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Some remarks on meromorphic functions on open Riemann surfaces

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

By

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Introduction. The generalization of Abel's theorem to open Riemann surfaces in the form analogous to the classical one has been studied by many authors (for recent references, Maitani [4], Sainouchi [6], [7], Watanabe [8]). In this paper, we define a class $\mathscr{M}$ ($\mathscr{M}_0$) of meromorphic functions on an arbitrary open Riemann surface (for the definition of $\mathscr{M}$ ($\mathscr{M}_0$), see section **2**) and give a necessary and sufficient condition for the existence of a meromorphic function of $\mathscr{M}$ ($\mathscr{M}_0$) which has the given divisor. It has a similar formulation to classical Abel's theorem, but we do not assume the finiteness of divisor. In the last section, for certain class of Riemann surfaces with a metric condition, we give some sufficient conditions in order that a meromorphic function should belong to the prescribed class.

1. We shall consider an arbitrary open Riemann surface R and denote its genus by g $(0<g\leqq+\infty)$. Let $\{\Omega_n\}$ $(n=1, 2,\dots)$ be a canonical exhaution of R, then there exists a canonical homology basis $\{A_i, B_i\}$ $(i=1, 2,\dots, k(n),\dots)$ with respect to $\{\Omega_n\}$ such that $\{A_i, B_i\}$ $(i=1, 2,\dots, k(n))$ for a canonical homology basis of Ω_n (mod $\partial\Omega_n$). Let $\mathscr{D}$ be a class of an enumerable number of semiexact holomorphic differentials dw_i $(i=1, 2,\dots)$ on R such that $\int_{A_j} dw_i=\delta_{ij}$ (Kronecker's δ) and set $\int_{B_j} dw_i=B_{ij}=\xi_{ij}+i\tau_{ij}$ (ξ_{ij}, τ_{ij}; real). Also a class of the square integrable semiexact holomorphic differentials having the same property as above is denoted by $\mathscr{D}_0$. The class $\mathscr{D}$ ($\mathscr{D}_0$) always exists on an arbitrary open Riemann surface, but does not be determined uniquely for given canonical homology basis.

2. Let $\mathscr{M}$ be a class of meromorphic functions on R such that each function f belonging to $\mathscr{M}$ has the following two properties: (1) there exists an integer n_0 such that for all n $(\geqq n_0)$

$$\int_{\gamma_n^{(i)}} d\log f=0 \qquad (i=1, 2,\dots, m_n),$$

where $\gamma_n^{(i)}$ are the components of $\partial\Omega_n$ and do not contain the zeros and poles of f.

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$$\lim_{n\to\infty}\int_{\partial\Omega_n} w_i d\log f=0, \tag{2}$$

where $dw_i \in \mathscr{D}$ and $w_i=\int_{p_0}^{p} dw_i=w_i(p)$ with a fixed point $p_0 \in R$.

If $\mathscr{D}$ is replaced by $\mathscr{D}_0$ in (2), we use $\mathscr{M}_0$ in place of $\mathscr{M}$. We note that $\int_{\partial\Omega_n} w_i d\log f\ (n \geqq n_0)$ do not depend on the choice of branches of w_i and the number of zeros of f in Ω_n is equal to that of poles of f.

Now let $\delta=\prod a_i/\prod b_i$ be a divisor on R and $\delta_n=a_1\cdots a_{l(n)}/b_1\cdots b_{l(n)}$ its restriction to Ω_n, where we assume $\partial\Omega_n$ does not contain a_i and b_i $(i=1, 2,\dots, l(n))$. Also let us denote by γ_i a singular 1-chain in Ω_n such that $\partial\gamma_i=b_i-a_i$ and set $c(n)=\sum_{i=1}^{l(n)}\gamma_i$.

Proposition 1. *The necessary and sufficient condition for the existence of single valued meromorphic function f of $\mathscr{M}$ ($\mathscr{M}_0$) whose divisor is exactly δ is that the conditions*

$$\lim_{n\to\infty}\left\{\int_{c(n)} dw_i-(m_i-\sum_{j=1}^{k(n)} n_j B_{ij})\right\}=0 \tag{3}$$

hold for all differentials $dw_i \in \mathscr{D}$ ($\mathscr{D}_0$), where m_i and n_j are integers.

