y_n] \Rightarrow \operatorname{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$,

$E_2(\boldsymbol{G}) = \operatorname{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2) \Rightarrow H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$.

Clearly $\mathbf{P.S.}(H^*(\boldsymbol{BG};\boldsymbol{Z}_2)) \ll \mathbf{P.S.}(\boldsymbol{Z}_2[y_1, \cdots, y_n])$ so $\dim(H^*(\boldsymbol{BG};\boldsymbol{Z}_2);\boldsymbol{Z}_2) \leq n$ thus $l_2(\boldsymbol{G}) \leq n$. (6.2) clearly implies (6.1). So (6.1) and (6.2) are equivalent.

(6.3) implies (6.2):

Since $\boldsymbol{G}$ satisfies (1.3), $\operatorname{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2) = \boldsymbol{Z}_2[y_1, \cdots, y_n]$. But $\boldsymbol{G}$ satisfies (1.2) each y_i is a parmanent cycle and so $H^*(\boldsymbol{BG};\boldsymbol{Z}_2) = \boldsymbol{Z}_2[\bar{y}_1, \cdots, \bar{y}_n]$. Note that this is an easy proof of Borel's theorem (§ 9 of [9]). So we have (6.4).

Clearly (6.4) implies (6.3) and so (6.2) implies (6.3). Thus (6.2) and (6.3) are equivalent. Q.E.D.

Now we give some examples.

Let $l(\boldsymbol{G})$ be the rank of $\boldsymbol{G}$. Note that $l_p(\boldsymbol{G}) \geq l(\boldsymbol{G})$ for any p and $l_p(\boldsymbol{G}) = l(\boldsymbol{G})$ for almost any p. If $l_p(\boldsymbol{G}) > l(\boldsymbol{G})$ then $H_*(\boldsymbol{G};\boldsymbol{Z})$ has p-torsion.

Examples 6.4. i) *If $\boldsymbol{G}$ satisfies (1.1) $l_2(\boldsymbol{G}) = l(\boldsymbol{G})$. Note that any maximal elementary 2-group is contained in a maximal torus.*

ii) $\boldsymbol{G} = \boldsymbol{SO}(n)$ *satisfies* (1.2) *and* $s(\boldsymbol{SO}(n)) = l_2(\boldsymbol{SO}(n)) = n-1$,

iii) $\boldsymbol{G} = \boldsymbol{Spin}(n)$, $n \geq 10$ *does not satisfy (1.2)* (*[30]*),

iv) $\boldsymbol{G} = \boldsymbol{E}_6$ *does not satisfy (1.2) but* $l_2(\boldsymbol{E}_6) = l(\boldsymbol{E}_6)$

(*cf. Example 7.12*).

Proposition 6.5. i) $s(\boldsymbol{Spin}(n)) = (n-1) - [\log_2(n-1)]$,

ii) $l_2(\boldsymbol{Spin}(n)) = n - \log_2 R(n)$,

where $R(n)$ is the Radon-Hurwitz number (§ 6 of [30]).

Proof. i) is due to Borel [9] and ii) is due to Quillen [30]. Q.E.D.

By iii) *of Examples 6.4, $s(\boldsymbol{Spin}(n)) > l_2(\boldsymbol{Spin}(n))$ for $n \geq 10$.*

Due to Borel-Siebenthal [14] $\boldsymbol{E}_8$ contains a closed connected subgroup $\boldsymbol{H}$ of local type $\boldsymbol{D}_8$ and $\pi_1(\boldsymbol{H}) = \boldsymbol{Z}_2$. Then we can easily get the following:

Lemma 6.6. *$\boldsymbol{H}$ is $\boldsymbol{SO}(16)$ or $\boldsymbol{Ss}(16)$ ($\boldsymbol{Semi\text{-}Spin}(16)$)*

(*cf. p. 330 of [5]*).

Due to [26], $\boldsymbol{E}_8$ does not satisfy (1.2). Due to [3], $s(\boldsymbol{E}_8)=15$. So by Theorem 6.1, $l_2(\boldsymbol{E}_8)\leq 14$. Since $l_2(\boldsymbol{SO}(16))=15$, $\boldsymbol{H}$ is not $\boldsymbol{SO}(16)$.

Propositon 6.7. *$\boldsymbol{H}$ is $\boldsymbol{Ss}(16)$.*

The homogeneous space $\boldsymbol{E}_8/\boldsymbol{Ss}(16)$ is an irreducible symmetric space and denoted by **EVIII** ([18]). See also [20].

We use the following notations:

Notations 6.8.

i) $(a_1, \cdots, a_n)$ is a diagonal matrix $\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in \boldsymbol{Sp}(n)$.

ii) $\Delta(n)=\{\pm(1, \cdots, 1)\in \boldsymbol{Sp}(n)\}$.

iii) $P(n_1, \cdots, n_j)=\boldsymbol{Sp}(n_1)\times\cdots\times\boldsymbol{Sp}(n_j)/\Delta(n_1+\cdots+n_j)$.

iv) Δ_j is the j-hold diagonal map.

v) $\Delta_k\boldsymbol{Sp}(1)=\{(\alpha, \cdots, \alpha)\in \boldsymbol{Sp}(k)\,;\,\alpha\in \boldsymbol{Sp}(1)\}$.

