J. Math. Kyoto Univ. (JMKYAZ)
26-1 (1986) 95-99

Local normality of a meromorphic function and a Picard type theorem

Dedicated to Prof. Yukio Kusunoki on his 60th birthday

By

Tatsuo FUJI'I'E

Introduction. For a meromorphic function in the unit disk problems of singularity at boundary points have been studied in various manner. We are concerned, in this paper, with the treatment of Picard type theorem by V. I. Gavrilov, P. M. Gauthier and others. A sequence $\{z_n\}$ in the unit disk D is called a sequence of P-points for a meromorphic function f in D if, for every $\varepsilon>0$ and every subsequence $\{z_{n_k}\}$, f assumes every value, except at most two, infinitely often in the set $\bigcup_{k=1}^{\infty}\{z;\ \rho(z, z_{n_k})<\varepsilon\}$, where ρ denotes the hyperbolic metric in D. A normal meromorphic function in D has no such sequences.

In this paper, we construct an analogue of a sequence of P-points under some local normality condition and obtain a Picard type theorem at a regular boundary point of D in the maximal ideal space of H^{∞}.

Preliminaries. Let H^{∞} denote the Banach algebra of bounded analytic functions in D. The maximal ideal space M of H^{∞} is a compact Hausdorff space with Gel'fand topology which contains D as an open dense subset. Each point of $\Delta=M-D$ is classified into Gleason parts, that is, each $m\in\Delta$ belongs to some Gleason part $P(m)$. $P(m)$ is either a one-to-one continuous image of D or a singleton. In the former case, m is called a regular point. A regular point is captured in the closure of an interpolating sequence in D. It was shown by L. Brown and P. M. Gauthier ([1]) that a normal meromorphic function in D is extended continuously to each reqular point $m\in\Delta$. For details of topological structure of Δ, we refer to [5].

1. We denote by ρ and χ the hyperbolic metric on the unit disk $D=(|z|<1)$ and spherical metric, respectively. A meromorphic function f in D is said $\rho-\chi$ continuous at a point ζ on the unit circle if, for arbitrary sequences $\{a_n\}$ and $\{b_n\}$ tending to ζ, $\chi(f(a_n), f(b_n))\to 0(n\to\infty)$ provided that $\rho(a_n, b_n)\to 0$. In addition, M_{ζ} denotes the fiber of Δ over ζ, G the set of regular points of Δ and $L_a=(z+a)/(1+\bar{a}z)$ the conformal mapping of D onto itself.

Most part of the following theorem is due to [1], [2] and [3].

Received October 31, 1984.

Theorem 1. *For a meromorphic function f the followings are equivalent.*

1) *f is not $\rho-\chi$ continuous at $\zeta \in \partial D$.*
2) *There is a sequence a_n tending to ζ for which $\{f \circ L_{a_n}\}$ is not a normal family. We can take as $\{a_n\}$ an interpolating sequence.*
3) *There exists a point $m \in M_\zeta \cap G$ at which f has no continuous extension.*
4) *There is an interpolating sequence of P-points converging to ζ.*

Proof 1)→2) It is enough to prove that $\{f \circ La_n\}$ is not normal at $z=0$ by assuming 1) is true.

If f is not $\rho-\chi$ continuous at ζ there exist sequences $\{a_n\}$ and $\{b_n\}$ converging to ζ such that $\rho(a_n), b_n) \to 0$ but $\limsup \chi(f(a_n), f(b_n)) = \varepsilon > 0$. This means that there exists an arbitrary large number m such that $\chi(f(a_m), f(b_m)) = \chi(f \circ L_{a_m}(0), f \circ L_{a_m}(z_m)) > \frac{\varepsilon}{2}$, where $L_{a_m}(z_m) = b_m$. Therefore, the family $\{f \circ La_n\}$ is not spherically equicontinuous at $z=0$, and hence it is not normal at $z=0$ ([4] p. 244)

Since $\{f \circ La_n\}$ was shown to be non normal, it contains a subsequence $\{f \circ L_{a_k}\}$ every subsequence of which is not convergent locally uniformly. We can choose an interpolating subsequence $\{a_j\}$ of $\{a_k\}$ such that $\{f \circ L_{a_j}\}$ is not normal.

