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Graded factorial rings of dimension 3 of a restricted type

By

Masa-Nori ISHIDA

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§1. Introduction

We fix an algebraically closed field k of an arbitrary characteristic in this paper.

It is shown in [1] that if X is a normal projective variety over k such that $\mathrm{Cl}\, X \simeq \boldsymbol{Z}$, then $R(X^{\circ}, L)=\oplus_{i=0}^{\infty} H^{0}(X^{\circ}, L^{\otimes i})$ is a graded factorial ring, where X° is the open subvariety of X consisting of smooth points, L is the ample generator of $\mathrm{Pic}\, X^{\circ} \simeq \mathrm{Cl}\, X \simeq \boldsymbol{Z}$ and $\mathrm{Cl}\, X$ is the group of linear equivalence classes of Weil divisors. Graded k-algebras isomorphic to such ones are called geometric graded factorial rings. All the graded factorial k-algebras A with non-negative degrees such that $k \subsetneq A_0$ are completely classified once geometric factorial rings are classified [1]. The classification of such graded k-algebras of dimension ≤ 2 is given in [1]. However in the case of dimension 3, it seems rather difficult to classify geometric graded factorial rings because even non-singular projective surfaces X over k such that $\mathrm{Pic}\, X \simeq \boldsymbol{Z}$ are not classified at all. In view of this fact, it seems worthwhile to restrict ourselves to normal projective surfaces X such that $\mathrm{Cl}\, X \simeq \boldsymbol{Z}$ which admits a non-trivial action of G_m. This is equivalent to considering geometric graded factorial rings of dimension 3 which admit a nondegenerate bigradation (for the definition, see §2).

Our main result is the following.

Theorem 1.1. *Let R be a geometric graded factorial ring of dimension 3 which admits a nondegenerate bigradation. Then there exists one and only one index Φ (see below) such that R is isomorphic to R_{Φ} given in the following examples.*

Example I. Let $\Phi=(e_1, e_2, e_3)$ be a triple of pairwise relatively prime positive integers with $e_1 \geq e_2 \geq e_3$. Then

$$R_{\Phi}=k[x_1, x_2, x_3] \quad \deg x_i=e_i \quad (i=1, 2, 3),$$

is a geometric graded factorial ring which admits a nondegenerate bigradation.

Example II. Let $r\geq 3$, $e_1>\cdots>e_r>1$ and m be positive integers such that $e_1, \cdots, e_r$ are pairwise relatively prime and $(e, m)=1$ where $e=\prod_{i=1}^{r}e_i$. Let $a_3=1, \cdots, a_r$ be mutually distinct elements of k^*. Then for $\Phi=(\boldsymbol{e}, \boldsymbol{a}, m)$, $\boldsymbol{e}=(e_1, \cdots, e_r)$ and $\boldsymbol{a}=(a_3, \cdots, a_r)$,

$$R_\Phi=k[x_1, \cdots, x_r, u]/I,$$

where $$I=(x_1^{e_1}+a_3x_2^{e_2}+x_3^{e_3}, \cdots, x_1^{e_1}+a_rx_2^{e_2}+x_r^{e_r}),$$

$$\deg x_i=e/e_i \quad (i=1, \cdots, r), \text{ and } \deg u=m,$$

is a geometric graded factorial ring which admits a nondegenerate bigradation.

Example III. Let $r\geq 2$, $e_1>\cdots>e_r>1$, $c\leq d$, l, m and p be positive integers such that

1) $(e_i, e_j)=1$ $(1\leq i<j\leq r)$, $(c, d, e)=1$ where $e=\prod_{i=1}^{r}e_i$,
2) l, m, p are prime to each other, moreover $l\leq m$ if $c=d$.
3) $lc+md=pe$.

Let $b_3=1, \cdots, b_r, b$ be mutually distinct elements of k^* if $r\geq 3$ and $b=1$ if $r=2$. Then for $\Phi=(\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{c})$, $\boldsymbol{e}=(e_1, \cdots, e_r)$, $\boldsymbol{b}=(b_3, \cdots, b_r, b)$ and $\boldsymbol{c}=(c, d, l, m, p)$,

$$R_\Phi=k[x_1, \cdots, x_r, u, v]/I$$

is a geometric graded factorial ring which admits a nondegenerate bigradation, where $I=(x_1^{e_1}+b_3x_2^{e_2}+x_3^{e_3}, \cdots, x_1^{e_1}+b_rx_2^{e_2}+x_r^{e_r}, x_1^{e_1}+bx_2^{e_2}+u^cv^d)$ if $r\geq 3$, $I=(x_1^{e_1}+x_2^{e_2}+u^cv^d)$ if $r=2$, $\deg x_i=pe/e_i$ $(i=1,\cdots,r)$, $\deg u=l$ and $\deg v=m$.

(The factoriality of the rings given in Example II and the geometricity of the rings given in Examples I, II and III follow easily from the results of [1]. The factoriality of the rings given in Example III is shown in §2.) We introduce a bigradation on the rings given above in §2. We study G_m-surfaces in §3 as a preliminary to later sections. In §4, we consider an arbitrary normal projective surface X with a nontrivial G_m-action (called a G_m-surface) such that $\mathrm{Cl}\,X\simeq\boldsymbol{Z}$. We take a desingularization $g:\tilde{X}\to X$ such that $\pi:\tilde{X}\to\boldsymbol{P}^1$ is a morphism ($\boldsymbol{P}^1$ is the projective line whose function field is $k(X)^{G_m}$). Then it is proved that the number i of exceptional curves for g which are not contained in the fibres of π is 1 or 2. The cases $i=1$ and 2 are treated as Cases 1 and 2, respectively. In §5, we shall prove that in Case 1 $R(X^\circ, L)$ is isomorphic to a graded ring given in Example I or II. In §6, it is proved that in Case 2 $R(X^\circ, L)$ is isomorphic to a graded ring given in Example I or III. Thus the existence of Φ in Theorem 1.1 is proved. §7 is devoted to the proof of the uniqueness of Φ.

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Notation and terminology. For an integral domain A, we denote by $Q(A)$ the quotient field of A. For a graded integral domain $A=\bigoplus_{i\in\boldsymbol{Z}}A_i$, we denote by $QH(A)$ the quotient ring $S^{-1}A$, where $S=\cup_{i\in Z}\{A_i-\{0\}\}$. Then $QH(A)$ has a natural gradation induced by A. All homomorphisms between graded rings are assumed to preserve gradation.

§2. Bigraded factorial rings

In this section, by a graded ring, we understand an almost geometric graded ring over k defined in [1].

Definition 2.1. Let $R=\oplus_{i\geq 0}R_i$ be a graded ring, and $R_{i,j}$ k-submodules of R $(i,j\in \boldsymbol{Z}, i\geq 0)$, and $R_i=\oplus_{j\in Z}R_{i,j}$. Then we call $R=\oplus_{i,j\in Z}R_{i,j}$ a bigraded ring if $R_{i,j}R_{i',j'}\subset R_{i+i',j+j'}$ for all $i, i', j, j'\in \boldsymbol{Z}$ $i, i'\geq 0$ and $R_{0,0}=k$.

For a bigraded ring R, we define a dual action of G_m, $\mu: R\to k[t, t^{-1}]\otimes_k R$ by $\mu(a)=t^j\otimes a$ $(a\in R_{i,j})$. This dual action of G_m on the graded ring R induces a G_m-action on $\mathrm{Proj}\,R$.

Lemma 2.2. *If a bigraded ring R is an integral domain and $R_{i,j}R_{i,j'}\neq 0$ for some i, j, j' $(j\neq j')$, then the action of G_m on* Proj R *is nontrivial.*

Proof. Take non-zero elements $f\in R_{i,j}$ and $g\in R_{i,j'}$. Then the open subscheme $D(f)=\mathrm{Spec}(R[f^{-1}])_0$ is G_m-stable, but the induced dual action sends $g/f\in R[f^{-1}]_0$ to $t^{j'-j}\otimes g/f$. Hence the G_m-action on $D(f)$ is nontrivial, and thus the action on $\mathrm{Proj}\,R$ is nontrivial. q.e.d.

Definition 2.3. We call a bigraded ring R nondegenerate, if there exist integers i, i', j, j' such that $i, i'\geq 0$, $ij'-i'j\neq 0$ and $R_{i,j}$, $R_{i',j'}\neq 0$.