Remark 6.9. $\Delta(n_1+\cdots+n_j)$ is contained in the center of $\boldsymbol{Sp}(n_1+\cdots+n_j)$. So $\boldsymbol{P}(n_1, \cdots, n_j)$ is a compact connected Lie group. Since $\pi_1(\boldsymbol{P}(n_1, \cdots, n_j))=\boldsymbol{Z}_2$, $\boldsymbol{P}(n_1, \cdots, n_j)$ does not satisfy (1.1).

We can construct examples satisfing (1.2) by making use of $\boldsymbol{P}(n_1, \cdots, n_j)$. Let

$$\begin{array}{ccccc} & & \Delta(n_1) & \xrightarrow{i} & \boldsymbol{Sp}(n_1) \\ & & \times & \vdots & \times \\ \Delta(n) & \xrightarrow[\bar{\Delta}_j]{} & \vdots & \vdots & \vdots \\ & & \times & \vdots & \times \\ & & \Delta(n_j) & \longrightarrow & \boldsymbol{Sp}(n_j) \end{array}$$

be the inclusion $(n=n_1+\cdots+n_j)$. Then as is well known:

(6.5) $$H^*(\boldsymbol{B}\Delta(n)\,;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[\mu],\ \deg\mu=1.$$

$$H^*(\boldsymbol{B}(\Delta(n_1)\times\cdots\times\Delta(n_j))\,;\boldsymbol{Z}_2))=\bigotimes_{k=1}^{j} H^*(\boldsymbol{B}\Delta(n_k)\,;\boldsymbol{Z}_2)$$
$$=\boldsymbol{Z}_2[t_1, \cdots, t_j],\ \deg t_k=1.$$
$$H^*(\boldsymbol{B}(\boldsymbol{Sp}(1)\times\cdots\times\boldsymbol{Sp}(1))\,;\boldsymbol{Z}_2)=\bigotimes_{j} H^*(\boldsymbol{BSp}(1)\,;\boldsymbol{Z}_2)$$
$$=\boldsymbol{Z}_2[q_{1,1}, \cdots, q_{1,j}],\ \deg q_{1,k}=4.$$

Note that $\bar{\Delta}_j{}^*(t_k)=\mu$ and $i^*(q_{1,k})=t_k{}^4$.

So we have the following:

Lemma 6.10. *If one of $\{n_1, \cdots, n_j\}$ is odd, $s(P(n_1, \cdots, n_j))=n+1$.*

Proof. Consider the Serre spectral sequence for the fibering

$$\boldsymbol{Sp}(n_1)\times\cdots\times\boldsymbol{Sp}(n_j)\to\boldsymbol{P}(n_1, \cdots, n_j)\to\boldsymbol{B}\Delta(n).$$

The above result follows from (6.5) and Lemma 10.1 of [8]. Since $\boldsymbol{V}(n)$ in [23] is contained in $\boldsymbol{P}(n_1, \cdots, n_j)$, $l_2(\boldsymbol{P}(n_1, \cdots, n_j))\geq n+1$ for any $n_1, \cdots, n_j$.

Thus we have the following:

Theorem 6.11. *If one of $\{n_1, \cdots, n_j\}$ is odd, $\boldsymbol{P}(n_1, \cdots, n_j)$ satisfies (1.2).*

Example 6.12. $H^*(\boldsymbol{BP}(1, 3)\,;\boldsymbol{Z}_2)\cong\boldsymbol{Z}_2[y_2, y_3, y_4, y_8, y_{12}]$.

Remark 6.13. $H^*(\boldsymbol{BPSp}(2n+1)\,;\boldsymbol{Z}_2)$ is determined in [23].

§ 7. Properties of compact connected Lie groups satisfing (1. 2)

Let $\boldsymbol{G}$ be a compact connected Lie group satisfing (1. 2). Then we get the following:

Theorem 7. 1. i) *Every maximal elementary 2-group is conjugate to each other.*

ii) *Let* $\boldsymbol{V}$ *be one of the maximal elementary 2-groups then the Serre spectral sequence for the fibering*

$$\boldsymbol{G}/\boldsymbol{V} \xrightarrow{p} \boldsymbol{BV} \xrightarrow{i} \boldsymbol{BG} \tag{7. 1}$$

with $\boldsymbol{Z}_2$ *coefficient collapses.*

iii) *In paticular* $i^*: H^*(\boldsymbol{BG};\boldsymbol{Z}_2) \to H^*(\boldsymbol{BV};\boldsymbol{Z}_2)$ *is injective,* $p^*: H^*(\boldsymbol{BV};\boldsymbol{Z}_2) \to H^*(\boldsymbol{G}/\boldsymbol{V};\boldsymbol{Z}_2)$ *is surjective and* $H^*(\boldsymbol{BV};\boldsymbol{Z}_2)$ *is a free* $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$*-module.*

Remark 7. 2. We can compute cohomology operation of $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$ and $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$ by iii) of Theorem 7. 1.

