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Trace moduli for quasifuchsian groups

Dedicated to Professor Yukio Kusunoki on his 60th birthday

By

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0. Introduction.

In earlier work [13, 14, 15] we consider the problem of finding moduli for the Teichmüller space of marked Fuchsian groups, $T(G)$. Our approach is to decompose the Fuchsian group G into two-generator subgroups. We construct fundamental polygons for these subgroups and then combine them into a single fundamental polygon P for the full group. Although the number of sides P has is bigger than the minimum number of sides a polygon must have, it is not much bigger. "Geometric" moduli are determined as lengths of certain of the sides of P, and distances between particular pairs of the other sides. These geometric moduli also have an interpretation as the traces of a particular set S of elements G. This parametrization determines a real analytic equivalence between $T(G)$ and a simply connected subset of Euclidean space R^p, for appropriate p.

In this paper, we generalize these results to a special class of quasifuchsian groups. In the space of marked quasifuchsian groups, stability considerations tell us that the same set of traces can be used as moduli in a neighborhood of the Fuchsian locus. The number of faces of the fundamental polyhedra in H^3 for these groups, however, can be arbitrarily large. Our class contains quasifuchsian groups that have fundamental polyhedra with a constant "small" number of sides and no interior vertices, and can be parametrized by this same set of traces. It extends out of the neighborhood of the Fuchsian locus and intersects the boundary in B-groups.

1. Preliminaries.

In this section we recall some results about Fuchsian groups and set our notation. Let G be a marked Fuchsian group of type (g, n). Denote the trace of an element $W \in G$ by tr W. tr W is real since G is Fuchsian. We will assume for simplicity of exposition that $(\mathrm{tr}\, W)^2 > 4$. If this is not the case, the constructions and theorems must be slightly modified. We say something about this in the last section. If U is the disk in $\hat{C}$, invariant under G, the attracting fixed point q_W and repelling fixed point p_W of W lie on its boundary, ∂U.

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A collection, $\{F_i\}_1^{2g-2+n}$, of two generator subgroups of G such that G can be reconstructed from the F_i by a series of free products with amalgamation is called a *subgroup decomposition* of G. The subgroups fall into two categories; the first, denoted by T, of type (1, 1), and the second, denoted by P, of type (0, 3).

For the given marked Fuchsian group, $G=\langle A_1, B_1,\dots, A_g, B_g, M_1,\dots, M_n; \prod_1^n M_j \prod_1^g (B_i^{-1}A_i^{-1}B_iA_i)=id\rangle$, we form the *canonical subgroup decomposition.* We have: $\{T_i=\langle A_i, B_i\rangle\ i=1,\dots, g,\ P_j=\langle C_j, C_{j+1}\rangle, j=1,\dots, g-2+n\}$ where the A_i and B_i are the marked elements and the C_j's are defined as follows:

$$\begin{array}{lll} \text{let} & K_i=B_i^{-1}A_i^{-1}B_iA_i, & i=1,\dots, g \\ & K_{i+g}=M_i & i=1,\dots, n, \\ \text{set} & C_1=K_1,\ C_j=C_{j-1}K_j & j=2,\dots, g+n-2 \\ & C_{g+n-1}=M_{n-1}. & \end{array}$$

We define trace moduli for the group G from the canonical subgroup decomposition. For each group T_i we have four parameters: $x_i=\operatorname{tr} A_i$, $y_i=\operatorname{tr} B_i$, $z_i=\operatorname{tr} A_iB_i$ and $k_i=2-\operatorname{tr} K_i$. (The last is chosen this way to facilitate computation). These satisfy the conditions:

$$x_i^2+y_i^2+z_i^2-x_iy_iz_i=4-k_i, \tag{1.1}$$

$$x_i>2,\ y_i>2,\ z_i>2,\ k_i>4. \tag{1.2}$$

For each group P_j we use three parameters: $r_j=\operatorname{tr} C_j$, $s_j=\operatorname{tr} C_{j+1}$, $t_j=\operatorname{tr} C_j^{-1}C_{j+1}$. These satisfy:

$$r_js_jt_j<0,\ |r_j|>2,\ |s_j|>2,\ |t_j|>2. \tag{1.3}$$

To reconstrust G from the subgroup decomposition, we perform a series of free products with amalgamation. We call these gluing operations. Each requires one parameter, or when using traces, two new traces and one new relation. Suppose we have formed $G_{k-1}=G_{k-2}*P_{j-1}\operatorname{am}\langle C_j\rangle$ and now want $G_k=G_{k-1}*P_j\operatorname{am}\langle C_{j+1}\rangle$. We introduce the traces $l_j=\operatorname{tr} C_{j-1}C_{j+1}$ and $m_j=\operatorname{tr} K_jC_{j+1}$. These satisfy the *gluing relation*

$$r_j^2+m_j^2+l_j^2-r_jm_jl_j+J_j^1r_j+J_j^2m_j+J_j^3l_j+J_j^4=0 \tag{1.2}$$

where

$$\begin{aligned} J_j^1&=s_jt_j+s_{j+1}t_{j+1}\\ J_j^2&=t_jt_{j+1}+s_js_{j+1}\\ J_j^3&=s_jt_{j+1}+s_{j+1}t_j\\ J_j^4&=s_j^2+t_j^2+s_{j+1}^2+t_{j+1}^2+s_jt_js_{j+1}t_{j+1}-4\\ &\quad l_j<-2,\ m_j<-2. \end{aligned}$$

In [13, 15] we prove

Theorem 1. *Let G be a marked Fuchsian group of type $(g; n)$. Let $\{T_i, P_j\}$ be a canonical subgroup decomposition for G. The set of trace parameters for these subgroups and the trace parameters for the gluing operations together with their relations form —taking repetitions into account— a set $\mathscr{T}(G)$ of $9g-9+4n$ moduli which satisfy $3g-3+n$ relations. These moduli determine a real analytic embedding of the Teichmüller space of marked surfaces of type $(g; n)$, $T(G)$, into a simply connected domain in $\boldsymbol{R}^{6g-6+3n}$.*

In the proof we take a point in the moduli space and write down generators for the group. We prove the group is Fuchsian by constructing a fundamental polygon for it. We construct the fundamental polygon from fundamental polygons for each of the subgroups in the decomposition.