It is easy to see that if a nondegenerate bigraded ring R is an integral domain, R satisfies the condition of lemma 2.2 for the integers ii', ij', and $i'j$. Thus the natural G_m-action on $\mathrm{Proj}\,R$ (introduced above) is nontrivial.

In the remainder of this section, we show that the examples in §1 are factorial nondegenerate bigraded rings.

Example I: If we define the bigradation of the graded ring $R_\Phi=k[x_1, x_2, x_3]$ by $\deg x_1=(e_1, 0)$, $\deg x_2=(e_2, 0)$, and $\deg x_3=(e_3, 1)$, then R_Φ becomes a nondegenerate bigraded ring. Thus $\mathrm{Proj}\,R_\Phi$ has a nontrivial G_m-action. In fact, $\mathrm{Proj}\,R_\Phi$ is a torus embedding of dimension 2.

Example II: If we define the bigradation on R_Φ $(\Phi=(\boldsymbol{e}, \boldsymbol{a}, m))$ by $\deg x_i=(e/e_i, 0)$ $(i=1, \cdots, r)$ $\deg u=(m, 1)$, then R_Φ becomes a nondegenerate bigraded ring.

Example III: If we define the bigradation on R_Φ $(\Phi=(\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{c}))$ by $\deg x_i=(pe/e_i, 0)$ $(i=1, \cdots, r)$, $\deg u=(l, -d)$, $\deg v=(m, c)$, then R_Φ becomes a nondegenerate bigraded ring. The factoriality of R_Φ follows from the following theorem.

Theorem 2.4. *Let r (>1) be an integer and $e_1>\cdots>e_r>1$ be positive integers such that $(e_i, e_j)=1$ for all i, j $(1\leq i<j\leq r)$. Let $c\geq d$ be positive*

integers such that $(c, d, e)=1$ *where* $e=\prod_{i=1}^{r}e_i$, *and let* $a_2=1, \cdots, a_r$ *be mutually distinct elements of* k^*. *Then the ring* $R=k[x_1, \cdots, x_r, u, v]/I$, *where* $I=(x_1^{e_1}+x_2^{e_2}+a_2u^cv^d, \cdots, x_1^{e_1}+x_r^{e_r}+a_ru^cv^d)$, *is a factorial ring.*

Proof. By $(c, d, e)=1$ there exist positive integers $\bar{l}$, $\bar{m}$ such that $(\bar{l}, \bar{m})=1$ and $(\bar{l}c+\bar{m}d, e)=1$. We consider the graded ring $S=k[x_1, u, v]$, where $\deg x_1=\bar{l}c+\bar{m}d$, $\deg u=\bar{l}e_1$, $\deg v=\bar{m}e_1$. Then since S is a graded factorial ring, it follows from Theorem 4.1 in [1] that R is also a graded factorial ring. In fact, it is easy to see that $\boldsymbol{e}=(e_2, \cdots, e_r)$ and $\boldsymbol{v}=(-x_1^{e_1}-a_2u^cv^d, \cdots, -x_1^{e_1}-a_ru^cv^d)$ form a ramification data defined in [1, §4]. Hence $S[\boldsymbol{v}^{1/e}]$ is a graded factorial ring by Theorem 4.1 in [1]. Thus $R=S[\boldsymbol{v}^{1/e}]$ is factorial. q.e.d.

It is easy to see that R_Φ ($\Phi=(\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{c})$) is isomorphic to a ring given in Theorem 2.4.

§3. G_m-surfaces

In this section, we shall study some properties of surfaces with a G_m-action. The results we obtain are essentially the same as in [4], [5].

Definition 3.1. If X is a surface and $\mu: G_m\times X\to X$ is a nontrivial G_m-action, we call $\{X, \mu\}$ (or, simply, X) a G_m-surface. Let $\{X, \mu\}$, $\{Y, \nu\}$ be G_m-surfaces. We call a morphism $f: X\to Y$ a G_m-morphism if f makes the following diagram commute.

$$\begin{array}{ccc} G_m\times X & \xrightarrow{1_{G_m}\times f} & G_m\times Y \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} \\ X & \xrightarrow{f} & Y \end{array} .$$

First, we shall study normal affine G_m-surface.

Let $\{X, \mu\}$ be a normal affine G_m-surface and $X=\operatorname{Spec} A$. Then there is a gradation $A=\bigoplus_{i\in \boldsymbol{Z}}A_i$ such that the dual action $\tilde{\mu}: A\to k[t, t^{-1}]\otimes_k A$ sends $a\in A_i$ to $t^i\otimes a$ for all $i\in\boldsymbol{Z}$ [SGAD]. Since A is an integral domain, $QH(A)=K[z, z^{-1}]$, where K is the field $QH(A)_0$ and z is a homogeneous element of degree $d>0$. As the induced dual action of G_m on $QH(A)$ sends z to $t^d\otimes z$, K is the field of all the G_m-invariant elements of the function field $Q(A)$ of X. Since $Q(A)=K(z)$, $\operatorname{tr.deg}_k K=1$. Hence we can take the nonsingular projective curve C whose function field is K. Let $\varphi: X\to C$ be the rational map associated to the inclusion $K\hookrightarrow Q(A)$.

Proposition 3.2. *If the rational map* $\varphi: X\to C$ *is a morphism, then* $Q(A_0)=K$ *and the general fibres of* φ *are smooth curves.*

Proof. Since $A^{G_m}=A_0$, the morphism $\pi: X\to\operatorname{Spec} A_0$ is the categorical quotient [GIT]. By the universality of the categorical quotient, there is a unique morphism $g: \operatorname{Spec} A_0\to C$ such that $\varphi=g\circ\pi$. But as A_0 is the subring of K, g is a birational morphism and $K=Q(A_0)$. Let $S=A_0-\{0\}$, then

$S^{-1}A$ is a normal K-subalgebra of $QH(A)=K[z, z^{-1}]$ whose quotient field is $Q(A)=K(z)$. Such a K-subalgebra must be $K[z]$, $K[z^{-1}]$, or $K[z, z^{-1}]$. In the case $S^{-1}A=K[z]$, let A be $A_0[u_1, \cdots, u_n]$, and $u_1, \cdots, u_n$ homogeneous elements of A. There exists a non-zero element $f\in A_0$ such that $A_f\ni z$, and $z^{r_i}(A_0)_f\ni u_i$ $(i=1, \cdots, n)$ where $r_i=\deg u_i/d$. Then $A_f=(A_0)_f[z]$. Hence the geometric fibres of the morphism $\pi: X\to \mathrm{Spec}\, A_0$ on $D(f)$ are isomorphic to $\boldsymbol{A}^1$. Similarly one can show that, in the case $S^{-1}A=K[z^{-1}]$ or $K[z, z^{-1}]$, there exists a non-zero element $f\in A_0$ such that the geometric fibres of the morphism on $D(f)$ are isomorphic to $\boldsymbol{A}^1$ or $\boldsymbol{A}^1-\{0\}$, respectively. In all the cases, general fibres of π are smooth curves. On the other hand, $g: \mathrm{Spec}\, A_0 \to C$ is a birational morphism and C is a nonsingular curve, hence g is an open immersion. Thus general fibres of $\varphi=g\circ\pi$ are smooth curves. q.e.d.

Proposition 3.3. *If the set X^{G_m} of G_m-invariant points of X is of dimension* 1, $X^{G_m}\simeq \mathrm{Spec}\, A_0$.

Proof. If m_x is the maximal ideal of A associated to a point $x\in X^{G_m}$, m_x is homogeneous and $A/m_x=k$. This implies that $m_x\supset A_i$ $(i\neq 0)$. Hence by Hilbert's zero point theorem, the ideal $\mathfrak{a}$ of the reduced closed subscheme X^{G_m} contains A_i $(i\neq 0)$. Thus the ring homomorphism $A_0\to A/\mathfrak{a}$ is a surjection, and X^{G_m} can be considered as a closed subscheme of $\mathrm{Spec}\, A_0$. But A_0 is integral and at most of dimension 1. Hence if $\dim X^{G_m}=1$, $X^{G_m}\simeq \mathrm{Spec}\, A_0$.

q.e.d.