Proof of Theorem 7. 1.

i) Since $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$ is a polynomial algebra, Spec$(H^*(\boldsymbol{BG};\boldsymbol{Z}_2))$ has a unique maximal point (i.e. $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$ has a unique minimal prime ideal) (0). By iii) of Theorem 5. 6 every conjugacy class of maximal elementary 2-groups of $\boldsymbol{G}$ corresponds to a maximal point of Spec$(H^*(\boldsymbol{BG};\boldsymbol{Z}_2))$. So we get i) of the above theorem.

ii) Consider the principal $\boldsymbol{G}$ bundle

$$\boldsymbol{G} \xrightarrow{\rho} \boldsymbol{G}/\boldsymbol{V} \xrightarrow{i} \boldsymbol{BV}. \tag{7. 2}$$

The Serre spectral sequence for (7. 2) with $\boldsymbol{Z}_2$ coefficient has the following E_2-term:

$$E_2 \cong H^*(\boldsymbol{BV};\boldsymbol{Z}_2) \otimes H^*(\boldsymbol{G};\boldsymbol{Z}_2) = H^*(\boldsymbol{BV};\boldsymbol{Z}_2) \otimes \Delta(x_1, \cdots, x_n)$$

where $\{x_1, \cdots, x_n\}$ is the simplesystem generators of $\boldsymbol{G}$ satisfing (1. 2) and $n = l_2(\boldsymbol{G})$. Since $\{x_1, \cdots, x_n\}$ is universally transgressive, $\{x_1, \cdots, x_n\}$ is transgressive with respect to (7. 2). Put $\tau(x_i) = y_i \in H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$ then $H^*(\boldsymbol{BG};\boldsymbol{Z}_2) = \boldsymbol{Z}_2[y_1, \cdots, y_n]$. But $l_2(\boldsymbol{G}) = n$ and so $H^*(\boldsymbol{BV};\boldsymbol{Z}_2) = \boldsymbol{Z}_2[t_1, \cdots, t_n]$ where $\deg t_i = 1$. By i) of Remark 5. 7 and Corollary 2. 6, $\{i^*(y_1), \cdots, i^*(y_n)\}$ is a regular sequence. Since x_i is transgressive with $\tau'(x_i) = i^*(x_i)$ where τ' is the transgression in (7. 2), $E_\infty^{p,q} = 0$ if $q \neq 0$. So ρ^* is surjective and we get ii) of above theorem. iii) is easy. Q.E.D.

Remark 7. 3. We can also prove ii) of Theorem 7. 1 by making use of the following fact:

$\{i^*(y_1), \cdots, i^*(y_n)\}$ is a regular sequence and so $H^*(\boldsymbol{BV};\boldsymbol{Z}_2)$ is a free $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$-module (cf. Corollary 2. 6). The result follows from the following spectral sequence:

$$E_2 = \mathrm{Tor}_{H^*(\boldsymbol{BG};\boldsymbol{Z}_2)}(\boldsymbol{Z}_2, H^*(\boldsymbol{BV};\boldsymbol{Z}_2)) \Rightarrow H^*(\boldsymbol{G}/V;\boldsymbol{Z}_2) \quad \text{(cf. [4])}.$$

Example 7. 4. *Compact connected simple* $\boldsymbol{G}$ *satisfing (1. 2) is clas-*

sified in [23].

i) *If* $\boldsymbol{G}$ *satisfies* (1. 1), $\boldsymbol{V}$ *is containd in a maximal torus.*

ii) *If* $\boldsymbol{G}=\boldsymbol{SO}(n)$ $\boldsymbol{V}$ *is as follows:*

$$\boldsymbol{V}=\left\{\begin{pmatrix}\varepsilon_1 & & 0\\ & \ddots & \\ 0 & & \varepsilon_n\end{pmatrix}\in \boldsymbol{SO}(n);\ \varepsilon_i=\pm 1\right\}\cong(\boldsymbol{Z}_2)^{n-1}. \quad (\textit{See } [12].)$$

iii) *If* $\boldsymbol{G}=\boldsymbol{G}_2$, $\boldsymbol{V}$ *is given in* § *10 of* [12].

iv) *If* $\boldsymbol{G}=\boldsymbol{Spin}(7)$, $\boldsymbol{Spin}(8)$, $\boldsymbol{Spin}(9)$ *or* $\boldsymbol{F}_4$, $\boldsymbol{V}$ *is given in* [11] *and* i) *of Theorem 7. 1 is proved in* [10] *by making use of the Lefschetz fixed point theorem and in* [11].

v) $\boldsymbol{G}=\boldsymbol{PSp}(2n+1)$ *then* $\boldsymbol{V}$ *is given in* [23].

Remark 7. 5. i) The above Theorem 7. 1 is pointed out by Borel for special cases ([9]).

ii) If $\boldsymbol{G}$ satisfies (1. 1) and maximal elementary 2-groups are replaced by maximal tori then the corresponding results are well known for any prime p.

iii) If $\boldsymbol{Z}_2$ is replaced by $\boldsymbol{Q}$, and maximal elementary 2-groups are replaced by maximal tori then the corresponding results are also well known for any compact connected Lie group. (See Borel [9]. See also [1].)

Let $(\boldsymbol{G},\boldsymbol{U})$ be a pair of a compact connected Lie group and its closed subgroup. We consider the following conditions for $(\boldsymbol{G},\boldsymbol{U})$

(7. 3) $\boldsymbol{G}$ *satisfies* (1. 2) *and* $H^*(\boldsymbol{BU};\boldsymbol{Z}_2)=\boldsymbol{Z}_2[t_1,\cdots,t_{n+h}]/(r_1,\cdots,r_h)$, *where* $n=l_2(\boldsymbol{G})$ *and* $\{r_1,\cdots,r_h\}$ *is a regular sequence in* $\boldsymbol{Z}_2[t_1,\cdots,t_{n+h}]$.