2.1. Quasifuchsian groups. Let Γ be a marked quasifuchsian group and let Ω_Γ be its domain of discontinuity. Let φ be an isomorphism, $\varphi: G \to \Gamma$, which takes the marking for G into the marking for Γ. If G is of type (g, n) so is Γ. (Note that we include quasifuchsian groups of the second kind in our discusion). Since G is quasiconformally stable, we have:

Theorem 2. *Given G, there is an $\varepsilon > 0$, such that if $|\mathrm{tr}\,\varphi(W) - \mathrm{tr}\,W| < \varepsilon$,* for all *$W \in \mathscr{T}(G)$, then $\mathscr{T}(\Gamma)$ is a set of moduli for Γ.*

Let φ^* be the map induced by φ on the traces of the elements of G. Below, we determine another set of necessary conditions which imply that $\mathscr{T}(\Gamma) = \varphi^*(\mathscr{T}(G))$ is a set of moduli for Γ.

Set $T_i = \varphi(T_i)$ and $P_j = \varphi(P_j)$. We call these two generator quasifuchsian groups, *basic quasifuchsian groups*, and construct a canonical fundamental polyhedron for each of them in hyperbolic 3-space, H^3. The subgroups $\{T_i, P_j\}$ form a subgroup decomposition for Γ, and the reconstruction of Γ from them by a series of free products with amalgamation determines a set of gluing procedures for the canonical polyhedra. Interpreting these gluing operations geometrically, we obtain a fundamental polyhedron for Γ.

Although the groups T and P are not distinct types of quasifuchsian groups, their Fuchsian preimages have different properties and we will use them differently. Each of them, acting on $\hat{C}$ represents a Riemann surface of genus 2, and the Weierstrass points of these surfaces play an important role in keeping our constructions intrinsic.

We carry out our construction for a group Γ of type (2, 2). This is the least complicated case for which all the different kinds of gluing occur. A convenient (though not quite standard) presentation of Γ is:

$$\Gamma = \langle A_1, B_1, A_2, B_2, C_1, C_2;\ K_1 = B_1^{-1} A_1^{-1} B_1 A_1,$$
$$K_2 = B_2^{-1} A_2^{-1} B_2 A_2,\ K_2 C_2 K_1 C_1 = id \rangle .$$

The subgroup decomposition we use is:

$$\{T_1=\langle A_1, B_1\rangle, T_2=\langle A_2, B_2\rangle, P_1=\langle C_1, (K_1C_1)^{-1}\rangle, P_2=\langle K_1C_1, C_2\rangle\}.$$

We perform the amalgamations in the following order:

$$\Gamma_1=T_1*P_1 \text{ am } \langle K_1\rangle; \qquad \Gamma_1 \text{ has type } (1, 2);$$

$$\Gamma_2=\Gamma_1*P_2 \text{ am } \langle K_1C_1\rangle; \qquad \Gamma_2 \text{ has type } (1, 3);$$

$$\Gamma=\Gamma_2*T_2 \text{ am } \langle K_2\rangle; \qquad \Gamma \text{ has type } (2, 2).$$

2.2. In this section we consider a group $P=\langle A, B\rangle$ and make special assumptions about it. We assume it is again Fuchsian, and as a Fuchsian group, is of type (0, 3). The trace moduli r, s, and t are therefore all real; we assume P is represented in SL(2, $\boldsymbol{C}$) so that they satisfy $r<-2$, $s>2$, and $t>2$. Such a group is a Schottky group and we form the fundamental domain Δ_P as the common exterior of the isometric circles of A, B, A^{-1}, B^{-1} respectively. We may noramlize so that P fixes the upper half plane, and so that the hyperelliptic involution is of the form, E: $z\to -z$ (see [17]). The lifted Weierstrass points are then 0, ∞, α, α', β, β'. They are colinear; 0 and ∞ lie interior to Δ_P and the others lie on its boundary. We extend Δ_P to H^3 by erecting hemispheres on the isometric circles and obtain a fundamental polyhedron which has no vertices. (See figure 1).

