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Holomorphic families of isomorphisms of Möbius groups*

To Yukio Kusunoki on his 60th birthday

By

Lipman BERS

This note contains a new proof and a slight extension of a result by Sullivan (see Proposition 1 below).

The group PSL (2, $\boldsymbol{C}$) may be identified with the group of all conformal self-maps of the Riemann sphere $\hat{\boldsymbol{C}}=\boldsymbol{C}\cup\{\infty\}$ (Möbius transformations). In what follows G denotes a subgroup of PSL (2, $\boldsymbol{C}$) and A a subset of $\hat{\boldsymbol{C}}$ invariant under G and containing at least 3 points. An isomorphism X: $G\rightarrow$PSL (2, $\boldsymbol{C}$) is said to be *induced* by an injection f: $A\rightarrow\hat{\boldsymbol{C}}$ if

$$f\circ g(a)=X(g)\circ f(a) \qquad \text{for} \quad g\in G,\ a\in A.$$

An isomorphism induced by a quasiconformal self-map of $\hat{\boldsymbol{C}}$ is called a *quasiconformal deformation.*

Let $D\subset\boldsymbol{C}$ be a domain. A *holomorphic family of injections* $\{f_\lambda\}$ of A, defined over D, is a rule assigning an injection f_λ: $A\rightarrow\boldsymbol{C}$ to every $\lambda\in D$ so that for every $a\in A$ the point $f_\lambda(a)$ depends holomorphically on λ. *A holomorphic family of isomorphisms* $\{X_\lambda\}$ of G, defined over D, is a rule assigning to each $\lambda\in D$ an isomorphisms X_λ: $G\rightarrow$PSL (2, $\boldsymbol{C}$) so that for every $a\in\hat{\boldsymbol{C}}$ and every $g\in G$ the point $X_\lambda(g)(a)$ depends holomorphically on λ. The family $\{X_\lambda\}$ is induced by the family $\{f_\lambda\}$ if f_λ induces X_λ for every $\lambda\in D$.

Proposition 1. *Let the holomorphic family $\{f_\lambda\}$ of injection of A induce the holomorphic family $\{X_\lambda\}$ of isomorphisms of G, both families being defined over a domain $D\subset\boldsymbol{C}$. If there is a $\lambda_0\in D$ such that X_{λ_0} is a quasiconformal deformation of G, so are all X_λ, $\lambda\in D$.*

For a finitely generated group G this assertion is proved (though not formulated as a special proposition) in §6 of Sullivan's paper [3]. Sullivan observes that the proof involves a "delicate point".

We establish fiirst a more precise result.

Proposition 2. *Under the hypotheses of Proposition* 1, *assume that D is the*

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unit disc $|\lambda|<1$, $\lambda_0=0$ *and* $f_0=id$ *(so that* $X_0=id$*). Then there is a holomorphic family* $\{F_\lambda\}$ *of quasiconformal self-maps of* $\hat{C}$ *defined over the disc* $|\lambda|<\frac{1}{3}$ *such that* $F_0=id$ *and* F_λ *induces* X_λ *(for* $|\lambda|<\frac{1}{3}$*).*

The proof is based on Theorem 3 in Bers-Royden [1]. According to this theorem there exists a *unique* holomorphic family $\{F_\lambda\}$ of quasiconformal self-maps of $\hat{C}$, defined over $|\lambda|<\frac{1}{3}$, such that (i) $F_0=id$, (ii) for $|\lambda|<\frac{1}{3}$ we have

$$F_\lambda \mid A=f_\lambda,$$

and (iii) the Beltrami coefficient of F_λ, i.e.,

$$\mu_\lambda=(\partial F_\lambda/\partial \bar{z})/(\partial F_\lambda/\partial z),$$

is harmonic in $\hat{C}\backslash\hat{A}$, where $\hat{A}$ denotes the closure of A in $\hat{C}$.

Condition (iii) means that in any component of $\hat{C}\backslash\hat{A}$.

$$\mu_\lambda(z)=\rho(z)^{-2}\overline{\psi_\lambda(z)}$$

where $\rho(z)|dz|$ is the Pioncaré metric in the component of $\hat{C}\backslash\hat{A}$ containing z and $\psi_\lambda(z)$ is holomorphic in that component. (The condition is vacuous if $\hat{A}=\hat{C}$).

Set, for some $g\in G$,

$$\tilde{F}_\lambda=X_\lambda(g)^{-1}\circ F_\lambda\circ g.$$

Then $\{\tilde{F}_\lambda\}$ is a holomorphic family of quasiconformal self-maps of $\hat{C}$, defined over $|\lambda|<\frac{1}{3}$, and $\tilde{F}_0=id$. Since

$$f_\lambda\circ g \mid A=X_\lambda(g)\circ f_\lambda$$

we have that

$$\begin{aligned}\tilde{F}_\lambda \mid A&=X_\lambda(g)^{-1}\circ F_\lambda\circ g \mid A\\&=X_\lambda(g)^{-1}\circ f_\lambda\circ g \mid A=f_\lambda.\end{aligned}$$

Finally, the Beltrami coefficient $\tilde{\mu}_\lambda$ of $\tilde{F}_\lambda$ (in $\hat{C}\backslash\hat{A}$) is easily computed to be

$$\begin{aligned}\tilde{\mu}_\lambda(z)&=\mu_\lambda(g(z))\overline{g'(z)}/g'(z)\\&=[\rho(g(z))|g'(z)|]^{-2}\overline{\psi_\lambda(g(z))g'(z)^2}\\&=\rho(z)^{-2}\overline{\varphi_\lambda(z)}\end{aligned}$$

where $\varphi_\lambda(z)$ is holomorphic in the component containing z. (Here we used the conformal invariance of the Poincaré metric).