Remark 7. 6. We can prove that $\dim(k[t_1,\cdots,t_{n+h}]/(r_1,\cdots,r_h);k)\leqq n$ if and only if $\{r_1,\cdots,r_h\}$ is a regular sequence in $k[t_1,\cdots,t_{n+h}]$.

Now we prove the following:

Theorem 7. 7. *If* $(\boldsymbol{G},\boldsymbol{U})$ *satisfies* (7. 3) *we have the following:*

i) *The Serre spectral sequence for the fibering*

$$\boldsymbol{G}/\boldsymbol{U}\xrightarrow{\rho}\boldsymbol{BU}\xrightarrow{i}\boldsymbol{BG} \tag{7. 4}$$

with $\boldsymbol{Z}_2$ *coefficient collapses.*

ii) *So* i^* *is injective,* ρ^* *is surjective and* $H^*(\boldsymbol{BU};\boldsymbol{Z}_2)$ *is a free* $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$*-module under* ρ^*.

Proof The proof is similar to the proof of Theorem 7. 1.

Proposition 7. 8. *If* $\boldsymbol{U}$ *is one of the following,* $(\boldsymbol{G},\boldsymbol{U})$ *satisfies* (7. 3).

i) *An extra special 2-group in* $\boldsymbol{G}$ *and* $l_2(\boldsymbol{U})=l_2(\boldsymbol{G})$,

ii) *A closed connected subgroup of* $\boldsymbol{G}$ *satisfing* (1. 2) *and* $l_2(\boldsymbol{U})=l_2(\boldsymbol{G})$.

For i) see [30] and ii) is clear.

Examples 7. 9. *The following* $(\boldsymbol{G},\boldsymbol{U})$ *satisfies* (7. 3)*:*

i) *If* $H_*(\boldsymbol{G};\boldsymbol{Z})$ *and* $H_*(\boldsymbol{U};\boldsymbol{Z})$ *are 2-torsion free and* $l(\boldsymbol{G})=l(\boldsymbol{U})$ *where* $l(\boldsymbol{G})$ *is the rank of* $\boldsymbol{G}$. *Note that in this case* $l(\boldsymbol{G})=l_2(\boldsymbol{G})$.

ii) $(\boldsymbol{F}_4, \boldsymbol{Spin}(9))$, $(\boldsymbol{F}_4, \boldsymbol{Spin}(8))$, $(\boldsymbol{Spin}(7), \boldsymbol{G}_2)$ *e.t.c.*
(See $\boldsymbol{V}$. *of Borel [8].)*

iii) $\boldsymbol{G}=\boldsymbol{Sp}(n)$ *and* $\boldsymbol{U}=\tilde{\boldsymbol{V}}(n)$ *in [23].*

Remark 7.10. Borel-Hirzebruch defined the 2-root of $\boldsymbol{G}$ by making use of $Ad|\boldsymbol{V}$, where $Ad:\boldsymbol{G}\to\mathrm{Aut}(\mathscr{G})$ is the adjoint representation of $\boldsymbol{G}$ ([12]). If $\boldsymbol{G}$ satisfies (1.2), by i) of Theorem 7.1, 2-root is an invariant of $\boldsymbol{G}$. But if $\boldsymbol{G}$ does not satisfy (1.2) we must consider all conjugacy classes of maximal elementary 2-groups.

Remark 7.11. If the pair $(\boldsymbol{G}, \boldsymbol{U})$ satisfies (7.3) we can compute the cohomology operations of $H^*(\boldsymbol{BG};\boldsymbol{Z}_2)$ by making use $\mathrm{Im}\, i^*$ and the cohomology operations of $H^*(\boldsymbol{BU};\boldsymbol{Z}_2)$.

Example 7.12. *E_6 does not satisfy (1.2) ([25]). Due to [2],* $s(\boldsymbol{E}_6)=7$ *and* $l(\boldsymbol{E}_6)=6$, so $l_2(\boldsymbol{E}_6)=6$. *Put $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ as follows:*

$$\boldsymbol{E}_6\supset\boldsymbol{F}_4\supset\boldsymbol{V}_1=(\boldsymbol{Z}_2)^5,$$
$$\boldsymbol{E}_6\supset\boldsymbol{T}^6\supset\boldsymbol{V}_2=(\boldsymbol{Z}_2)^6.$$

Then $\boldsymbol{V}_1$ is not contained in any maximal torus. So $\boldsymbol{E}_6$ has at least two different conjugacy classes of maximal elementary 2-groups.

Remark 7.13. Due to Kono-Mimura [25],

$$H^*(\boldsymbol{BE}_6;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_4, y_6, y_7, y_{10}, y_{18}, y_{32}, y_{34}, y_{48}]/I,$$

where I is the ideal generated by y_7y_{10}, y_7y_{18}, y_7y_{34} and $y_{34}^2+\cdots$, and $\deg y_i=i$. Then $\Phi(\boldsymbol{V}_1)=(y_{10}, y_{18}, y_{34})$ and $\Phi(\boldsymbol{V}_2)=(y_7)$.