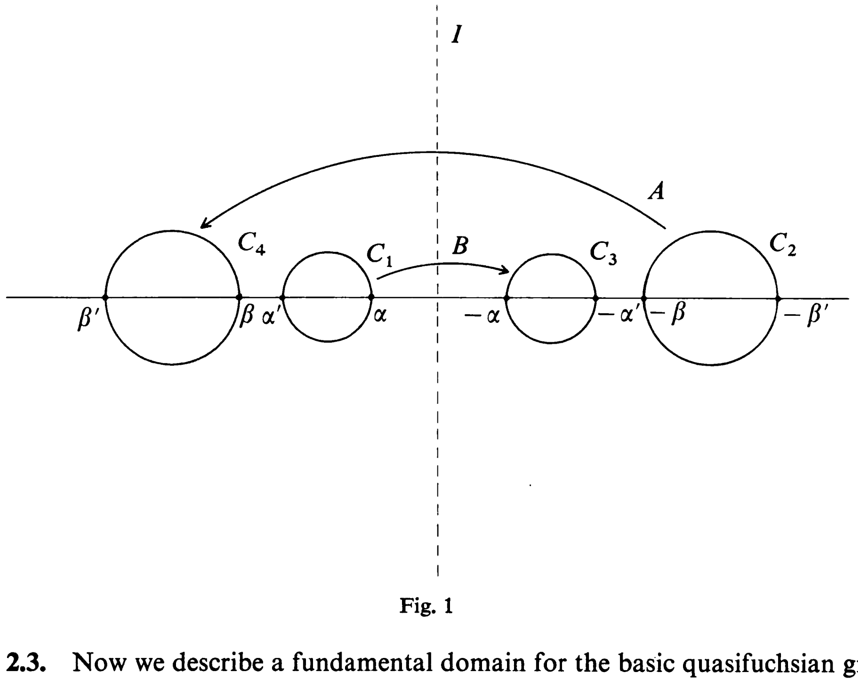

Fig. 1

2.3. Now we describe a fundamental domain for the basic quasifuchsian group $T=\langle A, B\rangle$. The special assumptions we make now are that the trace moduli x, y, z, k satisfy z real, $z>2$, k real and $k>4$. The philosophy behind the assumption that certain traces are real is the following. Each group element corresponds to a free homotopy class of curves on a Riemann surface associated to the group. The free product with amalgamation of two groups corresponds to gluing two surfaces together along a curve in the homotopy class corresponding to the generator of the amalgamated subgroup. We assume that the elements corresponding to curves we glue along are real. The group T is assumed to have been obtained from its subgroup $P=\langle AB, BA\rangle$ by gluing along the curves corresponding to AB and BA. This construction is similar to free product with amalgamation and is called the HNN construction. (See [20] for a discussion of HNN constructions).

Using the normalization described in the previous section we write down elements of T as follows:

$$A=\frac{1}{2}\begin{bmatrix} x & -Xs \\ -X/s & x \end{bmatrix} \qquad B=\frac{1}{2}\begin{bmatrix} y & Yt \\ Y/t & y \end{bmatrix}$$

$$AB=\frac{1}{2}\begin{bmatrix} z-i\sqrt{k} & (z^2+k-4)/z \\ z & z+i\sqrt{k} \end{bmatrix} \qquad BA=\frac{1}{2}\begin{bmatrix} z+i\sqrt{k} & (z^2+k-4)/z \\ z & z-i\sqrt{k} \end{bmatrix}$$

$$K=\frac{1}{2}\begin{bmatrix} 2-k+iz\sqrt{k} & (z^2+k-4)i\sqrt{k}/z \\ -iz\sqrt{k} & 2-k-iz\sqrt{k} \end{bmatrix} \quad \text{and}$$

$$L=ABA^{-1}B^{-1}=\frac{1}{2}\begin{bmatrix} 2-k-iz\sqrt{k} & (z^2+k-4)i\sqrt{k}/z \\ -iz\sqrt{k} & 2-k+iz\sqrt{k} \end{bmatrix}$$

$$X=\sqrt{x^2-4}, \quad Y=\sqrt{y^2-4}, \quad Z=\sqrt{z^2-4}$$

$$s=(2y-xz+ix\sqrt{k})/Xz \quad t=(yz-2x+iy\sqrt{k})/Yz.$$

where

$$p_A=-q_A=s, \quad p_B=-q_B=t \quad \text{and} \quad (p_K+q_K)/2=-1.$$

and

$$p_K=(-z+i\sqrt{k-4})/z, \quad q_K=(-z-i\sqrt{k-4})/z, \quad p_L=(z+i\sqrt{k-4})/z \quad \text{and}$$
$$q_L=(z-i\sqrt{k-4})/z.$$

The Weierstrass points α, α', β, β' are:

$$\alpha=(y-2)t/Y \quad \alpha'=(y+2)t/Y,$$

$$\beta=(x-2)s/X \quad \beta'=(x+2)s/X$$

We note for further use that

$$p_{AB}=(-Z-i\sqrt{k})/z, \quad q_{AB}=(+Z-i\sqrt{k})/z, \quad p_{BA}=(-Z+i\sqrt{k})/z, \quad q_{BA}=(Z+i\sqrt{k})/z$$

The points α, β, α' and β' again lie on a circle which we call C_{BA} since it also contains p_{BA} and q_{BA}. To see that this is so, we compute the cross-ratios:

$$(p_{BA}, q_{BA}, \alpha, \alpha')=(p_{BA}, q_{BA}, \beta, \beta')=-1, \quad (\alpha, \alpha', \beta, \beta')=(z+2)/4.$$

Similarly we call the circle through $-\alpha$, $-\alpha'$, $-\beta$, $-\beta'$, p_{AB} and q_{AB}, C_{AB}. A and B

map these circles onto one another; that is $A(C_{BA})=C_{AB}$, $B(C_{AB})=C_{BA}$. The involution E interchanges these circles. We can compute the centers of these circles easily. The center of C_{BA} is ic where $-c=\dfrac{|p_{BA}|^2-|\beta|^2}{2[\operatorname{Im}\beta-\operatorname{Im}p_{BA}]}=\dfrac{|p_{BA}|^2-|\beta'|^2}{2[\operatorname{Im}\beta'-\operatorname{Im}p_{AB}]}$ $=\dfrac{|p_{BA}|^2-|\alpha|^2}{2[\operatorname{Im}\alpha-\operatorname{Im}p_{BA}]}=\dfrac{|p_{BA}|^2-|\alpha'|^2}{2[\operatorname{Im}\alpha-\operatorname{Im}p_{BA}]}$. The center of C_{AB} is $-ic$. These circles are the natural generalization of the axes of the elements AB and BA when the group is Fuchsian. To keep the number of faces down, we require that the circles be disjoint. It is an easy exercise to compute that they will be disjoint if and only if $c>(z^2+k-4)/2\sqrt{k}z$. In terms of the moduli, this inequality becomes:

$$\left||x|^2+|y|^2-z\operatorname{Re}(\bar{x}y)+\frac{(+z^2-4+k)}{2\sqrt{k}}\operatorname{Im}(\bar{x}y)\right|< \frac{(z^2+k-4)}{2\sqrt{k}}|z\operatorname{Im}x-2\operatorname{Im}y-\sqrt{k}\operatorname{Re}x| \tag{2.1}$$

From the formulas in section 2 we see that the points p_K, q_K, p_{AB}, q_{AB}, p_L, q_L, p_{BA} and q_{BA} all lie on a circle $\mathscr{L}$ of radius $(z^2+k-4)^{1/2}/z$ centered at the origin. There is a circle E uniquely defined by the conditions:

i) E is orthogonal to C_{BA}.
ii) E is orthogonal to $\mathscr{L}$.
iii) p_K and p_{BA} are interior to E.

Similarly there is a unique circle D defined by the conditions:

i) D is orthogonal to C_{AB}.
ii) D is orthogonal to $\mathscr{L}$.
iii) q_K and p_{AB} are interior to D.

Let $E'=BA(E)$ and $D'=AB(D)$. By the symmetries of the normalization these circles occur as they are shown in figure 2. Also by these symmetries these four circles are all the same size and are therefore isometric circles of the respective transformations. If $x=\bar{y}$ then $I=A(E)=B(D)$ is the imaginary axis. It is invariant under the hyperelliptic involution and, more important for our purposes, it is orthogonal to both the circles C_{AB} and C_{BA}. Call the common exterior of the six circles D, C_{AB}, D', E, C_{BA} and E', Δ_T. We claim it is a fundamental domain for T. Looking at the identifications (see figure 2), we see that this follows from a standard argument on combining groups. (See for example, [1, p. 103]).

We extend this domain to H^3 in the usual way, by erecting hemispheres. Since both A and B^{-1} map C_{BA} onto C_{AB}, we see that (figure 2) the polyhedron we have constructed has 8 faces.

3.1. In this section we describe how we glue groups $T=\langle A_1, B_1\rangle$ and $\tilde{P}=\langle M, N\rangle$ together using a gluing parameter. The moduli x_1, y_1, z_1, k_1 for T satisfy the inequalities, $z_1>2$, $k_1>4$ and (2.1) while the moduli t_1, t_2, t_3 for $\tilde{P}$ satisfy $t_1>2$, $t_2>2$ and $t_3<-2$. In order to glue, we must also have $t_1=2-k_1$. We construct the normalized fundamental domains Δ_T and $\Delta_{\tilde{P}}$ as in sections 2.2 and 2.3.

These constructions use more normalization parameters than we are allowed. Therefore, we use our gluing parameter δ to determine the element R in $\mathrm{SL}(2, \boldsymbol{C})$

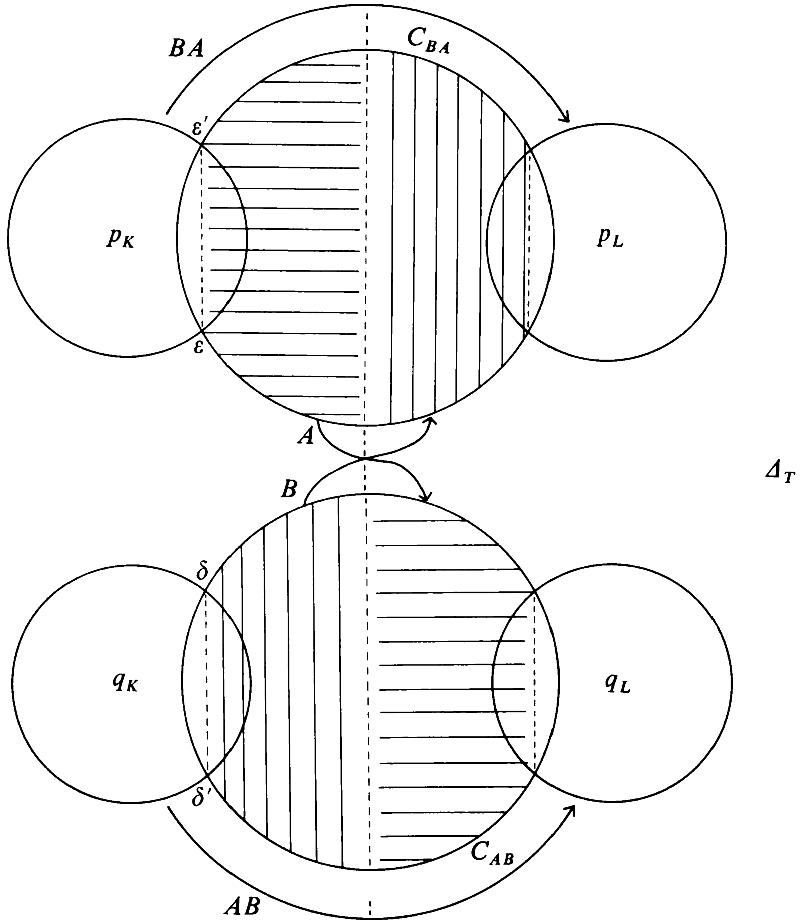

Fig. 2

which conjugates $\tilde{P}$ into an unnormalized group $P_1=\langle K_1, C_1\rangle$, so that we may form $\Gamma_1=T*P_1$ am $\langle K_1\rangle$. We take $\delta=q_C$, the attracting fixed point of the element C_1 in P_1, and determine R by the conditions:

$$R\colon p_M \longrightarrow q_{K_1}, \quad R\colon q_M \longrightarrow p_{K_1}, \quad R\colon q_N \longrightarrow \delta.$$

To form the new fundamental domain, we choose δ so that:

i) δ is interior to E, and p_C, the other fixed point of C_1 is exterior to E.

ii) The circle D separates the fixed points of K_1C_1.