2)→1) We suppose that f is $\rho-\chi$ continuous at ζ. Let $\{a_n\}$ a sequence converging to ζ and $\{b_n\}$ another one such that $\rho(a_n, b_n) \to 0$. By our assumption, we can find δ for $\varepsilon > 0$ so that $\chi(f(a_n), fb_n)) < \varepsilon$ whenever $\rho(a_n, b_n) < \delta$. This means that $\{f \circ L_{a_n}\}$ or $\{(f \circ L_{a_n})^{-1}\}$ is uniformly bounded in $D_\delta = (|z| < \delta)$ and that $\{f \circ L_{a_n}\}$ is normal in D_δ.

1)→3) If f is not $\rho-\chi$ continuous at ζ, we can find sequences $\{a_n\}$ and $\{b_n\}$ such that $\rho(a_n, b_n) \to 0$ but $\chi(f(a_n), f(b_n)) > \varepsilon$. By taking a subsequence, $\{a_n\}$ may be assumed an interpolating and $\overline{\{a_n\}} \cap \Delta \cap M_\zeta \cap G \neq \emptyset$. By the same proof as Theorem 4 in [1], we conclude that $C(f, m)$ is not a singleton for $m \in \overline{\{a_n\}}$.

3)→1) Suppose there exists a point $m \in M_\zeta \cap G$ such that the cluster set $C(f, m)$ contains two values w_1 and w_2. We obtain a contradiction in the same way as Brown Gauthier ([1], Theorem 4).

1)→4) f is not $\rho-\chi$ continuous at ζ if and only if there is a sequence of W-points [2] converging to ζ by definition. And existence of a sequence of W-points is equivalent to that of a sequence of P-points ([13]).

2. Let m be a regular point of Δ and $\{a_\nu\}$ a net converging to m.

Lemma 1. *A meromorphic function f is extendable continuously to m if the family $\{f \circ L_{a_\nu}\}$ is normal for all nets $\{a_\nu\}$ converging to m.*

Proof. We suppose contrary that f is not extendable continuously to m. Then there exist two nets $\{a_\nu\}$ and $\{b_\mu\}$ converging to m and $\lim f(a_\nu) = w_1$, $\lim f(b_\mu) = w_2$; $w_1 \neq w_2$. For neighborhoods $U(m)$, $V(w_1)$ and $V(w_2)$ there exists ν_0 and μ_0 such that a_ν, $b_\mu \in U(m)$ and $f(a_\nu) \in V(w_1)$ and $f(b_\mu) \in V(w_2)$ for $\nu > \nu_0$ and $\mu > \mu_0$, respectively. Here we choose $V(w_1)$ and $V(w_2)$ such as χ-distance of them is greater than $\delta > 0$. And we set $S = \{a_\nu;\ \nu > \nu_0\}$ and $T = \{b_\mu;\ \mu > \mu_0\}$, then the closure of these sets in the maximal ideal space M contains m in common, that is, $\bar{S} \cap \bar{T} \cap G \neq \emptyset$. Hence, by Theorem 3([1]), for any $\varepsilon_n > 0$ there exists r_n such that $\rho(S \cap (|z| > r_n),\ T \cap (|z| > r_n)) < \varepsilon_n$ and we can choose sequences $\{a_n;\ a_n \in S \cap (|z| > r_n)\}$ and $\{b_n \|\ b_n \in T \cap (|z| > r_n)\}$ such that $\rho(a_n, b_n) < 2\varepsilon_n$. Here, if the family $\{f \circ L_{a_\nu}\}$is normal, then $\{f \circ L_{a_n}\}$ is also normal as a subfamily and so spherically equicontinuous in $D_\kappa = (|z| < \kappa)$ for any $\kappa < 1$ ([4] p. 244). Therefore, if we take z_n so that$L_{a_n}(z_n) = b_n$, z_n tends to 0 because $L_{a_n}(0) = a_n$ and $\rho(a_n, b_n) \to 0 (n \to \infty)$, and by equicontinuity $\chi(fL_{a_n}(0), fL_{b_n}(z_n)) < \delta$ which contradicts to our first assumption.

In the proof of the Lemma we arrive at the same contradiction by assuming that $\{f \circ L_{a_\nu}\}_{\nu > \iota}$ is normal at $z = 0$ for some index ι. So, we have a stronger result.