§ 8. Cohomology mod 2 of some homogeneous spaces

In this section cohomology mod 2 of some homogeneous spaces are computed. In this section $\boldsymbol{G}_2$ (resp. $\boldsymbol{F}_4$) is a compact connected simple Lie group of type $\boldsymbol{G}_2$ (resp. $\boldsymbol{F}_4$). Then the following is well known ([7]):

$$(8.1)\quad \begin{cases} H^*(\boldsymbol{G}_2;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_3]/(x_3{}^4)\otimes\Lambda(x_5), \text{ where } \deg x_i=i \text{ and } Sq^2x_3=x_5,\\ \boldsymbol{G}_2 \text{ satisfies } (1.2),\\ H^*(\boldsymbol{BG}_2;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_4, y_6, y_7], \text{ where } \deg y_i=i, y_6=Sq^2y_4 \text{ and } y_7=\\ \quad Sq^3y_4.\end{cases}$$

$$(8.2)\quad \begin{cases} H^*(\boldsymbol{F}_4;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_3]/(x_3{}^4)\otimes\Lambda(x_5, x_{15}, x_{23}), \text{ where } \deg x_i=i\ \ x_5=\\ \quad Sq^2x_3 \text{ and } x_{23}=Sq^8x_{15},\\ \boldsymbol{F}_4 \text{ satisfies } (1.2),\\ H^*(\boldsymbol{BF}_4;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_4, y_6, y_7, y_{16}, y_{24}], \text{ where } \deg y_i=i, y_6=Sq^2y_4, y_7\\ \quad =Sq^3y_4 \text{ and } y_{24}=Sq^8y_{16}.\end{cases}$$

$$(8.3)\quad \begin{cases} H^*(\boldsymbol{SU}(n);\boldsymbol{Z})=\Lambda(e_3, e_5, \cdots, e_{2n-1}), \text{ where } \deg e_i=i,\\ H^*(\boldsymbol{BSU}(n);\boldsymbol{Z})=\boldsymbol{Z}[c_2, c_3, \cdots, c_n], \text{ where } \deg c_i=2i \text{ and } c_i=\tau(e_{2i-1}).\end{cases}$$

$$(8.4)\quad \begin{cases} H^*(\boldsymbol{Sp}(n);\boldsymbol{Z})=\Lambda(e_3, e_7, \cdots, e_{4n-1}), \text{ where } \deg e_i=i,\\ H^*(\boldsymbol{BSp}(n);\boldsymbol{Z})=\boldsymbol{Z}[q_1, \cdots, q_n] \text{ where } \deg q_i=4i \text{ and } q_i=\tau(e_{4i-1})\end{cases}$$

$$(8.5)\quad \begin{cases} H^*(\boldsymbol{BSO}(n);\boldsymbol{Z}_2)=\boldsymbol{Z}_2[w_2, w_3, \cdots, w_n], \text{ where } \deg w_i=i \text{ and cohomo-}\\ \quad \text{logy operations are determined by Wu's formula.}\end{cases}$$

Due to Borel-Siebenthal [14] $\boldsymbol{G}_2$ contains a closed connected subgroup $\boldsymbol{H}$ of type $\boldsymbol{A}_1\times\boldsymbol{A}_1$. The homogeneous space $\boldsymbol{G}_2/\boldsymbol{H}$ is an irreducible symmetric space and denoted by $\boldsymbol{G}$ (cf. Cartan [18]). Then we can easily get:

Lemma 8.1. *$\boldsymbol{H}$ is isomorphic to one of the following:*

(8.6) $\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1)$,

(8.7) $\boldsymbol{Sp}(1)\times\boldsymbol{SO}(3)=\boldsymbol{Ss}(4)$,

(8.8) $\boldsymbol{SO}(4)$,

(8.9) $\boldsymbol{SO}(3)\times\boldsymbol{SO}(3)$.

By the argument of 2-rank (8.9) is impossible. Since $H^*(\boldsymbol{B}(\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1));\boldsymbol{Z}_2)\cong\boldsymbol{Z}_2[q_1,q_1']$, where $\deg q_1=\deg q_1'=4$, $i^*(y_6)=i^*(y_7)=0$, where $i:\boldsymbol{H}\to\boldsymbol{G}$ is the inclusion. So it is impossible by Theorem 5.5. On the other hand if (8.7) or (8.8) is true then $(\boldsymbol{G},\boldsymbol{H})$ satisfies (7.3). So we have:

Lemma 8.2. *If (8.7) or (8.8) is true then $\{i^*(y_4),i^*(y_6),i^*(y_7)\}$ is a regular sequence.*

Note that $H^*(\boldsymbol{B}(\boldsymbol{Sp}(1)\times\boldsymbol{SO}(3));\boldsymbol{Z}_2)=H^*(\boldsymbol{BSp}(1);\boldsymbol{Z}_2)\otimes H^*(\boldsymbol{BSO}(3);\boldsymbol{Z}_2)\cong\boldsymbol{Z}_2[q_1]\otimes\boldsymbol{Z}_2[w_2,w_3]$ as algebra over $\mathscr{A}_2$. So if (8.8) is true then $i^*(y_4)=\alpha q_1+\beta w_2^2$ for $\alpha,\beta\in\boldsymbol{Z}_2$. Then $i^*(y_7)=Sq^3i^*(y_4)=0$. So it is impossible. Thus we have the following:

Lemma 8.3. *$H=\boldsymbol{SO}(4)$ as Lie group.*[i)]

By (8.1) and (8.5) $i^*(y_4)=\alpha w_4+\beta w_2^2$, $i^*(y_6)=\alpha w_4w_2+\beta w_3^2$ and $i^*(y_7)=\alpha w_4w_3$ for $\alpha,\beta\in\boldsymbol{Z}_2$. By Lemma 8.2, $\alpha,\beta=1$.