R then maps the real axis R onto a circle X through the points p_K, q_K and δ. Conditions (i) and (ii) imply that the circles E, $E_1=C_1^{-1}(E)$, D and $D_1=(K_1C_1)^{-1}(D)$ are mutually disjoint and bound a fundamental domain Δ_{P_1} for P_1.

To obtain a fundamental domain for Γ_1, we choose a circle, I_1, which is orthogonal to X and which separates the circles D and E from D_1 and E_1. I_1 is then interior to both Δ_{P_1} and Δ_T; moreover, so is the circle $I_1'=B_1A_1C_1(I_1)$. Again, using standard combination arguments, we see that the domain Δ_{Γ_1}, bounded by the circles $C_{A_1B_1}$, D, $A_1B_1(D)$, $C_{B_1A_1}$, E, $B_1A_1(E)$, I_1 and I_1' is indeed a fundamental domain for Γ_1. (See figure 3).

The polyhedron in H^3 bounded by hemispheres over these circles has two new faces (the hemispheres over I_1 and I_1'). It has no interior vertices.

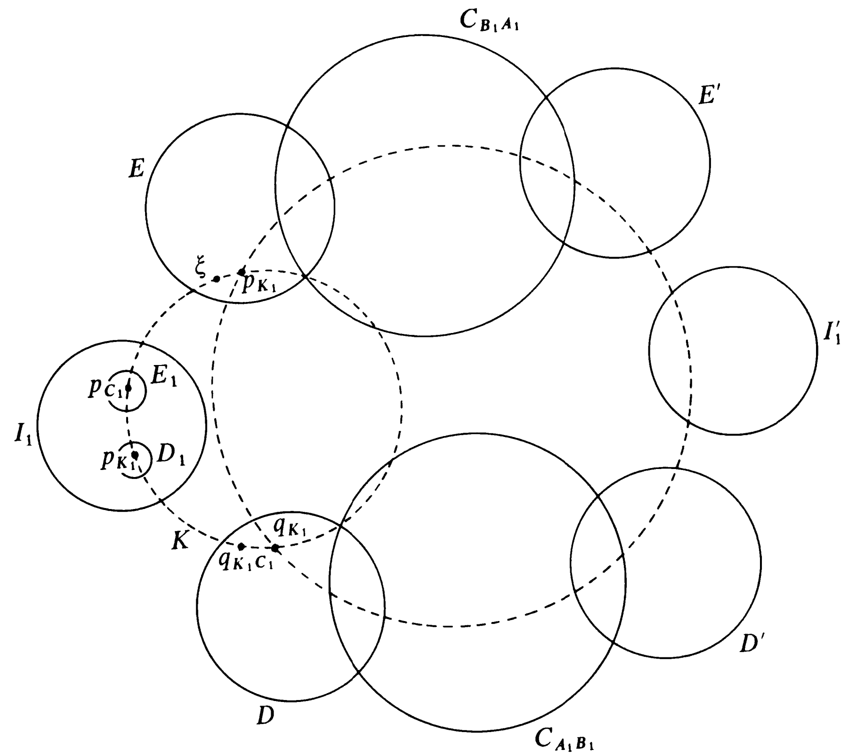

Fig. 3

3.2. Next, we use moduli, t_2, t_3, t_4 where $t_2>2$, $t_3>2$, $t_4<-2$, to form another normalized group $\tilde{P}=\langle M, N\rangle$. Again, we are not allowed to normalize, so we use the gluing parameter to conjugate $\tilde{P}$ into a group $P_2=\langle K_1C_1, C_2\rangle$, which we amalgamate to Γ_1; that is, we form $\Gamma_2=\Gamma_1 * P_2 \text{ am } \langle K_1C_1\rangle$. As above, the gluing parameter v determines the element R in $\mathrm{SL}(2, \boldsymbol{C})$ which performs the conjugation. We take v as the attracting fixed point q_C of the element C_2 in P_2, and find R by the conditions:

$$R: p_{MN} \longrightarrow q_{K_1C_1}\ R: q_{MN} \longrightarrow p_{K_1C_1}\ R: p_N \longrightarrow v.$$

To form the new fundamental domain, we choose v so that:

i) The circle E separates the fixed points of C_2 and v is not in Δ_{Γ_1} while q_C is.

ii) The circle I_1 separates the fixed points of $C_2K_1C_1$, and $q_{C_2K_1C_1}$ is inside Δ_{Γ_1} while $p_{C_2K_1C_1}$ is not.

R then maps the real axis R onto a circle X' through the points $p_{K_1C_1}$, $q_{K_1C_1}$ and v. Conditions (i) and (ii) imply that the circles D, $D_2=C_2(D)$, I_1 and $I_1''=C_2K_1C_1(I_1)$

are mutually disjoint and bound a fundamental domain Δ_{P_2} for P_2.