The uniqueness statement of Theorem 3 in [1] implies that $\tilde{F}_\lambda=F_\lambda$ for every g. Thus

$$X_\lambda(g)=F_\lambda\circ g\circ F_\lambda^{-1} \qquad \text{for} \quad |\lambda|<\frac{1}{3}.$$

Corollary 1. *Under the hypotheses of Proposition* 1, *assume that* D *is the unit disc and* $\lambda_0=0$. *For* $|\lambda|<\frac{1}{3}$ *the isomorphism* X_λ *is a quasiconformal deformation.*

Proof. Set $\tilde{A}=f_0(A)$, $\tilde{G}=X_0(G)$, $\tilde{f}_\lambda=f_\lambda \circ f_0^{-1}$, and $\tilde{X}_\lambda=X_\lambda \circ X_0^{-1}$. Now apply Proposition 2 to the holomorphic families $\{\tilde{f}_\lambda\}$ and $\{\tilde{X}_\lambda\}$ of injections of $\tilde{A}$ and of isomorphisms of $\tilde{G}$, respectively.

Corollary 2. *Under the hypotheses of Proposition* 1, *assume that* D *is simply connected and* $D\neq \boldsymbol{C}$. *For every* λ *inside the Poincaré disc* (*in* D) *with center* λ_0 *and radius* $\log 2$ *the isomorphism* X_λ *is a quasiconformal deformation.*

Proof. Map D conformally onto the unit disc, taking λ_0 into the origin. Then apply Corollary 1 and note that the Poincaré distance (in the unit disc) from 0 to a point ξ with $|\xi|=\frac{1}{3}$ is $\log 2$.

Proof of Proposition 1. It suffices to prove the proposition for the case when D is simply connected and $D\neq \boldsymbol{C}$ since given any λ_1 in D there is a bounded simply connected domain $D_0\subset D$ containing λ_0 and λ_1.

If $D\neq \boldsymbol{C}$ and D is simply connected, let Θ denote the set of all λ in D for which X_λ is a quasiconformal deformation. Then $\lambda_0\in\Theta$, by hypothesis, and Θ is open and closed, by Corollary 2. Hence $\Theta=D$, q. e. d.

Here is an application of Proposition 1.

Theorem. *Let* G *contain two loxodromic* (*including hyperbolic*) *elements with* 4 *distinct fixed points.*

Let $\{X_\lambda\}$ *be a holomorphic family of isomorphisms of* G *defined over* D, *with* X_{λ_0} *a quasiconformal deformation for some* $\lambda_0\in D$. *Assume that, for all* $\lambda\in D$, (i) $X_\lambda(G)$ *is discrete and* (ii) $X_\lambda(g)$ *is parabolic if and only if* $g\in G$ *is. Then each* X_λ *is a quasiconformal deformation.*

Proof. By (i) and (ii), $X_\lambda(g)$ is elliptic if and only if $g\in G$ is. Let A be the set of fixed points of loxodromic elements of G. Recalling that two loxodromic Möbius transformations with precisely 3 distinct fixed points generate a non-discrete group, we conclude from (i) and (ii) that the map f_λ which takes an attracting fixed point of a loxodromic element $g\in G$ into the attracting fixed point of $X_\lambda(g)$ is a well-defined injections of A.

Next we verify that f_λ induces X_λ. Let us denote the attracting fixed point of a loxodromic $h\in G$ by $\alpha[h]$. If g is any element of G, then

$$\begin{aligned} f_\lambda\circ g(\alpha[h]) &= f_\lambda(\alpha[g\circ h\circ g^{-1}]) \\ &= \alpha[X_\lambda(g\circ h\circ g^{-1})]=\alpha[X_\lambda(g)\circ X_\lambda(h)\circ X_\lambda(g)^{-1}] \\ &= X_\lambda(g)(\alpha[X_\lambda(h)])=X_\lambda(g)\circ f_\lambda(\alpha[h]) \end{aligned}$$

as required.

Clearly, $f_{\lambda_0}=id.$ and $f_\lambda(\alpha[h])=\alpha[X_\lambda(h)]$ depends holomorphically on λ, for a fixed loxodromic h. Now apply Proposition 1.

(Stronger results hold for finitely generated G, see [2]).

DEPARTMENT OF MATHEMATICS,
COLUMBIA UNIVERSITY

References

[1] L. Bers and H. S. Royden, Holomorphic families of injections, to appear.

[2] D. Sullivan, Quasiconformal homeomorphisms and dynamics II. Structural stability implies hyperbolicity for Kleinian groups, Acta Math. 155 (1985), 243–260.

Added in proof. Hypothesis (ii) in the theorem follows easily from the others (as observed by C. McMullen).