Thus by Theorem 7.7, we have the following:

Theorem 8.4. *$H^*(\boldsymbol{G};\boldsymbol{Z}_2)=\boldsymbol{Z}_2[\overline{w}_2,\overline{w}_3]/(\overline{w}_2^3+\overline{w}_3^2,\overline{w}_2^2\overline{w}_3)$, where $\overline{w}_i=\rho^*w_i$ for $\rho:\boldsymbol{G}\to\boldsymbol{BSO}(4)$, and so $Sq^1\overline{w}_2=\overline{w}_3$.*

Let $u(\boldsymbol{G})$ be the total *Wu* class of $\boldsymbol{G}$ and $W(\boldsymbol{G})$ be the total Stiefel-Whitney class of $\boldsymbol{G}$. Then by *Wu's* formula [29], we have the following:

Theorem 8.5. i) $u(\boldsymbol{G})=1+\overline{w}_2+\overline{w}_3+\overline{w}_2^2$,
ii) $W(\boldsymbol{G})=1+\overline{w}_2^2+\overline{w}_3^2+\overline{w}_2^4$.

Remark 8.6. The above Theorem 8.4 and ii) of Theorem 8.5 are proved in [12] by making use of Caylay number and the 2-root of $\boldsymbol{G}_2$.

Due to Borel-Siebenthal [14] $\boldsymbol{F}_4$ contains a closed connected subgroup, $\boldsymbol{K}$ of local type $\boldsymbol{A}_1\times\boldsymbol{C}_3$. The homogeneous space $\boldsymbol{F}_4/\boldsymbol{K}$ is an irreducible symmetric space denoted by $\boldsymbol{FI}$ [18]. By the similar argument we have:

Theorem 8.7. i) *$\boldsymbol{K}=\boldsymbol{P}(1,3)=\boldsymbol{Sp}(1)\cdot\boldsymbol{Sp}(3)$ for $\boldsymbol{Sp}(1)\cap\boldsymbol{Sp}(3)=\boldsymbol{Z}_2$,*
ii) $\mathrm{P.S.}(\boldsymbol{FI})=\mathrm{P.S.}(\boldsymbol{G})\cdot(1+t^8)\cdot(1+t^{12})$.

Remark 8.8. To determine $H^*(\boldsymbol{FI};\boldsymbol{Z}_2)$, $u(\boldsymbol{FI})$, and $W(\boldsymbol{FI})$ we need cohomology operations of $H^*(\boldsymbol{BK};\boldsymbol{Z}_2)$.

i) Due to Baum-Browder (p. 330 of [5]) $\boldsymbol{Ss}(4)=\boldsymbol{SO}(4)$ as Lie group. But it is not true.

Now we consider an example of different type. $\boldsymbol{F}(n)=\boldsymbol{Sp}(n)/\boldsymbol{SU}(n)$, $n\geqq 2$. Then $\boldsymbol{Sp}(n)$ and $\boldsymbol{SU}(n)$ satisfy (1.2) but $(\boldsymbol{Sp}(n), \boldsymbol{SU}(n))$ does not satisfy (7.3), since $l_2(\boldsymbol{Sp}(n))=n$ and $l_2(\boldsymbol{SU}(n))=n-1$. Let $i:\boldsymbol{SU}(n)\to\boldsymbol{Sp}(n)$ be the inclusion. Note that

(8.10) $$i^*(q_i)=(-1)^i\sum_{j+k=2i}(-1)^j c_j c_k \ \textit{for}\ c_1=0 \ ([7]).$$

So $i^*(q_1)=-2c_2$. By the Serre exact sequence [35] for the fibering $\boldsymbol{F}(n)\to\boldsymbol{BSU}(n)\to\boldsymbol{BSp}(n)$, we have the following

Lemma 8.9.
$$H^i(\boldsymbol{F}(n);\boldsymbol{Z})\cong\begin{cases}0, & 0<i\leq 3\\ \boldsymbol{Z}_2, & i=4.\end{cases}$$

Now we consider the Serre spectral sequence for the fibering

(8.12) $$\boldsymbol{Sp}(n)\to\boldsymbol{F}(n)\to\boldsymbol{BSU}(n)$$

with $\boldsymbol{Z}_2$ coefficient.
$$E_2=\boldsymbol{Z}_2[c_2, c_3, \cdots, c_n]\otimes\Lambda(e_3, e_7, \cdots, e_{4n-1}),$$
where c_i and e_{4j-1} are the mod 2-reduction of c_i and e_{4j-1}. Since e_{4j-1} is universally transgressive, e_{4j-1} is transgressive with respect to this fibering. $\tau(e_{4j-1})=i^*(q_j)=c_j^2$ by the mod 2 reduction of (8.10) for $j=2, 3, \cdots, n$, and $\tau(e_3)=0$. Clearly $\{\tau(e_7), \cdots, \tau(e_{4n-1})\}$ is a regular sequence. Thus we have:
$$E_\infty=\Lambda(e_3, c_2, c_3, \cdots, c_n).$$
Thus $H^*(\boldsymbol{F}(n);\boldsymbol{Z}_2)$ is generated by $\bar{e}_3\in H^3(\boldsymbol{F}(n);\boldsymbol{Z}_2)$ and $\bar{c}_i=\rho^*c_i$, $2\leq i\leq n$. Clearly $\bar{c}_i^2=0$.