Now to find a fundamental domain for Γ_2, we choose a circle, I_2, which is orthogonal to X' and which separates the circles D_2 and I_1'' from D and I_1. I_2 must be interior to Δ_Γ; moreover, so is the circle $I_2'=A_1B_1C_1^{-1}(I_2)$. Again, using standard combination arguments, we see that the domain Δ_{Γ_2}, defined as the intersection of Δ_{Γ_1} and the common exteriors of I_2 and I_2' is a fundamental domain for Γ_2.

The new polyhedron in H^3 bounded by these circles has two new faces and no new vertices.

3.3. The last amalgamation we perform in the construction of Γ, is of another group $T=\langle A_2, B_2\rangle$. Having done this, it will be clear how to amalgamate any number of basic groups, P, or T, to obtain a quasifuchsian group of type (g, n) (except for the case (2, 0) which we treat in the next section).

The trace moduli, x_2, y_2, z_2, k_2, satisfy $z_2>2$, $k_2>4$, and inequality 2.1. They determine a normalized group, $\tilde{T}=\langle A, B\rangle$, with fundamental domain $\Delta_{\tilde{T}}$ as in section 2.2. The gluing parameter ζ ,is chosen as the repelling fixed point, $p_{B_2A_2}$, of B_2A_2. It determines the conjugating transformation R, $T_2=RTR^{-1}=\langle A_2, B_2\rangle$ so that $RKR^{-1}=C_2K_1C_1$, provided that $t_4=2-k_2$. We choose ζ so that:

i) I_2 separates the fixed points of B_2A_2 in such a way that ζ does not lie in Δ_Γ, and $q_{B_2A_2}$ does.

ii) I_1 separates the fixed points of A_2B_2 so that $p_{A_2B_2}$ lies in Δ_{Γ_2}, and $q_{A_2B_2}$ does not.

iii) The image of the imaginary axis $I_3=R(I)$ does not intersect any of the circles bounding Δ_{Γ_2} except I_1 and I_2.

The circles $C_{A_2B_2}$ and $C_{B_2A_2}$ are canonically defined; the circles I_1 and I_2 are not. They are therefore not necessarily the images of the circles D and E of $\Delta_{\tilde{T}}$. We note, however, that the circles $B_2^{-1}(I_3)$ and $A_2^{-1}(I_3)$ intersect the circles $C_{A_2B_2}$ and $C_{B_2A_2}$ orthogonally. Moreover, we may replace the circles I_1 and I_2 by these circles, and I_1' and I_2' by their images under $B_1A_1C_1$ and $A_1B_1C_2^{-1}$ respectively, without changing the rest of the configuration. Call the replaced circles, I_1, I_2, I_1' and I_2' again. It is clear that I_3 does not intersect either I_1 or I_2 now, and furthermore, that the construction is now canonical.

We complete our fundamental domain for Γ by adding the circles $C'_{A_2B_2}=B_1A_1C_1(C_{A_2B_2})$ and $C'_{B_2A_2}=A_1B_1C_2^{-1}(C_{B_2A_2})$. Comparing the identifications on the circles, we again apply standard combination techniques to conclude that the common exterior of all of the circles is a fundamental domain for $\Gamma=\Gamma_2 * T_2$ am $\langle K_2\rangle$.

The new polyhedron in H^3 bounded by hemispheres over the circles bounding Δ_Γ has eight faces more than that of Δ_{Γ_2}, for a total of twenty. It has *no* interior vertices.

3.4. The final combination construction is to form a group of type (2, 0) from moduli, x_i, y_i, z_i, k_i, $i=1, 2$ which satisfy z_i real, $z_i>2$, k_i real, $k_1=k_2>4$. Moreover, each quadruple satisfies relation 1.1 and inequality 2.1.

We form the normalized groups $\tilde{T}_i=\langle A_i, B_i\rangle$, $i=1, 2$. We set $T_1=\tilde{T}_1$. Next, we use a gluing parameter δ to find a transformation $R=R(\delta)$ and set $T_2=R\hat{T}_1R^{-1}$. δ is the repelling fixed point $p_{A_2B_2}$ of A_2B_2. R is defined by:

$$R\colon p_{K_2} \longrightarrow q_{K_1}, \quad q_{K_2} \longrightarrow p_{K_1}, \quad p_{A_2B_2} \longrightarrow \delta.$$

We choose δ inside the circle E of Δ_{T_1} subject to the conditions:

i) E separates the fixed points of A_2B_2,

ii) D separates the fixed points of B_2A_2,

iii) $C_{B_2A_2}$ and $C_{A_2B_2}$ are disjoint from $C_{A_1B_1}$ and $C_{B_1A_1}$.

We can find a one-parameter family of circles orthogonal to both $C_{B_1A_1}$ and $C_{A_2B_2}$; denote this family $E(\tau)$. Set $I(\tau)=A_2(E(\tau))$ and $D(\tau)=B_2^{-1}(I(\tau))$. $D(\tau)$ must separate the various fixed points of elements of T_1 and T_2 the same way D does. $D(\tau)$ is orthogonal to $C_{B_2A_2}$. We can choose $\tau=\tau_0$ so that $D(\tau_0)$ is also orthogonal to $C_{A_1B_1}$. Replace the circles D and E by $D(\tau_0)$ and $E(\tau_0)$. Let $I'(\tau_0)=B_1A_1(A_2B_2)^{-1}(I(\tau_0))$. Replace D' and E' by $D'(\tau_0)=A_1B_1(D(\tau_0))$ and $E'(\tau_0)=B_1A_1(E(\tau_0))$. We now drop the τ_0 from our notation. Set $C'_{A_2B_2}=A_1B_1(C_{B_2A_2})=B_1A_1A_2^{-1}(C_{A_2B_2})$ and $C'_{B_2A_2}=B_1A_1(C_{A_2B_2})=A_1B_1B_2(C_{B_2A_2})$.