Proof. Take a simply connected region U_i containing a_i and b_i and set $U=\bigcup_{i=1}^{l(n)} U_i$. Since the square integrable analytic and anti-analytic differentials are mutually orthogonal, we have

$$(dw_i, {}^*\overline{d\log f})_{\Omega_n-U}=0.$$

On the other hand, by the Green's formula we have

$$(dw_i, {}^*\overline{d\log f})_{\Omega_n-U}=\sum_{j=1}^{k(n)}\left(\int_{A_j} dw_i\int_{B_j} d\log f-\int_{A_j} d\log f\int_{B_j} dw_i\right)-\int_{\partial(\Omega_n-U)} w_i d\log f. \tag{4}$$

Set $\int_{A_j} d\log f=2\pi i n_j$ and $\int_{B_j} d\log f=2\pi i m_j$ (n_j, m_j; integers), then the right hand side of (4) becomes to

$$\begin{aligned}&2\pi i\sum_{j=1}^{k(n)}(\delta_{ij}m_j-n_jB_{ij})-\int_{\partial\Omega_n} w_i d\log f+\int_{\partial U} w_i d\log f\\ &=2\pi i(m_i-\sum_{j=1}^{k(n)} n_jB_{ij})-\int_{\partial\Omega_n} w_i d\log f+\int_{\partial U} w_i d\log f.\end{aligned}$$

By the calculation of residue we have

$$\int_{\partial U} w_i d\log f=2\pi i\sum_{j=1}^{l(n)}(w_i(a_j)-w_i(b_j))=-2\pi i\int_{c(n)} dw_i.$$

Hence we obtain

$$\int_{c(n)} dw_i - (m_i - \sum_{j=1}^{k(n)} n_j B_{ij}) = -\frac{1}{2\pi i}\int_{\partial\Omega_n} w_i d\log f. \tag{5}$$

Therefore by (2) we have the desired result (3).

Conversely, let $d\varphi$ be a meromorphic differential which has a simple pole of residue 1 (-1) at a_i (b_i) $(i=1, 2, \ldots)$, respectively and add to $d\varphi$ an appropriate holomorphic differential, then we can get the meromorphic differential $d\psi$ such that

$$\int_{A_j} \psi = 2\pi i n_j, \quad \int_{B_j} d\psi = 2\pi i m_j \quad \text{and} \quad \int_{\gamma_n^{(i)}} d\psi = 0.$$

Set $\psi = \int d\psi$ and $f = \exp\psi$, then f is a single valued meromorphic function with given divisor and $\int_{\gamma_n^{(i)}} d\log f = 0$. Hence if (3) is satisfied, it follows from (5) that $\lim_{n\to\infty}\int_{\partial\Omega_n} w_i d\log f = 0$. Thus f belongs to $\mathscr{M}$ and the proof of proposition is completed.

3. When dlog f is the distinguished harmonic differential, a single valued meromorphic function f is defined to be quasi rational (Ahlfors-Sario [1]).

Proposition 2. *The quasi rational function belongs to class* $\mathscr{M}_0$.

Proof. Since a distinguished harmonic differential is semiexact outside a sufficiently large regular region, the property (1) is clear. Also we can put

$$d\log f = \omega_{hm} + \omega_{eo} + \tau,$$

where $\omega_{hm} \in \Gamma_{hm}$, $\omega_{eo} \in \Gamma_{eo} \cap \Gamma^1$ and τ is zero in each component of $R - \Omega_n$ for a large n. Since $\Gamma_{hm} \perp {}^*\Gamma_{hse}$ and $\Gamma_{eo} \perp \Gamma_h$, we have

$$(d\log f - \tau, {}^*\overline{dw_i}) = (\omega_{hm} + \omega_{eo}, {}^*\overline{dw_i}) = 0.$$

While $d\log f - \tau$ is exact, so if we set $dh = d\log f - \tau$, then

$$(d\log f - \tau, {}^*\overline{dw_i}) = \lim_{n\to\infty}(dh, {}^*\overline{dw_i})_{\Omega_n}$$

$$= -\lim_{n\to\infty}\int_{\partial\Omega_n} h dw_i = \lim_{n\to\infty}\int_{\partial\Omega_n} w_i dh = \lim_{n\to\infty}\int_{\partial\Omega_n} w_i d\log f.$$

Thus the quasi rational function has the property (2).

A quasi rational function has the same finite number of zeros as poles and dlog f is exact in $R - \Omega_n$ for a sufficiently large n, and so there exists an integer k such that

$$n_j = \frac{1}{2\pi i}\int_{A_j} d\log f = 0 \quad (j > k).$$

Corollary. *Let* $\delta = a_1 \cdots a_l / b_1 \cdots b_l$ *be the divisor of a quasi rational function, then*

$$\int_c dw_i = m_i - \sum_{j=1}^{k} n_j B_{ij} \qquad \textit{for all} \quad dw_i \in \mathscr{D}_0,$$

where c is a finite cahin $\sum_{j=1}^{l} \gamma_j\ (\partial\gamma_j = b_j - a_j)$.