Definition 3.4. If a smooth projective G_m-surface S is a ruled surface over a curve C, the G_m-action on S induces a G_m-action on C. We call S a G_m-ruled surface if the G_m-action on C is trivial.

Proposition 3.5. *Let S be a G_m-ruled surface, then there are two sections C_0 and C_1 of the structure morphism $\pi: S\to C$ such that $S^{G_m}=C_0\cup C_1$ and $C_0\cap C_1=\emptyset$.*

Proof. Since G_m acts on each fibre $\boldsymbol{P}^1$ of π, there are at least two invariant points on each fibre of π. Hence we can take a nonempty G_m-stable affine open subvariety U of S such that general fibres of the natural morphism $U\to C$ have G_m-invariant points [3]. Then $\dim U^{G_m}\neq 0$, hence by Proposition 3.3 $U^{G_m}\simeq \mathrm{Spec}\, A_0$, where $A=\bigoplus_{i\in Z}A_i$ and $U=\mathrm{Spec}\, A$. On the other hand by Proposition 3.2, $\mathrm{Spec}\, A_0$ is birational to C. Hence there is a rational map $s_1: C\to U^{G_m}\to S$. Since S is complete and C is nonsingular, s_1 is a morphism. Let $s_1(C)=C_1$, $C_1\subset S^{G_m}$. Taking another G_m-stable affine open subvariety U' disjoint from C_1 [3], we obtain another section $s_0: C\to S$ such that $C_0=s_0(C)\subset S^{G_m}$. Since Proposition 3.3 implies that no two G_m-invariant curves intersect with each other [3], $C_0\cap C_1=\emptyset$ and S^{G_m} contains no fibre of π. Hence there are exactly two invariant points on each fibre of π, thus $S^{G_m}=C_0\cup C_1$.

q.e.d.

Proposition 3.6. *Let X be a nonsingular projective G_m-surface. Then there are a nonsingular projective G_m-surface $\tilde{X}$, a G_m-ruled surface S and birational G_m-morphisms $f:\tilde{X}\to X$, $h:\tilde{X}\to S$.*

Proof. Let L be the function field of X and K the field of all the G_m-invariant elements of L. Let C be the nonsingular projective curve whose function field is K ($\mathrm{tr.deg}_k K=1$ by the result in the affine case). Let $g_0: X\to C$ be the rational map associated to the inclusion $K\hookrightarrow L$. If g_0 is not a morphism, then every fundamental point x_1 of g is G_m-invariant and the blowing-up X_1 of X_0 at x_1 is a G_m-surface, and the morphism $X_1\to X$ is a G_m-morphism. If $g_1: X_1\to C$ is not a morphism, the same operation can be repeated. Thus there is a sequence of G_m-morphisms $\tilde{X}=X_n\to X_{n-1}\to\cdots\to X_0=X$ such that $g_n:\tilde{X}\to C$ is a morphism (elimination of indeterminacy). Since $\tilde{X}$ can be covered by a finite number of G_m-stable affine open subvarieties, by Proposition 3.2 general fibres of g_n are nonsingular curves. But general fibres of g_n are complete curves with a nontrivial G_m-action, so they are isomorphic to $\boldsymbol{P}^1$. Then it is well-known that, by successive contraction of exceptional curves of the first kind on special fibres, one obtains a ruled surface S over C and g_n factors through $S\to C$, namely $g_n:\tilde{X}\xrightarrow{h}S\to C$. Since exceptional curves of the first kind on G_m-surfaces are G_m-stable, S is a G_m-surface and h is a G_m-morphism. q.e.d.

Proposition 3.7. *Let X be a normal projective G_m-surface. Then there are a nonsingular projective G_m-surface $\tilde{X}$, a G_m-ruled surface S and birational G_m-morphisms $f:\tilde{X}\to X$ and $h:\tilde{X}\to S$.*

Proof. The desingularization of a surface is done by finite repetition of blowing up along an isolated singular point and the normalization. Thus the G_m-action on X can be extended to them. Hence there exist a nonsingular projective G_m-surface X', and a birational G_m-morphism $g: X'\to X$. By Proposition 3.6 there are $\tilde{X}$ and S and $f':\tilde{X}\to X'$, $h:\tilde{X}\to S$. Set $f=g\circ f'$. q.e.d.

In the remainder of this section, we quote some well-known lemmas, for the reader's convenience, which are necessary in the following section.

Lemma 3.8. *Let X and Y be nonsingular projective surfaces, and $\varphi: X\to Y$ a birational morphism which induces an isomorphism $X-\varphi^{-1}(P)\to Y-\{P\}$ for a point $P\in Y$. Set $\varphi^{-1}(P)=\cup_{i=1}^{n}X_i$, where X_i $(i=1,\cdots,n)$ is an irreducible curve of X. Then if we consider the injection $\varphi^*:\mathrm{Cl}\,Y\to\mathrm{Cl}\,X$,*

$$\mathrm{Cl}\,X=\varphi^*(\mathrm{Cl}\,Y)\oplus\sum_{i=1}^{n}\boldsymbol{Z}\,\mathrm{cl}(X_i).$$

Proof. By the factorization theorem for birational morphisms of nonsingular projective surfaces, we may assume that φ is a quadratic transformation. In this case, the assertion is obvious. q.e.d.

Lemma 3.9. *Let X be a nonsingular projective surface, Y a normal projective surface, and $\varphi: X \to Y$ a birational morphism which induces an isomorphism $X - \varphi^{-1}(P) \to Y - \{P\}$ for a point $P \in Y$. Set $\varphi^{-1}(P) = \cup_{i=1}^{n} X_i$, where X_i $(i=1, \cdots, n)$ is an irreducible curve on X. Then $\mathrm{cl}(X_1), \cdots, \mathrm{cl}(X_n)$ are linearly independent over $\boldsymbol{Z}$ and*

$$\mathrm{Cl}\, X/(\sum_{i=1}^{n} \boldsymbol{Z}\, \mathrm{cl}(X_i)) \simeq \mathrm{Cl}\, Y.$$

In particular, if $\mathrm{Cl}\, Y$ is a finitely generated $\boldsymbol{Z}$-module, so is $\mathrm{Cl}\, X$, and

$$\mathrm{rank}\ \mathrm{Cl}\, X = \mathrm{rank}\ \mathrm{Cl}\, Y + n.$$

Proof. Let $X^\circ = X - \cup_{i=1}^{n} X_i$, then $\mathrm{Cl}\, X^\circ \simeq \mathrm{Cl}\, Y$ and $f: \mathrm{Cl}\, X \to \mathrm{Cl}\, X^\circ$ is surjective and $\mathrm{Ker}\, f$ is generated by $\mathrm{cl}(X_i)$ $(i=1, \cdots, n)$. The linear independence of $\mathrm{cl}(X_i)$ $(i=1, \cdots, n)$ follows from [2]. Hence $\mathrm{Cl}\, X/(\sum_{i=1}^{n} \boldsymbol{Z}\, \mathrm{cl}(X_i)) \simeq \mathrm{Cl}\, X^\circ \simeq \mathrm{Cl}\, Y$. q.e.d.

Lemma 3.10. *Let F be a free $\boldsymbol{Z}$-module of rank n, and $a_1, \cdots, a_r$ be elements of F. Then if $F/\langle a_1, \cdots, a_r\rangle$ is a free $\boldsymbol{Z}$-module of rank $n-r$, $F/\langle a_1\rangle$ is a free $\boldsymbol{Z}$-module of rank $n-1$.*

Proof. The exact sequence

$$0 \longrightarrow \langle a_1, \cdots, a_r\rangle \longrightarrow F \longrightarrow F/\langle a_1, \cdots, a_r\rangle \longrightarrow 0$$

splits, since $F/\langle a_1, \cdots, a_r\rangle$ is a free $\boldsymbol{Z}$-module. Hence there is a free submodule E of rank $n-r$ of F such that $F = \langle a_1, \cdots, a_r\rangle \oplus E$. Since $\langle a_1, \cdots, a_r\rangle$ is of rank r, $a_1, \cdots, a_r$ are linearly independent over $\boldsymbol{Z}$ and $F/\langle a_1\rangle \simeq \langle a_2, \cdots, a_r\rangle \oplus E$ is a free $\boldsymbol{Z}$-module of rank $n-1$. q.e.d.