Since $H^4(\boldsymbol{F}(n);\boldsymbol{Z})=\boldsymbol{Z}_2$, $Sq^1\bar{e}_3\neq 0$. But $H^4(\boldsymbol{F}(n);\boldsymbol{Z}_2)=\boldsymbol{Z}_2$ generated by $\bar{c}_2$ so $Sq^1\bar{e}_3=\bar{c}_2$. By the dimensional reason $Sq^2\bar{e}_3=0$ since $H^5(\boldsymbol{F}(n);\boldsymbol{Z}_2)=0$. Thus $\bar{e}_3^2=Sq^3\bar{e}_3=Sq^1Sq^2\bar{e}_3=0$. Now we have the following:

Theorm 8.10. $H^*(\boldsymbol{F}(n);\boldsymbol{Z}_2)=\Lambda(\bar{e}_3, \bar{c}_2, \bar{c}_3, \cdots, \bar{c}_n)$.
Cohomology operations are computed by Wu's formula and $Sq^1\bar{e}_3=\bar{c}_2$ *and* $Sq^2\bar{e}_3=0$.

Remark 8.11. Note that ρ^* is not surjective. We can also compute $H^*(\boldsymbol{F}(n);\boldsymbol{Z}_2)$ by making use of the spectral sequence in Remark 7.3. In fact this spectral sequence collapses (see Baum [4]).

§ 9. Cohomology operations of $H^*(\boldsymbol{BF}_4;\boldsymbol{Z}_2)$

The purpose of this section is to determine the cohomology operations of $H^*(\boldsymbol{BF}_4;\boldsymbol{Z}_2)$. Since the pair $(\boldsymbol{F}_4, \boldsymbol{Spin}(9))$ satifies (7.3) we can use Remark 7.11.

First we determine the cohomology operations of $H^*(\boldsymbol{BSpin}(9);\boldsymbol{Z}_2)$. Let $\pi:\boldsymbol{Spin}(n)\to\boldsymbol{SO}(n)$ be the covering projection and $\Delta:\boldsymbol{Spin}(n)\to\boldsymbol{O}(2^h)$ be the spin representation, where 2^h is the Radon-Hurwitz number (Quillen [30]).

Then the following is due to Quillen [30].

Theorem 9.1. i) $H^*(\boldsymbol{BSpin}(n);\boldsymbol{Z}_2)=\mathrm{Im}\,\pi^*\otimes\boldsymbol{Z}_2[e_{2^h}]$, *where* $e_{2^h}=e=w_{2^h}(\Delta)$.

ii) $w_i(\Delta) \neq 0$ *if and only if* $i = 0, 2^h - 2^{h-1}, 2^h - 2^{h-2}, \cdots, 2^h - 2^r, 2^h$, *where* $r = 0, 1$ *or* 2.

iii) $\{w_{2^h}(\Delta), w_{2^h - 2^{h-1}}(\Delta), \cdots, w_{2^h - 2^r}(\Delta), w_{2^h}(\Delta)\}$ *is a regular sequence in* $H^*(\boldsymbol{BSpin}(n); \boldsymbol{Z}_2)$.

For details see § 6 of [30].

The following Theorem 9. 2 is easily proved:

Theorem 9. 2. $Sq^1(e) = w_i(\Delta) \cdot e$. *So* $Sq^i(e) \neq 0$ *if and only if* $i = 0$, $2^h - 2^{h-1}, \cdots, 2^h - 2^r, 2^h$.

Proof. Since $H^*(\boldsymbol{BO}(2^h); \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[w_1', \cdots, w'_{2^h}]$, where w'_i is the i-th universal Stiefel-Whitney class. By Wu's formula [8], $Sq^i(w'_{2^h}) = w'_i w'_{2^h}$. But $e = \Delta^*(w'_{2^h})$ and $w_i(\Delta) = \Delta^* w_i'$. So $Sq^i(e) = Sq^i(\Delta^* w'_{2^h}) = \Delta^* Sq^i w'_{2^h} = \Delta^*(w'_i w'_{2^h}) = (\Delta^* w'_i)(\Delta^* w'_{2^h}) = w_i(\Delta) e$. Q.E.D.

Pemark 9. 3. If $x \in \operatorname{Im} \pi^*$ then we can compute $Sq^i(x)$ by Wu's formula.

Example 9. 4. $n = 8$ then $h = 3$ and $r = 0$.