Let $\mathscr{D}_\chi$ be the set of square integrable analytic differentials φ such that the real part of φ has the Γ_χ-behavior, where $\Gamma_{hm} \subset \Gamma_\chi \subset \Gamma_{he}$ (Kusunoki [3], Yoshida [9]). Now we denote by $\mathscr{M}_\chi$ the class of meromorphic function f such that f satisfies (1) in section **2** and

$$\lim_{n\to\infty} \mathrm{Im} \int_{\partial\Omega_n} d \log f = 0,$$

where $\Phi = \Phi(p) = \int_{p_0}^{p} \varphi\ (\varphi \in \mathscr{D}_\chi)$.

Proposition 3. *The necessary and sufficient condition for the existence of a mpromorphic function f such that f belongs to* $\mathscr{M}_\chi$ *and its divisor is exactly* δ *is that the conditions*

$$\lim_{n\to\infty} \mathrm{Re} \int_{c(n)} \varphi_{A_i} = integer$$

$$\lim_{n\to\infty} \mathrm{Re} \int_{c(n)} \varphi_{B_i} = integer$$

hold for the differentials φ_{A_i}, φ_{B_i} $(\in \mathscr{D}_\chi)$, *where* φ_{A_i} *and* φ_{B_i} *are the fundamental differentials associated to the homology basis* A_i *and* B_i *respectively.*

The proof is obtained easily by the same way as in section **2** and so we shall omit it. Also let $\mathscr{E}_\chi$ be the class of meromorphic function f such that Re log f has Γ_χ-behavior, then $\mathscr{E}_\chi \subset \mathscr{M}_\chi$.

4. We take mutually disjoint annuli $D_n^{(i)}$ $(i=1,\dots, m_n)$ each of which includes exactly one contour of $\partial\Omega_n$. Let $D_n = \bigcup_{i=1}^{m_n} D_n^{(i)}$ and assume that D_n $(n=1,\dots)$ are disjoint each other. If, in the definition of $\mathscr{M}$ $(\mathscr{M}_0)$, (2) is replaced by the following (2′), then we shall use $\mathscr{M}'$ $(\mathscr{M})$ in place of $\mathscr{M}$ $(\mathscr{M}_0)$: (2′) there exists a canonical exhaustion $\{\Omega_{n'}\}$ such that

$$\partial\Omega_{n'} \subset D_{n'} \text{ and } \lim_{n'\to\infty} \int_{\partial\Omega_{n'}} w_i d \log f = 0,$$

where the sequence $\{\Omega_{n'}\}$ is the subsequence of $\{\Omega_n\}$ and depends of f and $dw_i \in \mathscr{D}$ $(\mathscr{D}_0)$.

By the same way as in the proof of proposition **1**, we have

Proposition 4. *The necessary and sufficient condition for the existence of a single valued meromorphic function f such that f belongs to* $\mathscr{M}'$ $(\mathscr{M}'_0)$ *and its divisor is exactly* δ *is that the condition*

$$\text{(3}'\text{)} \qquad \lim_{n'\to\infty} \left\{ \int_{c(n')} dw_i - (m_i - \sum_{j=1}^{k(n')} n_j B_{ij}) \right\} = 0$$

holds for each differential $dw_i \in \mathscr{D}$ $(\mathscr{D}_0)$, *where the support of* δ *is contained in* $R - \cup D_n$.

Now let $v_n^{(i)}$ be the harmonic modulus of $D_n^{(i)}$. If

$$\sum_n \min_i v_n^{(i)} = \infty, \tag{6}$$

then the Riemann's period relation holds for any two differentials $(\in \Gamma_{hse})$ and so the class $\mathscr{D}_0$ is determined uniquely by the canonical homology basis and $B_{ij} = \int_{B_j} dw_i = \int_{B_i} dw_j = B_{ji}$ $(dw_i, dw_j \in \mathscr{D}_0)$, and for each n the matrix $(\operatorname{Im} B_{ij})_{i,j=1,2,\dots,n}$ is positive definite (Sainouchi [5], Kobori-Sainouchi [2]).

Proposition 5. *Let* $\sum_n \min_i v_n^{(i)}$ *be divergent and* f *be a meromorphic function on* R *such that* f *satisfies* (1) *and*

$$\|\dot{d} \log f\|_{\cup D_n} < +\infty, \tag{7}$$

then f *belongs to* $\mathscr{M}'_0$ *and so*

$$\lim_{n' \to \infty} \left\{ \int_{c(n')} dw_i - (m_i - \sum_{j=1}^{k(n')} n_j B_{ij}) \right\} = 0$$

for each differential $dw_i \in \mathscr{D}_0$.