Lemma 3.11. *Let X be a nonsingular G_m-surface, and L an invertible sheaf on X such that $\dim H^0(X, L) < \infty$. If a linear subspace $V \subset \boldsymbol{P}(H^0(X, L)^\vee)$ contains all the points of $\boldsymbol{P}(H^0(X, L)^\vee)$ which correspond to G_m-stable divisors of X, then $V = \boldsymbol{P}(H^0(X, L)^\vee)$.*

Proof. The condition implies that V contains all the G_m-invariant points of the induced G_m-action on $\boldsymbol{P}(H^0(X, L)^\vee)$. Since every G_m-action on a projective space is diagonalizable, G_m-invariant points of $\boldsymbol{P}(H^0(X, L)^\vee)$ span $\boldsymbol{P}(H^0(X, L)^\vee)$. Hence $V = \boldsymbol{P}(H^0(X, L)^\vee)$. q.e.d.

§4. The ring $R(X)$

Let X be a normal projective G_m-surface such that $\mathrm{Cl}\, X \simeq \boldsymbol{Z}$. Then by Proposition 3.7, there are a nonsingular projective G_m-surface $\tilde{X}$, a G_m-ruled surface S and birational G_m-morphisms $f: \tilde{X} \to X$ and $g: \tilde{X} \to S$. From now on, we fix such a 4-ple $\{\tilde{X}, S, f, g\}$ for X.

Definition 4.1. Let X° be a nonsingular open subvariety of X such that $X - X^\circ$ is a finite set. Let $L \in \mathrm{Pic}\, X^\circ \simeq \mathrm{Cl}\, X \simeq \boldsymbol{Z}$ be the ample generator of $\mathrm{Pic}\, X^\circ$. Then we call $R(X^\circ, L) = \oplus_{i=0}^{\infty} H^0(X^\circ, L^{\otimes i})$ the canonical homogeneous coordinate ring of X, and denote it by $R(X)$.

$R(X)$ is a geometric graded factorial ring of dimension 3, and independent of the choice of X° (see [1]).

Proposition 4.2. *S is a rational ruled surface.*

Proof. Since $\mathrm{Cl}\, X \simeq \boldsymbol{Z}$ and $f: \tilde{X} \to X$ is a birational morphism, $\mathrm{Cl}\, \tilde{X}$ is a finitely generated $\boldsymbol{Z}$-module by Lemma 3.9. Since $g: \tilde{X} \to S$ is also a birational morphism, $\mathrm{Cl}\, S$ is a finitely generated $\boldsymbol{Z}$-module. Thus the base curve of S is rational. q.e.d.

Let $\pi': S \to \boldsymbol{P}^1$ be the structure morphism of the rational G_m-ruled surface S, and let $\pi = \pi' \circ g: \tilde{X} \to \boldsymbol{P}^1$. Since general fibres of π are isomorphic to $\boldsymbol{P}^1$, we can take $a_1, \cdots, a_n \in \boldsymbol{P}^1$ $(n \geq 3)$ such that for every closed point $a \in \boldsymbol{P}^1 - \{a_1, \cdots, a_n\}$ $\pi^{-1}(a)$ is isomorphic to $\boldsymbol{P}^1$. Let $\pi^{-1}(a_i) = \sum_{j=1}^{r_i} e_{i,j} X_{i,j}$ $(i=1, \cdots, n)$, where $e_{i,j}$, $X_{i,j}$ $(i=1, \cdots, n, 1 \leq j \leq r_i)$ are positive integers and irreducible curves on $\tilde{X}$, respectively. By Lemma 3.5, there are two sections C_0' and C_1' of π' such that $C_0' \cap C_1' = \emptyset$ and $S^{G_m} = C_0' \cup C_1'$. Let C_0 and C_1 be the proper transforms of C_0' and C_1' by g, respectively. C_0 and C_1 are G_m-invariant curves on $\tilde{X}$.

Proposition 4.3. *Every G_m-stable irreducible divisor of $\tilde{X}$ defined over k is one of the following curves.*

1) *$\pi^{-1}(a)$, where a is a closed point of $P^1 - \{a_1, \cdots, a_n\}$,*
2) *$X_{i,j}$ $i=1, \cdots, n, 1 \leq j \leq r_i$,*
3) *C_0 and C_1.*

Proof. Let C be a G_m-stable irreducible divisor of $\tilde{X}$. If C is contained in $\pi^{-1}(a_i)$ for some i, then $C \subset \bigcup_{j=1}^{r_i} X_{i,j}$ and $C = X_{i,j}$ for some $1 \leq j \leq r_i$. If $C \not\subset \pi^{-1}(a_i)$ for all i and $C \not\subset C_0 \cup C_1$, there is a closed point $x \in C$ such that x is not G_m-invariant and $\pi(x) \notin \{a_1, \cdots, a_n\}$. Then C contains the closure of the orbit of x, which is equal to $\pi^{-1}(\pi(x))$. Hence $C = \pi^{-1}(\pi(x))$. This is the case (1). q.e.d.

Theorem 4.4. *X is obtained from $\tilde{X}$ by contracting $(\sum_{i=1}^{n} r_i) - n + 1$ components of*

$$\Big(\bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq r_i}} X_{i,j}\Big) \cup C_0 \cup C_1.$$

If one denotes by R the union of the remaining $n+1$ components, then one of the following two cases occurs:

Case 1. *There are an integer i $(i=0$ or $1)$ and n integers $m_1, \cdots, m_n$ $(1 \leq m_i \leq r_i$ for all $i)$ such that*

$$R = C_i \cup \Big(\bigcup_{i=1}^{n} X_{i,m_i}\Big).$$

Case 2. *There are n integers $m_1, \cdots, m_n$ $(1 \leq m_i \leq r_i$ for all $i)$ such that*

$$(\bigcup_{i,j} X_{i,j}) \supset R \supset \bigcup_{i=1}^{n} X_{i,m_i}.$$

Proof. Since rank $\mathrm{Cl}\, S=2$, rank $\mathrm{Cl}\, \tilde{X}=\sum_{i=1}^{n} r_i-n+2$ by Lemma 3.9. On the other hand, rank $\mathrm{Cl}\, X=1$. Hence by Lemma 3.9, $f: \tilde{X} \rightarrow X$ contracts exactly $\sum_{i=1}^{n} r_i-n+1$ curves of $\tilde{X}$. But since f is a G_m-morphism, those curves are G_m-stable. By Proposition 4.3, there are only three types of G_m-stable curves. But since the self-intersection number of a curve $\pi^{-1}(\alpha)$ $\alpha \notin \{\alpha_1, \cdots, \alpha_n\}$ is 0, $\pi^{-1}(\alpha)$ is not an exceptional curve of the morphism f [2]. Thus the first assertion is proved. Since the self-intersection number of $\pi^{-1}(\alpha_i)=\sum_j e_{i,j} X_{i,j}$ is 0 for each i, there are n integers $m_1, \cdots, m_n$ $(1 \leq m_i \leq r_i$ for all $i)$ such that $R \supset \bigcup_{i=1}^{n} X_{i,m_i}$. Then the case 1 occurs if $R \not\subset \bigcup_{i,j} X_{i,j}$, and the case 2 occurs if $R \subset \bigcup_{i,j} X_{i,j}$. q.e.d.

Definition 4.5. We call the components of R given in Theorem 4.4 remaining curves.

We will determine the canonical homogeneous coordinate ring $R(X)$ in each case in the following sections.

§ 5. Case 1

By renumbering the curves in Theorem 4.4, we may assume that $R=C_0 \cup (\bigcup_{i=1}^{n} X_{i,1})$ and $e_{1,1} \geq \cdots \geq e_{n,1}$. For simplicity, we denote, in this section, $X_{i,1}$ and $e_{i,1}$ by X_i and e_i, respectively $(i=1, \cdots, n)$. We can take elements $s_1, \cdots, s_n$ of $H^0(\boldsymbol{P}^1, O_{\boldsymbol{P}}(1))$ such that $(s_i)_0=\alpha_i$ and $s_1+a_i s_2+s_i=0$ $(i=3, \cdots, n)$, where $a_3=1, \cdots, a_n$ are mutually distinct elements of k^* uniquely determined by $\alpha_1, \cdots, \alpha_n$. Let r be the largest number $(1 \leq r \leq n)$ such that $e_r>1$, if $e_1>1$, and let r be 0 if $e_1=1$.