Let Δ_Γ be the exterior of the domain bounded by the twelve circles (see figure 4) E, $C_{B_1A_1}$, E', $C'_{B_2A_2}$, I', $C'_{A_2B_2}$, D', $C_{A_1B_1}$, D, $C_{B_2A_2}$, I, $C_{A_2B_2}$. The region has two connected components.

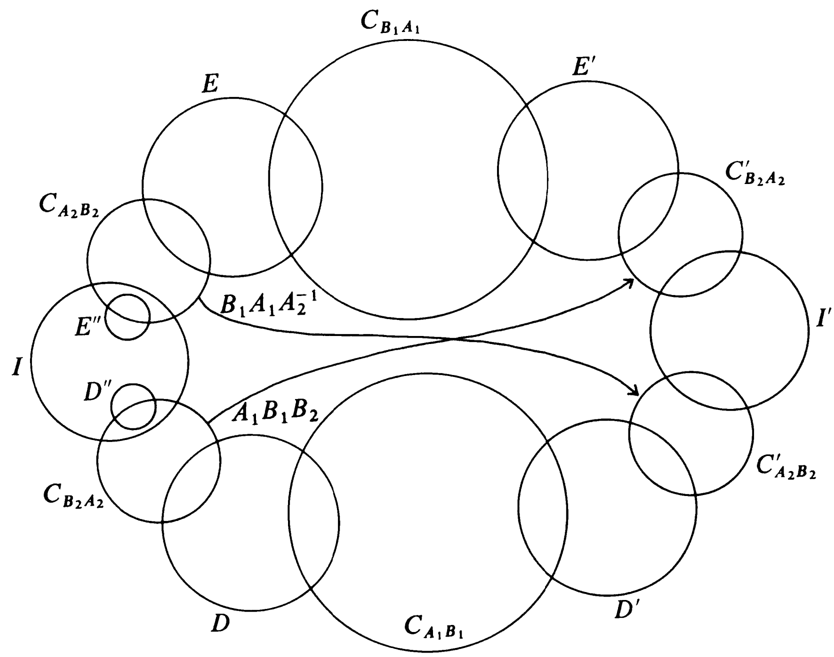

Fig. 4

Again combination arguments show that Δ_Γ is a fundamental domain for Γ. Erecting hemispheres on these circles we obtain a polyhedron with 16 faces and no interior vertices.

4.1. In the previous sections we gave geometric conditions on the gluing parameters, δ, v, and ζ so that the constructions would work. One can explicitly write them down as inequalities. Each condition involves all of the traces and gluing parameters used up to that point. They are however, not very instructive.

An examination of the conditions shows that only those which insist on the reality of a trace cut down the local dimension of the space of groups we have defined.

4.2. Gluing Moduli as Trace Parameters. In [13] we showed how to translate the twist parameters for Fuchsian groups into trace moduli. Although all the computations assumed that the variables were real they are valid if the variables are complex. The twist parameters of that work are algebraically the same as our gluing moduli. In the generic case we had two groups $G_1=\langle C_1, C_2\rangle$ and $G_2=\langle C_3, C_4\rangle$ and formed the product $G=G_1*G_2$ am $\langle C_1C_2=(C_3C_4)^{-1}\rangle$. The normalization put the fixed points of C_1C_2 at 0, ∞ and the fixed point of C_2 at 1. The fixed point ξ of C_3 was the gluing parameter. The trace parameters were $x_i=\operatorname{tr} C_i$, $i=1,\dots, 4$, $k=\operatorname{tr} C_1C_2$, $l=\operatorname{tr} C_1C_3$ and $m=\operatorname{tr} C_2C_3$. These satisfy a relation just like 1.2. l and m can be expressed as linear functions of ξ and ξ^{-1} so that if ξ is real and negative, l and m are complex if and only if ξ is. Also, as ξ varies in the simply connected domain defined by the inequalities in chapter 3, l and m vary in a simply connected domain M in C^2. Therefore, the set of moduli for $T(\Gamma)$ defined by the trace parameters and gluing parameters is equivalent to a set of moduli consisting only of traces.

Main Theorem. *Let Γ be a marked quasifuchsian group of type $(g; n)$ with a standard subgroup decomposition $\{\varphi_i\}$. Let $\mathscr{T}(\Gamma)$ be the set of traces for Γ defined in section* 2.1. *We suppose these $9g-9+4n$ traces satisfy the following constraints: we assume the traces of the generators of the basic quasifuchsian groups P_i are real and satisfy conditions* 1.3; *we also assume those of the groups T_i satisfy conditions* 1.1, *that z_i, k_i are real and that inequality* 2.1 *holds; we further assume that those corresponding to gluing moduli satisfy the inequalities described in section* 4.1. *The set N in $\mathscr{T}(\Gamma)$ so defined is real analytically embedded in the space of quasifuchsian groups $T(\Gamma)$. It contains the Fuchsian locus and has real codimension $3g-3+n$.*

Proof. Given a particular subgroup decomposition $\{\varphi_i\}$, the conditions above determine a set N_0 in $\mathscr{T}(\Gamma)$. N is the union of the sets N_0 defined for all possible subgroup decompositions. By the constructions of section 3, to each point in N, we can assign a group in $T(\Gamma)$. This group has a fundamental polyhedron with no interior vertices and small number of faces (depending only on the type).