Corollary. *If f is not continuously extendable to m, then there exists a net converging to m such that $\{f \circ L_{a_\nu}\}_{\nu > \iota}$ is not normal at $z = 0$ for any ι.*

Lemma 2. *Let $\{a_\nu\}$ be a net converging to a regular point $m \in \Delta$. If a meromorphic function f is continuously extendable to m, then there exists a net index ν_0 such that the family $\{f \circ L_{a_\nu}\}$ is normal at $z = 0$ for $\nu > \nu_0$.*

Proof. We may assume the extended value $\hat{f}(m)$ is finite, otherwise we consider $1/f$ instead of f. Then there exists a neighborhood U of m such that $|f(z) - \hat{f}(m)| < \varepsilon$ and so, $|f(z)| < |\hat{f}(m)| + \varepsilon$ in $U \cap D$. And the family $\{L_{a_\nu}\}$ of analytic maps of D is convergent to the non constant map L_m of D onto $P(m)$. And, we can find a net index ν_0 and a disk $D_\xi = (|z| < \xi)$ such that $L_{a_\nu}(D_\xi) \subset U$ for $\nu > \nu_0$ ([5] p. 84). This means $|f \circ L_{a_\nu}(z)| < |\hat{f}(m)| + \varepsilon$ in D_ξ for $\nu > \nu_0$, that is, $\{f \circ L_{a_\nu}\}_{\nu > \nu_0}$ is uniformly bounded. Therefore, $\{f \circ L_{a_\nu}\}$ is normal in D_ξ.

By Lemma 1, 2 and Corollary to Lemma 1 we obtain the following

Theorem 2. *A meromorphic function f is not continuously extendable to a regular point m if and only if there exists a net $\{a_\nu\}$ converging to m for which the family $\{f \circ L_{a_\nu}\}_{\nu > \iota}$ is not normal at $z = 0$ for any index ι.*

3. Now we study a Picard type theorem at a regular point m of Δ. As before, $\{a_\nu\}$ denotes a net converging to m.

Theorem 3. *Let f be a meromorphic function such that $\{f \circ L_{a_\nu}\}_{\nu > \iota}$ is not normal at $z = 0$ for any index ι and $\{b_n\}$ an interpolating sequence which contains m in its closure. Then, f assumes every value, except at most two, infinitely often in the set $\cup_{n=1}^{\infty} \{z;\ \rho(z, b_n) < \eta\}$ for each $\eta > 0$.*

Proof. Let A denote the Blaschke product with zeros at $\{b_n\}$, then $U(\varepsilon) = \{p \in M;\ |\hat{A}(p)| < \varepsilon\}$ is a neighborhood of m, where $\hat{A}$ denotes the continuous extension of A to M. $U(\varepsilon) \cap D = \cup R_n$ and R_n is contained in the set $\{z;\ \rho(z, b_n) < \eta\} = L_{b_n}(D_\eta)$; $D_\eta = (|z| < \eta)$ ([5], [6]). And for each $\varepsilon > 0$, as stated before, there exist $D_\xi = (|z| < \xi)$ and a net index ν_0 such that

$$L_{a_\nu}(D_\xi) \subset \cup R_n = \{z;\ |A(z)| < \varepsilon\} \cap D \qquad \text{for} \quad \nu > \nu_0$$

Since $\{f \circ L_{a_\nu}\}_{\nu > \nu_0}$ is not normal in D_ξ by our assumption, $\{f \circ L_{a_\nu}\}$ assumes every value, except possibly two, infinitely often in D_ξ by Montel's theorem and so does f in $\cup R_n$.

Remark. In the proof of the theorem, if ε decreases to 0 η decreases to 0 at the same time. So, f assumes every value infinitely often except at most two in $\cup_n \{z; \rho(z, b_n)<\eta\}$ for every $\eta>0$.

By Theorem 1 and 2, we obtain the following Picard type theorem at a regular point m.

Theorem 4. *If a meromorphic function f in D is not continuously extendable to a regular point m of Δ, f assumes every value, except possibly two, infinitely often in every neighborhood of m.*

In Theorem 3, $\{b_n\}$ was an arbitrary interpolating sequence containing m in its closure. Let $\{b_\nu\}$ be a subnet of $\{b_n\}$ which converges to m. The set $T=\{b_\nu\}$ is considered as an interpolating subsequence $\{b_{n_k}\}$ of $\{b_n\}$ which contains m in its closure, and it has the same property as $\{b_n\}$ in Theorem 3 if f is not continuously extendable to m.