Theorem 5.1. *$(e_i, e_j)=1$ for all i and j such that $1 \leq i<j \leq n$, and there is a positive integer m with the following property: If $r=0$, 1, or 2.*

$$R(X) \simeq k[x_1, x_2, u], \text{ where } \deg x_i=e_i \ (i=1, 2) \text{ and } \deg u=m.$$

If $r \geq 3$,

$$R(X) \simeq k[x_1, \cdots, x_r, u]/I, \text{ where } \deg x_i=e/e_i \ (i=1, \cdots, r),$$

$\deg u=m$, $e=\prod_{i=1}^{r} e_i$ and

$$I=(x_1^{e_1}+a_3 x_2^{e_2}+x_3^{e_3}, \cdots, x_1^{e_1}+a_r x_2^{e_2}+x_r^{e_r}).$$

Proof. Since S is a rational ruled surface, $\mathrm{Cl}\, S$ is generated by $n+2$ divisors $Y_i=\pi'^{-1}(\alpha_i)$ $(i=1, \cdots, n)$ and C_0' and C_1'. The relations among them are generated by n divisors Y_1-Y_i $(i=2, \cdots, n)$ and $C_0'-C_1'+kY_1$, where k is the self-intersection number of C_1'. By Lemma 3.8, $\mathrm{Cl}\, \tilde{X}$ is generated by $X_{i,j}$ $(i=1, \cdots, n,\ 1 \leq j \leq r_i)$ and C_0 and C_1. Again by Lemma 3.8, the relations among them are generated by the total transforms of the n divisors Y_1-Y_i $(i=2, \cdots, n)$ and $C_0'-C_1'+kY_1$. If we set

$$g^*(C_0'-C_1')=\sum_{i,j} d_{i,j}X_{i,j}+C_0-C_1,$$

then we have

$$g^*(Y_1-Y_i)=\sum_{j=1}^{r_1} e_{1,j}X_{1,j}-\sum_{j=1}^{r_i} e_{i,j}X_{i,j},$$
$$g^*(C_0'-C_1'+kY_1)=C_0-C_1+\sum_{i,j} d_{i,j}X_{i,j}+k\sum_{j=1}^{r_1} e_{1,j}X_{1,j}.$$

Since

$$\mathrm{Cl}\,X\simeq \mathrm{Cl}\,\tilde{X}/\boldsymbol{Z}\,\mathrm{cl}(C_1)+\sum_{X_{i,j}\not\subset R}\boldsymbol{Z}\,\mathrm{cl}(X_{i,j}),$$

$\mathrm{Cl}\,X$ is generated by $n+1$ remaining curves X_i $(i=1, \cdots, n)$ and C_0. The relations among them are generated by n divisors, $e_1X_1-e_iX_i$ $(i=2, \cdots, n)$, and $C_0-\sum_{i=1}^n d_{i,1}X_i+ke_1X_1$. Thus one has

$$\mathrm{Cl}\,X\simeq\sum_{i=1}^n \boldsymbol{Z}u_i/\sum_{i=1}^n \boldsymbol{Z}(e_1u_1-e_iu_i),$$

where $u_1, \cdots, u_n$ are linearly independent over $\boldsymbol{Z}$. Then noting that $\mathrm{Cl}\,X\simeq\boldsymbol{Z}$, one sees easily that $(e_i, e_j)=1$ if $i\neq j$ by Lemma 3.10. Since $e_i\mathrm{cl}(X_i)\in e_j\mathrm{Cl}\,X$ for every j, one can take a divisor E such that $(e/e_i)\mathrm{cl}\,E=\mathrm{cl}\,X_i$ for every i $(e=\prod_{i=1}^n e_i)$. Then $\mathrm{cl}\,E$ generates $\mathrm{Cl}\,X$. Let $X^\circ=\tilde{X}-\cup_{i=1}^n\cup_{j=2}^{r_i}X_{i,j}-C_1$, then X° is a smooth open subvariety of X and $X-X^\circ$ is a finite set. For simplicity, we denote the restriction to X° of a divisor on $\tilde{X}$ by the same symbol (if there is no danger of confusion). We set $L=O_{X^\circ}(E)$. Since $L^{\otimes e}\simeq O_{X^\circ}(e_1X_1)$, we fix an isomorphism $\pi^*O_{\boldsymbol{P}}(1)|_{X^\circ}\simeq L^{\otimes e}$ and identify them from now on in this section. Since $(\pi^*(s_i))_0=e_iX_i$ and $H^0(X^\circ, O_{X^\circ})=k$, there is an element $\bar{x}_i$ of $H^0(X^\circ, L^{\otimes e/e_i})$ such that $(\bar{x}_i)_0=X_i$ and $\bar{x}_i^{e_i}=\pi^*(s_i)$ for every i. Since $\mathrm{Cl}\,X$ is generated by $\mathrm{cl}\,E$, $C_0\sim mE$ for some integer m. One has $m>0$ because $C_0>0$ and $\mathrm{Cl}\,X\simeq\boldsymbol{Z}$. Hence there is an element $\bar{u}$ of $H^0(X^\circ, L^{\otimes m})$ such that $(\bar{u})_0=C_0$. It follows from Lemma 4.3 that G_m-stable irreducible divisors of X° are $X_1, \cdots, X_n, C_0$, and $\pi^{-1}(\alpha)$, where α is an arbitrary closed point of $\boldsymbol{P}^1-\{\alpha_1, \cdots, \alpha_n\}$. For $\alpha\neq\alpha_1, \alpha_2$ there is an element $a\in k^*$ such that $\alpha=(s_1+as_2)_0$, hence $\pi^{-1}(\alpha)=(\pi^*(s_1+as_2))_0=(\bar{x}_1^{e_1}+a\bar{x}_2^{e_2})_0$. Thus all the G_m-stable effective divisors on X° equivalent to $i\,\mathrm{cl}\,E$ are zeros of sections of $H^0(X^\circ, L^{\otimes i})$ which are homogeneous polynomials of degree i in $\bar{x}_1, \cdots, \bar{x}_n, \bar{u}$ $(i\in\boldsymbol{Z}, i\geq 0)$. Hence $R(X)$ is generated by $\bar{x}_1, \cdots, \bar{x}_n, \bar{u}$ as a k-algebra, by Lemma 3.11. On the other hand, since $s_1+a_is_2+s_i=0$ $(i=3, \cdots, n)$, one has $\bar{x}_1^{e_1}+a_i\bar{x}_2^{e_2}+\bar{x}_i^{e_i}=0$ $(i=3, \cdots, n)$. If i satisfies the conditions $r<i\leq n$ and $2<i$, $\bar{x}_i$ can be written as a polynomial in $\bar{x}_1$ and $\bar{x}_2$. We define a morphism of graded k-algebras $h: k[x_1, \cdots, x_n, u]\to R(X)$ by $h(x_i)=\bar{x}_i$ $(i=1, \cdots, n)$ and $h(u)=\bar{u}$, where $\deg x_i=e/e_i$ $(i=1, \cdots, n)$ and $\deg u=m$. When $r=0$, 1, or 2, $R(X)$ is the image of $k[x_1, x_2, u]$. Since $\dim R(X)=3$, $R(X)\simeq k[x_1, x_2, u]$, where $\deg x_i=e_i$ $(i=1, 2)$ and $\deg u=m$. When $r\geq 3$, h induces a surjection $k[x_1, \cdots, x_r, u]/I\to R(X)$, where $I=(x_1^{e_1}+a_3x_2^{e_2}+x_3^{e_3}, \cdots, x_1^{e_1}+a_rx_2^{e_2}+x_r^{e_r})$. Since $k[x_1, \cdots, x_r, u]/I$ is a factorial ring of dimension 3 by [1], and $\dim R(X)$

$=3$, we obtain $R(X)\simeq k[x_1, \cdots, x_r, u]/I$, where $\deg x_i=e/e_i$ $(i=1, \cdots, n)$ and $\deg u=m$. q.e.d.

Remark 5.2. Since $R(X)$ is a geometric graded factorial ring, $(e, m)=1$ by [1, Corollary 2.9]. Hence $R(X)$ is isomorphic to a graded ring given in Example I or II.