N clearly contains the Fuchsian locus. We may allow each coordinate to vary in an open neighborhood of this locus by choosing an appropriate subgroup decomposition (depending on the point). Each point of N is defined by $3g-3+n$ real analytic equalities, therefore N is a submanifold of $\mathscr{T}(\Gamma)$ of real codimension $3g-3+n$.

5. Concluding Remarks.

The groups in the domain N are quasifuchsian groups which have particularly simple geometric properties. We call them *nice groups*. We describe these properties as follows:

Each nice group Γ has a fundamental polyhedron which has no interior vertices. For each subgroup φ_i of Γ we defined points α_i, α'_i, β_i, β'_i, γ_i and γ'_i, which are the preimages of Weierstrass points. These points, which lie on $\boldsymbol{C}$, can be connected in pairs, α_i to α'_i, β_i to β'_i, γ_i to γ'_i by geodesics in $\boldsymbol{H}$. These geodesics lie on faces of the canonically defined fundamental polyhedron for φ_i. Since the factor space $(\boldsymbol{H}^3 \cup \Omega_{\varphi_i})/\varphi_i = S \times [0, 1]$, there is a natural identification of the surfaces $S \times \{0\}$ and $S \times \{1\}$ which is determined by the marking. With this identification we can think of the geodesics joining the Weierstrass points as forming a generalized braid. It is clear from the existence of the canonical fundamental polyhedron that for nice groups this braid is trivial. These braids exist not only for the elementary groups φ_i but for the full group Γ. In the latter situation they have many strands, in fact three for each basic subgroup. It follows from the constructions that if Γ is nice these many-stranded braids are trivial.

Jørgensen and Marden [11] have considered these braids in their work on Kleinian groups. They have shown that when the group is geometrically infinite these braids are infinite. Since the geometrically infinite groups are the most difficult to understand, one can use these braids as a measure of the complexity of these groups. The nice groups are the simplest in this scheme. It would be interesting to characterize further the relationship between the trace moduli and these braids.

Consider the gluing parameters as fixed points. Denote a generic one by ξ. We have seen that when ξ is real we obtain a Fuchsian subgroup in Γ bigger than any we began with. This real ξ can be throught of as follows. In the gluing procedure we amalgamate over a cyclic subgroup of Γ which we denote here by $\langle K \rangle$. $\langle K \rangle$ determines a free homotopy class on each of the boundary surfaces of the quotient. Let S be one of these surfaces and let γ be a geodesic in this free homotopy class. Let S be cut along γ, then twisted and glued back together. If ξ is real, it measures this twist. The imaginary part of ξ corresponds to a bending of the surface before it is reglued. Changing only the parameter ξ changes the conformal structure of both of the boundary surfaces. This follows easily from the formulas in [24] and [25].

Using the fundamental polyhedra we can obtain an explicit construction of some groups which are on the boundary of $T(\Gamma)$. Let Γ_0 be a fixed nice group. If the trace of the element $A_1 B_1$ of Γ_0 is z_0, set $z_t = (1-t) z_0 + 2t$, $t \in [0, 1)$. Varying the other moduli continuously subject to the conditions which make the group nice and letting the trace of $A_1 B_1$ take the values z_t we obtain a path in $T(\Gamma)$. At the endpoint of the path $(t=1)$ $z_1 = 2$ and the element $A_1 B_1$ is parabolic. Each point of the path corresponds to a nice group and the boundary point is a cusp (see [5, 22]). In the fundamental regions Δ_{Γ_t} the circles E and E' (of the construction of Δ_{T_1})

become tangent at the now single fixed point of B_1A_1. Similarly the circles D and D' become tangent at the single fixed point of A_1B_1. On one of the boundary surfaces of $(\boldsymbol{H} \cup \Omega_{\Gamma_t})/\Gamma_t \cong S_t \times [0, 1]$ a curve on a handle has become pinched. Bers describes this as "acquiring a node" (see [6]). We can perform an analogous deformation to a cusp on the boundary by starting with any element of Γ_0 which has a real trace. We must make sure that our normalization does not fix the fixed points of the element we are making parabolic. If we insist on keeping the path on the Fuchsian locus $\mathscr{F}\mathscr{L}$ we will "lose" part of the group — that is, the limit group will no longer be isomorphic to Γ_0. However we can vary through quasifuchsian nice groups to obtain a cusp group which is isomorphic to Γ_0.

These groups on the boundary are again nice groups since they have fundamental polyhedra with no interior vertices. Suppose now we fix such a group Γ_1 on the boundary — that is Γ_1 is an isomorphic image of Γ_0 and there is a parabolic element in Γ_1 whose preimage in Γ_0 is loxodromic. We can vary the parameters of Γ_1 so that we remain on the boundary of $T(\Gamma)$ and obtain a subspace of nice groups in this boundary. In the main body of this paper we assumed that all the group elements were loxodromic. If our original marked group had parabolic elements all the constructions would take place in this subspace. The basic canonical fundamental domains for the basic groups of the subgroup decomposition would be modified somewhat. The constructions for these groups are easily adapted from those for punctured tori and degenerate punctured tori described in [16] and those described as canonical polygons without accidental vertices in [12]. The latter can also be adapted if the original marked group contains elliptic elements.

The trace moduli we have defined are global in the following sense. The same set of traces can be used for any "nice" group. The number of traces we have had to use is greater than the dimension. Therefore our space is defined in terms of traces and relations. In [18], we indicated why, for Fuchsian groups, starting with a standard marking, this must be so. Recently, Wolpert [26] has shown that, for Fuchsian groups of type $(g, 0)$, no set of $6g-6$ trace can serve as global moduli. It follows then that the same is true for quasifuchsian groups.

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