We obtain the following corollary as an anlogy of a sequence of P-points.

Corollary. *If a meromoephic function f in D is not extendable continuously to a regular point $m \in \Delta$, then every interpolating sequence $\{b_n\}$ containing m in its closure has the following property: For each $\eta>0$ and each subnet $\{b_\nu\}$, f assumes every value, except at most two, infinitely often in $\cup_\nu \{z; \rho(z, z_\nu)<\eta\}$*

The following Weierstrass type theorem is a corollary to Theorem 4 but we give another proof.

Corollary. *If a meromorphic function f in D is not continuously extendable to a regular point m of Δ, then every complex number α belongs to the cluster set $C(f, m)$ of f at m, that is, $C(f, m)$ is total.*

Proof. By our assumption, $C(f, m)$ contains at least two values w_1 and w_2, and there exist two nets $\{a_\nu\}$ and $\{b_\mu\}$ converging to m such that $\lim f(a_\nu)=w_1$ and $\lim f(b_\mu)=w_2$, respectively. We choose neighborhoods $V(w_1)$ and $V(w_2)$ so that $\chi(V(w_1), V(w_2))>\varepsilon>0$. And then we choose net indices ν_0 and μ_0 so that $f(a_\nu)\in V(w_1)$ and $f(b_\mu)\in V(w_2)$ for $\nu>\nu_0$ and $\mu>\mu_0$, respectively.

Now we suppose $\alpha \notin C(f, m)$, then $g(z)=1/(f(z)-\alpha)$ is bounded in $U\cap D$ for a neighborhood U of m. And, as stated before, there exist $r>0$, net indices ν_1 and μ_1 such that $L_{a_\nu}(D_r)$, $L_{a_\mu}(D_r)\subset U$ for $\nu>\nu_1$ and $\mu>\mu_1$, respectively. Therefore, the family $\{g\circ L_{a_\nu}\}_{\nu>\nu_1}$ of holomorphic functions is uniformly bounded, and hence equicontinuous in D_r, that means, for all $\nu>\nu_1$, $|g\circ L_{a_\nu}(z_n)-g\circ L_{a_\nu}(0)|\to 0$ if $z_n\to 0$.

We set $S=\{a_\nu; \nu>\nu_0, \nu_1\}$ and $T=\{b_\mu; \mu>\mu_0, \mu_1\}$, then $\bar{S}\cap\bar{T}\cap G$ contains m and is not empty. Hence, as stated before, there are sequences $\{a_n; a_n\in S\}$ and $\{b_n; b_n\in T\}$ such that $\rho(a_n, b_n)\to 0\,(n\to\infty)$. Put $a_n=L_{a_{\nu_n}}(0)$ and choose z_n so that $L_{a_{\nu_n}}(z_n)=b_n$, then we obtain

$|g\circ L_{a_{\nu_n}}(0)-g\circ L_{a_{\nu_n}}(z_n)|\to 0$ $(n\to\infty)$, which contradicts the first setting $\chi(V(w_1), V(w_2))>\varepsilon>0$.

DEPARTMENT OF MATHEMATICS
COLLEGE OF LIBERAL ARTS
KYOTO UNIVERSITY

References

[1] L. Brown and P. M. Gauthier, Behavior of normal meromorphic functions on maximal ideal space of H^∞, Michigan. Math. J., **18** (1971), 365–371.

[2] D. M. Campbell and G. Wickes, Characterization of normal meromorphic functions, Complex analysis, Joensuu 1978. Lecture Notes Math. 747, Berlin-Heidelberg-New York, 1979, 55–72.

[3] P. M. Gauthier, A criterion for normalcy, Nagoya Math. J., **32** (1968), 277–282.

[4] E. Hille, Analytic function theory vol. 2, Boston, 1962.

[5] K. Hoffman, Bounded analytic functions and Gleason parts, Ann. of Math., (2) **86** (1967), 74–111.

[6] A. Kerr-Lawson, A filter description of the homomorphism of H^∞, Canad. J. Math., **17** (1965), 734–757.