§6. Case 2

Again by renumbering the curves, we may assume that $R=(\cup_{i=1}^{n}X_{i,1})\cup X_{n,2}$ and $e_{1,1}\geq\cdots\geq e_{n-1,1}$, $e_{n,1}\leq e_{n,2}$. In the case $e_{n,1}=e_{n,2}$, we may furthermore assume that the divisor $f(X_{n,2}-X_{n,1})$ is linearly equivalent to 0 or some positive multiple of it is ample (note that $\mathrm{Cl}\,X\simeq\boldsymbol{Z}$ and X is projective). For simplicity, we denote, in this section, $X_{i,1}$, $e_{i,1}$, $X_{n,1}$, $X_{n,2}$, $e_{n,1}$, and $e_{n,2}$ by X_i, e_i, U, V, c, and d, respectively $(i=1, \cdots, n-1)$. We can take elements $s_0, \cdots, s_{n-1}$ of $H^0(\boldsymbol{P}^1, O_{\boldsymbol{P}}(1))$ such that $(s_i)_0=\alpha_i$ $(i=1, \cdots, n-1)$, $(s_0)_0=\alpha_n$, $s_1+b_is_2+s_i=0$ $(i=3, \cdots, n-1)$ and $s_1+bs_2+s_0=0$, where $b_3=1, \cdots, b_{n-1}, b$ are mutually distinct elements of k^* uniquely determined by $\alpha_1, \cdots, \alpha_n$. Let r be the largest number $(1\leq r\leq n-1)$ such that $e_r>1$, if $e_1>1$, and let r be 0 if $e_1=1$.

Theorem 6.1. *$(e_i, e_j)=1$ for all $1\leq i<j\leq n-1$, and $(c, d, e)=1$, where $e=\prod_{i=1}^{n-1}e_i$. There are positive integers l, m, p with the following property: If $r=0$ or 1,*

$$R(X)\simeq k[x_1, u, v],\ \textit{where}\ \deg x_1=p,\ \deg u=l\ \textit{and}\ \deg v=m.$$

If $r\geq 2$,

$$R(X)\simeq k[x_1, \cdots, x_r, u, v]/I,$$
$$I=(x_1^{e_1}+b_3x_2^{e_2}+x_3^{e_3}, \cdots, x_1^{e_1}+b_rx_2^{e_2}+x_r^{e_r}, x_1^{e_1}+bx_2^{e_2}+u^cv^d),$$

where $\deg x_i=pe/e_i$ $(i=1, \cdots, n-1)$, $\deg u=l$ and $\deg v=m$.

Proof. As in Case 1, $\mathrm{Cl}\,\tilde{X}$ is generated by $X_{i,j}$ $(i=1, \cdots, n, 1\leq j\leq r_i)$, C_0 and C_1, and the relations among them are generated by

$$\sum_{j=1}^{r_1}e_{1,j}X_{1,j}-\sum_{j=1}^{r_i}e_{i,j}X_{i,j}\quad (i=2, \cdots, n),$$
$$C_0-C_1+\sum_{i,j}d_{i,j}X_{i,j}+k\sum_{j=1}^{r_1}e_{1,j}X_{1,j}.$$

Since by Lemma 3.9,

$$\mathrm{Cl}\,X\simeq\mathrm{Cl}\,\tilde{X}/\sum_{j=0}^{1}\boldsymbol{Z}\,\mathrm{cl}(C_i)+\sum_{X_{i,j}\not\subset R}\boldsymbol{Z}\,\mathrm{cl}(X_{i,j}),$$

$\mathrm{Cl}\,X$ is generated by the classes of the remaining curves $X_1, \cdots, X_{n-1}, U, V$, and the relations among them are generated by n divisors $e_1X_1-e_iX_i$ $(i=2, \cdots, n-1)$, $e_1X_1-cU-dV$ and $\sum_{i=1}^{n}d_{i,1}X_i+d_{n,1}U+d_{n,2}V+ke_1X_1$. It follows from Lemma 3.10 that $(e_i, e_j)=1$ for all $1\leq i<j\leq n-1$ and $(c, d, e)=1$. Let

$$X^{\circ}=\tilde{X}-\bigcup_{i=1}^{n-1}\bigcup_{j=2}^{r_i}X_{i,j}-\bigcup_{j=3}^{r_n}X_{n,j}-C_0-C_1,$$

then X° is a nonsingular open subvariety of X and $X-X^{\circ}$ is a finite set. Let L be the ample generator of $\mathrm{Pic}(X^{\circ})\simeq \mathrm{Cl}\, X\simeq \mathbf{Z}$. Then there is a positive integer p' such that $L^{\otimes p'}\simeq \pi^{*}O_{P}(1)|_{X^{\circ}}$. We fix such an isomorphism and identify them. Since $L^{\otimes p'}\simeq O_{X^{\circ}}(e_{i}X_{i})$, e_{i} divides p' $(i=1, \cdots, n-1)$. Hence there is a positive integer p, such that $p'=pe$. Since $(\pi^{*}(s_{i}))_{0}=e_{i}X_{i}$ $(i=1, \cdots, n-1)$ and $H^{0}(X^{\circ}, O_{X^{\circ}})=k$, there is an element $\bar{x}_{i}$ of $H^{0}(X^{\circ}, L^{\otimes pe/e_{i}})$ such that $(\bar{x}_{i})_{0}=X_{i}$ and $\bar{x}_{i}^{e_{i}}=\pi^{*}(s_{i})$ $(i=1, \cdots, n-1)$. Since $(\pi^{*}(s_{n}))_{0}=cU+dV$, there are elements $\bar{u}\in H^{0}(X^{\circ}, L^{\otimes l})$ and $\bar{v}\in H^{0}(X^{\circ}, L^{\otimes m})$ for some positive integers l and m such that $(\bar{u})_{0}=U$ and $(\bar{v})_{0}=V$ and $\bar{u}^{c}\bar{v}^{d}=\pi_{*}(s_{n})$. If $c=d$, then $l\leq m$ by our definition of $e_{n,1}$ and $e_{n,2}$. As in Case 1, $R(X)$ is generated by $\bar{x}_{1}, \cdots, \bar{x}_{n-1}, \bar{u}, \bar{v}$ as a k-algebra. On the other hand, since $s_{1}+b_{i}s_{2}+s_{i}=0$ $(i=3, \cdots, n-1)$ and $s_{1}+bs_{2}+s_{0}=0$, one has $\bar{x}_{1}^{e_{1}}+b_{i}\bar{x}_{2}^{e_{2}}+\bar{x}_{i}^{e_{i}}=0$ $(i=3, \cdots, n-1)$ and $\bar{x}_{1}^{e_{1}}+b\bar{x}_{2}^{e_{2}}+\bar{u}^{c}\bar{v}^{d}=0$. If i satisfies the conditions $r<i\leq n-1$ and $1<i$, $\bar{x}_{i}$ can be written as a polynomial in $\bar{x}_{1}$, $\bar{u}$ and $\bar{v}$. We define a morphism of graded k-algebras $h: k[x_{1}, \cdots, x_{n}, u, v]\rightarrow R(X)$ by $h(x_{i})=\bar{x}_{i}$ $(i=1, \cdots, n-1)$ and $h(u)=\bar{u}$ and $h(v)=\bar{v}$, where $\deg x_{i}=pe/e_{i}$ $(i=1, \cdots, n-1)$, $\deg u=l$, and $\deg v=m$. When $r=0$, or 1, $R(X)$ is the image of $k[x_{1}, u, v]$. Since $\dim R(X)=3$, $R(X)\simeq k[x_{1}, u, v]$, where $\deg x_{1}=p$, $\deg u=l$, and $\deg v=m$. When $r\geq 2$, h induces a surjection $k[x_{1}, \cdots, x_{r}, u, v]/I\rightarrow R(X)$, where $I=(x_{1}^{e_{1}}+b_{3}x_{2}^{e_{2}}+x_{3}^{e_{3}}, \cdots, x_{1}^{e_{1}}+b_{r}x_{2}^{e_{2}}+x_{r}^{e_{r}}, x_{1}^{e_{1}}+bx_{2}^{e_{2}}+u^{c}v^{d})$. Since $k[x_{1}, \cdots, x_{r}, u, v]/I$ is a factorial ring of dimension 3 by the results of §2 and $\dim R(X)=3$, we obtain $R(X)\simeq k[x_{1}, \cdots, x_{r}, u, v]/I$, where $\deg x_{i}=pe/e_{i}$ $(i=1, \cdots, n-1)$, $\deg u=l$, and $\deg v=m$. q.e.d.