$H^*(\boldsymbol{BSpin}(8); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_4, w_6, w_7, w_8, e_8]$, where w_i is the π^* image of the i-th universal Stiefel Whitney class in $H^*(\boldsymbol{BSO}(8); \boldsymbol{Z}_2)$ and $\deg e_8 = 8$. Since $w_4(\Delta) \neq 0$ so $w_4(\Delta) = w_4$ and so we have the following:

$$w_0(\Delta) = 1,\ w_4(\Delta) = w_4,\ w_6(\Delta) = w_6,\ w_7(\Delta) = w_7 \ \textit{and}\ w_8(\Delta) = e_8.$$

Example 9. 5. $n = 9$ then $h = 4$ and $r = 0$.

$H^*(\boldsymbol{BSpin}(9); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_4, w_6, w_7, w_8, e_{16}]$, where w_i, e_{16} are as above.

We may put $w_8(\Delta) = \alpha w_8 + \beta w_4^2$ for $\alpha, \beta \in \boldsymbol{Z}_2$.

Then
$$w_{12}(\Delta) = \alpha w_8 w_4 + \beta w_6^2,$$
$$w_{14}(\Delta) = \alpha w_8 w_6 + \beta w_7^2,$$
and
$$w_{15}(\Delta) = \alpha w_8 w_7.$$

By iii) of Theorem 9. 1, $\alpha = \beta = 1$ and so we have the following:

In $H^*(\boldsymbol{BSpin}(9); \boldsymbol{Z}_2)$

$$Sq^i(e_{16}) = \begin{cases} e_{16} & i = 0 \\ e_{16}(w_8 + w_4^2) & i = 8 \\ e_{16}(w_8 w_4 + w_6^2) & i = 12 \\ e_{16}(w_8 w_6 + w_7^2) & i = 14 \\ e_{16} w_8 w_7 & i = 15 \\ e_{16}^2 & i = 16 \\ 0 & \textit{others}. \end{cases} \tag{9.1}$$

Let $i: \boldsymbol{Spin}(9) \to \boldsymbol{F}_4$ be the inclusion. Then the following is well known:

Lemma 9. 6. $H^*(\boldsymbol{BF}_4; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[y_4, y_6, y_7, y_{16}, y'_{24}]$ *where* $\deg y_i^{(\prime)} = i$, $y_6 = Sq^2 y_4$, $y_7 = Sq^3 y_4$ *and* $y'_{24} = Sq^8 y_{16}$.

By ii) of Theorem 7. 7, i^* is injective. So $i^*(y_4) = w_4$ and so $i^*(y_6) = w_6$ and $i^*(y_7) = w_7$. Thus we may assume that $i^*(y^{16}) = \alpha e_{16} + \beta w_8^2 + \gamma w_8 w_4^2$ for

$\alpha, \beta, \gamma \in \boldsymbol{Z}_2$.

Since $Sq^4 y_{16}$ is decomposable, $i^*(Sq^4 y_{16}) = Sq^4 i^*(y_{16}) = \gamma w_8 w_4^3 + \gamma w_8 w_6^2$ is also decomposable in $\mathrm{Im}\, i^*$. So $\gamma = 0$. On the other hand $i^*(y'_{24}) = \alpha e_{16}(w_8 + w_4^2) + \beta w_8^2 w_4^2$. By i) of Remark 5.7 and Corollary 2.6, $\{i^*(y_{16}), i^*(y'_{24})\}$ is a regular sequence. So $\alpha = \beta = 1$.

Definition 9.7. Put $y_{24} = y'_{24} + y_{16} y_4^2$.

Then $i^*(y_{24}) = e_{16} w_8 + e_{16} w_4^2 + w_8^2 w_4^2 + e_{16} w_4^2 + w_8^2 w_4^2 = e_{16} w_8$.

Now we can easily get the following:

Theorem 9.10. *($Sq^i y_{16}$ and $Sq^i y_{24}$.)*

i	$Sq^i y_{16}$	$Sq^i y_{24}$
0	y_{16}	y_{24}
1	0	0
2	0	0
3	0	0
4	0	$y_{24} y_4$
5	0	0
6	0	$y_{24} y_6$
7	0	$y_{24} y_7$
8	$y_{24} + y_{16} y_4^2$	$y_{24} y_4^2$
9	0	0
10	0	0
11	0	0
12	$y_{24} y_4 + y_{16} y_4^2$	$y_{24}(y_6^2 + y_4^3)$
13	0	0
14	$y_{24} y_6 + y_{16} y_7^2$	$y_{24}(y_7^2 + y_6 y_4^2)$
15	$y_{24} y_7$	$y_{24} y_7 y_4^2$
16	y_{16}^2	$y_{24}(y_{16} + y_6^2 y_4)$
17	0	0
18	0	$y_{24}(y_6^3 + y_7^2 y_4)$
19	0	$y_{24} y_6^2 y_7$
20	0	$y_{24}(y_{16} y_4 + y_7^2 y_6)$
21	0	$y_{24} y_7^3$
22	0	$y_{24} y_{16} y_6$
23	0	$y_{24} y_{16} y_7$
24	0	y_{24}^2

Proof. $i^*(Sq^4 y_{24}) = Sq^4(e_{16} w_8) = e_{16} w_8 w_4$.

$$i^*(y_{24} y_4) = e_{16} w_8 w_4.$$

Since i^* is injective, the result follows. Q.E.D.

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