Remark 6.2. Since $R(X)$ is a geometric graded factorial ring, l, m, p are pairwise relatively prime by [1, Corollary 2.9]. Hence $R(X)$ is isomorphic to a graded ring given in Example I or III.

§7. The proof of Theorem 1.1.

Let R be a graded ring in Theorem 1.1, and fix a nondegenerate bigradation. Then $X=\mathrm{Proj}\, R$ is a normal projective surface such that $\mathrm{Cl}\, X\simeq \mathbf{Z}$ [1, §1], and X has a structure of G_{m}-surface $\{X, \mu\}$ induced by the bigradation. It was proved in previous sections that $R(X)\simeq R_{\Phi}$ for an index Φ. Since $R\simeq R(X)$ [1, §2], $R\simeq R_{\Phi}$. Thus the existence of Φ is proved.

Now it is sufficient to prove that if $R_{\Phi}\simeq R_{\Phi'}$ then $\Phi=\Phi'$. The indices Φ given in Examples I, II, and III are called of types I, II, and III, respectively.

Definition 7.1. For integers $a_{1}, \cdots, a_{n}, b_{1}, \cdots, b_{m}$, we denote $[a_{1}, \cdots, a_{n}]=[b_{1}, \cdots, b_{m}]$ if and only if $n=m$ and there is a permutation $s\in S_{n}$ such that $b_{i}=a_{s(i)}$ $(i=1, \cdots, n)$.

Let $R=\bigoplus_{i=0}^{\infty}R_{i}$ be a graded ring with non-negative degrees and of finite type over k such that $R_{0}=k$. Let $\{u_{1}, \cdots, u_{n}\}$ be a minimal basis of $R_{+}=\bigoplus_{i>0}R_{i}$ as an R-module consisting of homogeneous elements. Then $[\deg u_{1}, \cdots, \deg u_{n}]$ is uniquely determined by R (independent of the choice of

$u_1, \cdots, u_n$ [1, §2]).

Definition 7.2. Under the above notation, we denote $[\deg u_1, \cdots, \deg u_n]$ and n by $D(R)$ and $n(R)$, respectively.

Of course, if $R \simeq R'$, $D(R)=D(R')$ and $n(R)=n(R')$.

Proposition 7.3. *Let Φ and Φ' be two indices such that $R_\Phi \simeq R_{\Phi}'$. If Φ is of type I, then $\Phi=\Phi'$.*

Proof. Since $n(R_{\Phi'})=n(R_\Phi)=3$, it is obvious that Φ' is of type I. Let $\Phi=(e_1, e_2, e_3)$ and $\Phi'=(e_1', e_2', e_3')$. Then $D(R_\Phi)=[e_1, e_2, e_3]$ and $D(R_{\Phi'})=[e_1', e_2', e_3']$. Since $e_1 \geq e_2 \geq e_3$, $e_1' \geq e_2' \geq e_3'$ and $D(R_\Phi)=D(R_{\Phi'})$, $(e_1, e_2, e_3)=(e_1', e_2', e_3')$. Hence $\Phi=\Phi'$. q.e.d.

It is easy to see that the automorphism group $\mathrm{Aut}(R)$ of R (given in Definition 7.2) as a graded k-algebra is a linear algebraic group over k. We denote by $\mathrm{rank}\,\mathrm{Aut}^\circ(R)$ the rank of the 0-component $\mathrm{Aut}^\circ(R)$ of $\mathrm{Aut}(R)$, the dimension of a maximal torus of $\mathrm{Aut}^\circ(R)$.

Proposion 7.4. *Let Φ be an index of type II or III. Then* $\mathrm{rank}\,\mathrm{Aut}^\circ(R_\Phi)=2$.

Proof. Let $T \subset \mathrm{Aut}^\circ(R_\Phi)$ be the 2-dimensional torus associated to the bigradation $R_\Phi=\bigoplus_{i,j} R_{i,j}$ of R_Φ given in §2*). It is sufficient to prove that the centralizer $Z(T)$ is of dimension 2. By the definition of the centralizer, every element of $Z(T)$ preserves the bigradation of R.

First, let us assume that Φ is of type II, and set $\Phi=(\boldsymbol{e}, \boldsymbol{a}, m)$. Then, every element of $Z(T)$ induces automorphisms of

$$\bigoplus_{i\geq 0} R_{i,0}=k[x_1, \cdots, x_r]/(x_1^{e_1}+a_3x_2^{e_2}+x_3^{e_3}, \cdots, x_1^{e_1}+a_rx_2^{e_2}+x_r^{e_r})$$

and $$\bigoplus_{i\geq 0} R_{im,i}=k[u].$$

Since $x_1, \cdots, x_r, u$ generate R_Φ over k, the kernel of the homomorphism

$$Z(T) \longrightarrow \mathrm{Aut}(\bigoplus_{i\geq 0} R_{i,0}) \times \mathrm{Aut}(\bigoplus_{i\geq 0} R_{im,i})$$

of algebraic groups is of dimension 0. By the exact sequence of [1, Theorem 4.5], $\mathrm{Aut}(\bigoplus_{i\geq 0} R_{i,0}) \simeq G_m$. Since $\mathrm{Aut}(\bigoplus_{i\geq 0} R_{im,i}) \simeq G_m$, $Z(T)$ is of dimension 2.

Next, let Φ be of type III, and set $\Phi=(\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{c})$. We admit the following lemma and continue the proof.

Lemma 7.5. $R_{pe/e_i}=kx_i$ $(i=1, \cdots, r)$.

Thus every element of $Z(T)$ induces automorphisms of

$$\begin{aligned} R' &= k[R_{pe/e_1,0}, \cdots, R_{pe/e_r,0}] \\ &= k[x_1, \cdots, x_r]/(x_1^{e_1}+b_3x_2^{e_2}+x_3^{e_3}, \cdots, x_1^{e_1}+b_rx_2^{e_2}+x_r^{e_r}) \end{aligned}$$

and $$R''=k[R_{i,-d}, R_{m,d}]=k[u, v].$$

Since R_Φ is generated by $x_1, \cdots, x_r, u, v$ over k, the kernel of the homomorphism

$$Z(T) \longrightarrow \mathrm{Aut}(R') \times \mathrm{Aut}_{bi-gr.}(R'')$$

of algebraic groups is of dimension 0, where $\mathrm{Aut}_{bi-gr.}(R'')$ denotes the algebraic subgroup of $\mathrm{Aut}(R'')$ consisting of elements preserving bigradation. On the other hand, if $r \geq 3$ $\mathrm{Aut}(R') \simeq G_m$ and $\mathrm{Aut}_{bi-gr.}(R'') \simeq G_m{}^2$. Thus one sees that $Z(T)$ is of dimension 2, in view of the equation $x_1{}^{e_1}+bx_2{}^{e_2}+u^c v^d=0$. If $r=2$, $\dim Z(T)=2$ easily follows from the equation $x_1{}^{e_1}+x_2{}^{e_2}+u^c v^d=0$.

q.e.d.

Proof of Lemma 7.5. Since $R_{pe/e_i,0}$ is generated by monomials in $x_1, \cdots, x_r, u, v$, it is sufficient to prove the following assertion: If non-negative integers $l_1, \cdots, l_r, q, q'$ satisfies the condition $x_1{}^{l_1} \cdots x_r{}^{l_r} u^q v^{q'} \in R_{pe/e_i,0}$, then $l_j=0$ $(j \neq i)$ and $q=q'=0$. The condition $x_1{}^{l_1} \cdots x_r{}^{l_r} u^q v^{q'} \in R_{pe/e_i}$ implies

$$\sum_{j=1}^{r} l_j pe/e_j + ql + q'm = pe/e_i, \tag{1}$$

$$-qd+q'c=0 \tag{2}$$

Thus there is a non-negative integer l_0 such that $q=l_0 c/e_0$ and $q'=l_0 d/e_0$, where $e_0=(c, d)$. Hence we have

$$ql+q'm=l_0(lc+md)/e_0=l_0 pe/e_0,$$

$$\sum_{j=0}^{r} l_j e'/e_j = e'/e_i \tag{3}$$

by (1), where $e'=e_0 e$. By the equation (3), we have $l_j e'/e_j \equiv 0 \pmod{e_j}$ $(0 \leq j \leq r, j \neq i)$. Since $(e_0, e)=(c, d, e)=1$, we have $(e'/e_j, e_j)=1$ and $l_j \equiv 0 \pmod{e_j}$ for every $j \neq i$. Hence we have $l_j=0$ $(0 \leq j \leq r, j \neq i)$ by (3), and $q=q'=0$ by $l_0=0$.

q.e.d.

Definition 7.6. For a bigraded ring $R=\bigoplus_{i,j \in Z} R_{i,j}$, we set $M(R)=\{(i, j); R_{i,j} \neq \{0\}\}$.

Definition 7.7. For a bigraded ring R, let q (resp. q') be the minimum (resp. the maximum) of $\{j/i; (i, j) \in M(R)-(0, 0)\}$ (which exists since R is assumed to be of finite type over k). Then we define $R^l = \bigoplus_{\substack{i,j \\ iq=j}} R_{i,j}$ (resp. $R^r = \bigoplus_{\substack{i,j \\ iq'=j}} R_{i,j}$).

Let Φ be an index of type II. Then, with respect to the bigradation of R_Φ given in §2,

$$R_\Phi{}^l = k^l[x_1, \cdots, x_r]/(x_1{}^{e_1}+a_3 x_2{}^{e_2}+x_3{}^{e_3}, \cdots, x_1{}^{e_1}+a_r x_2{}^{e_2}+x_r{}^{e_r}),$$

$$R_\Phi{}^r = k[u].$$

Let Φ be an index of type III. Then $R_\Phi{}^l=k[u]$ and $R_\Phi{}^r=k[v]$, with respect to the bigradation given in §2.

Let Φ and Φ' be indices of type II or III such that $R_\Phi \simeq R_{\Phi'}$. Let $R_\Phi=\bigoplus_{i,j \in Z} R_{i,j}$ and $R_{\Phi'}=\bigoplus_{i,j \in Z} R'_{i,j}$ be the bigradation given in §2. There are the 2-dimensional tori T and T' of $\mathrm{Aut}^\circ(R_\Phi)$ and $\mathrm{Aut}^\circ(R_{\Phi'})$ associated to the above bigradation of R_{Φ} and $R_{\Phi'}$, respectively. Since $R_{\Phi}\simeq R_{\Phi'}$ and rank $\mathrm{Aut}^{\circ}(R_{\Phi})=2$, there is an isomorphism $\varphi: R_{\Phi}\xrightarrow{\sim} R_{\Phi'}$ such that T is mapped onto T' by the induced isomorphism $\tilde{\varphi}:\mathrm{Aut}^{\circ}(R_{\Phi})\xrightarrow{\sim}\mathrm{Aut}^{\circ}(R_{\Phi'})$. Hence there are rational numbers $\delta\neq 0$ and ν such that $\delta j+\nu i\in \boldsymbol{Z}$ for all $(i, j)\in M(R_{\Phi})$ and $\varphi(R_{i,j})=R'_{i,\delta j+\nu i}$. It is clear that $R_{\Phi}{}^{l}$ is mapped isomorphically onto $R_{\Phi'}^{l}$ if $\delta>0$ and $R_{\Phi'}^{r}$ if $\delta<0$ by φ.

First, let us assume that Φ is of type II. Then Φ' is also of type II and $\delta>0$, because $R^{l}=k[x_1, \cdots, x_r]/(x_1{}^{e_1}+a_3x_2{}^{e_2}+x_3{}^{e_3}, \cdots, x_1{}^{e_1}+a_rx_2{}^{e_2}+x_r{}^{e_r})$ is not generated by one homogeneous element over k. Thus we have $R_{\Phi}^{l}\simeq R_{\Phi'}^{l}$. Let $\Phi=(\boldsymbol{e}, \boldsymbol{a}, m)$ and $\Phi'=(\boldsymbol{e}', \boldsymbol{a}', m')$. Then one obtains $\boldsymbol{e}=\boldsymbol{e}'$ and $\boldsymbol{a}=\boldsymbol{a}'$ by [1, Theorem 5.1], and $m=m'$ by $D(R_{\Phi})=(e/e_1, \cdots, e/e_r, m)$ and $D(R_{\Phi'})=(e/e_1, \cdots, e/e_r, m')$. Thus we obtain:

Proposition 7.8. *Let Φ and Φ' be two indices such that $R_{\Phi}\simeq R_{\Phi'}$. If Φ is of type II, then $\Phi=\Phi'$.*

Thus it remains to treat the case where Φ and Φ' are of type III. Let $\Phi=(\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{c})$ and $\Phi'=(\boldsymbol{e}', \boldsymbol{b}', \boldsymbol{c}')$. We have shown that

$$\varphi(u)=\alpha u \text{ and } \varphi(v)=\beta v \text{ for some } \alpha,\beta\in k^{*} \text{ if } \delta>0,$$
$$\varphi(u)=\alpha v \text{ and } \varphi(v)=\beta u \text{ for some } \alpha,\beta\in k^{*} \text{ if } \delta<0.$$

In each case, it holds that

$$\varphi(uR_{\Phi}+vR_{\Phi})=uR_{\Phi'}+vR_{\Phi'}.$$

Hence φ induces an isomorphism

$$R_{\Phi}/(u, v) \xrightarrow{\sim} R_{\Phi'}/(u, v).$$

Thus it is obvious that $\nu=0$, because $M(R_{\Phi}/(u,v))$, $M(R_{\Phi'}/(u,v))\subset\{(i,0); i\in \boldsymbol{Z}\}$. From $D(R_{\Phi}/(u, v))=[pe/e_1, \cdots, pe/e_r]$ and $D(R_{\Phi'}/(u, v))=[p'e'/e'_1, \cdots, p'e'/e'_{r'}]$, it follows that $r=r'$, $p=p'$ and $\boldsymbol{e}=\boldsymbol{e}'$. Since the two graded k-algebras

$$\begin{aligned} &k[R_{pe/e_1,0}, \cdots, R_{pe/e_r,0}] \\ &\quad =k[x_1, \cdots, x_r]/(x_1{}^{e_1}+b_3x_2{}^{e_2}+x_3{}^{e_3}, \cdots, x_1{}^{e_1}+b_rx_2{}^{e_2}+x_r{}^{e_r}),\end{aligned}$$

and

$$\begin{aligned} &k[R'_{pe/e_1,0}, \cdots, R'_{pe/e_r,0}] \\ &\quad =k[x_1, \cdots, x_r]/(x_1{}^{e_1}+b_3'x_2{}^{e_2}+x_3{}^{e_3}, \cdots, x_1{}^{e_1}+b_r'x_2{}^{e_2}+x_r{}^{e_r})\end{aligned}$$

are isomorphic with each other, one obtains $(b_3, \cdots, b_r)=(b_3', \cdots, b_r')$ by [1, Theorem 5.1]. In view of Lemma 7.5, $R_{\Phi}/(x_1, \cdots, x_r)\simeq R_{\Phi'}/(x_1, \cdots, x_r)$, one has $k[u, v]/(u^{c}v^{d})\simeq k[u, v]/(u^{c'}v^{d'})$. Hence $c=c'$, $d=d'$, $l=l'$, and $m=m'$. From these facts, it follows easily that $b=b'$. Hence we have $\Phi=\Phi'$. Thus we obtain:

Proposition 7.9. *Let Φ and Φ' be two indices such that $R_{\Phi}\simeq R_{\Phi'}$. If Φ is of type III, then $\Phi=\Phi'$.*

Thus Theorem 1.1 is completely proved, by Propositions 7.3, 7.8, and 7.9.

*) There is a $G_m{}^2$-action on R defined by $(\alpha, \beta)(r)=\alpha^i\beta^j r$ for every $\alpha,\beta\in k^*$ and $r\in R_{i,j}$. T is the image of this action in $\mathrm{Aut}^\circ(R)$.

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI, 980 JAPAN

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