Proof. If $\sum \min_i v_n^{(i)} = \infty$, then there exists a canonical exhaustion $\{\Omega_{n'}\}$ such that (2′) holds for $dw_i (\in \mathscr{D}_0)$ and f satisfying (7), and so we obtain the desired result by use of proposition **4.**

5. Next we shall consider the converse of above proposition. Here the finiteness of divisor is assumed.

Proposition 6. *If* $\sum_n \min_i v_n^{(i)} = \infty$ *and* $\lim_{p \to \infty} \sum_{i,j=1}^{p} n_j n_j \tau_{ij}$ *converges and moreover*

$$\sum_{j=1}^{l} \int_{a_j}^{b_j} dw_i = m_i - \lim_{n \to \infty} \sum_{j=1}^{k(n)} n_j B_{ij} \tag{8}$$

for each $dw_i (\in \mathscr{D}_0)$, *then there exists a single valued meromorphic function* f *such that* (i)*the divisor of* f *is exactiy* $\delta = a_1 \cdots a_l / b_1 \cdots b_l$ (ii) $d \log f$ *is semiexact and its norm is finite outside of a compact subset of* R, *where* m_i *and* n_j *are integers. The function* f *is determined uniquely up to the multiplicative constant.*

Proof. We use the Abelian differential $dw(a_j, b_j)$ which has the following properties (i) $dw(a_j, b_j)$ has two simple poles a_j and b_j with residue 1 and -1, respectively (ii) the norm of $dw(a_j, b_j)$ is finite outside of an arbitrary region containing a_j and b_j (iii) it is semixact outside of a suitable curve joinning a_j and b_j (iv) all A-periods of $dw(a_j, b_j)$ vanish. The existence of $dw(a_j, b_j)$ has been proved in [5] and in the present case it is determined uniquely.

B_k-period of $dw(a_j, b_j)$ is obtained by the application of period relation and we have

$$\int_{B_k} dw(a_j, b_j) = 2\pi i(w_k(b_j) - w_k(a_j)). \tag{9}$$

Put $dw' = \sum_{j=1}^{l} dw(a_j, b_j)$ and $dw''_p = \sum_{j=1}^{p} n_j dw_j$, where n_j are integers in the assumption. Then

$$\begin{aligned}
\|dw''_p\|^2 &= (dw''_p, dw''_p) = \sum_{i,j=1}^{p} n_i n_j (dw_i, dw_j) \\
&= -i \sum_{i,j=1}^{p} n_i n_j \left\{ \lim_{n''+\infty} \sum_{A_k, B_k \subset \Omega_{n''}} \left(\int_{A_k} dw_i \int_{B_k} \overline{dw_j} - \int_{B_k} dw_i \int_{A_k} \overline{dw_j} \right) \right. \\
&= -i \sum_{i,j=1}^{p} n_i n_j (\bar{B}_{ji} - B_{ij}) = -2 \sum_{i,j=1}^{p} n_i n_j \tau_{ij},
\end{aligned}$$

therefore, if $\lim_{p\to\infty} \sum_{i,j=1}^{p} n_i n_j \tau_{ij}$ is convergent, dw''_p converges to $dw''(\in \Gamma_{hse})$ in norm and we have

$$\int_{B_j} dw'' = \lim_{p\to\infty} \sum_{i=1}^{p} n_i B_{ij} \quad \text{and} \quad \int_{A_j} dw'' = n_j. \tag{10}$$

Set $dw = dw' + 2\pi i dw''$, then by (8), (9) and (10) we get

$$\int_{A_k} dw = 2\pi i n_k$$

and

$$\begin{aligned}
\int_{B_k} dw &= 2\pi i \sum_{j=1}^{l} (w_k(b_j) - w_k(a_j)) + 2\pi i \lim_{p\to\infty} \sum_{i=1}^{p} n_i B_{ik} \\
&= 2\pi i m_k.
\end{aligned}$$

Now set $f = \exp \int dw$, then f is a single valued meromorphic function on R and its divisor is exactly δ and $d \log f = dw$ is semiexact outside of a compact subset of R and has a finite norm. Let g be an arbitrary meromorphic function with the same properties as f, then by the application of period relation to the square integrable harmonic differential $d \log |f/g|$ $(\in \Gamma_{he} \cap {}^*\Gamma_{hse})$ we have $|f/g| = \text{constant}$ and so f is determined uniquely up to the multiplicative constant.

6. Finally, we shall give a sufficient condition in order that a meromorphic function should belong to the class $\mathscr{M}'$. Let ω be a fixed square integrable holomorphic differential and set $dw_j = g_j \omega$ $(dw_j \in \mathscr{D})$. Also we denote by $u + iv$ (v; conjugate of u) the function mapping $\cup D_n$ onto a strip domain; $0 < u < R' = \sum_{n=1}^{\infty} \nu_n$, $0 < v < 2\pi$, where ν_n is the harmonic modulus of D_n.

Proposition 7. *Let f be a meromorphic function such that $\|df\|_{\cup D_n} < +\infty$ and f satisfies* (1), *and set*

$$M_n^{(i)}(g_j)=\max_{p\in D_n^{(i)}}|g_j(p)|,\quad M_n^{(i)}(1/f)=\max_{p\in D_n^{(i)}}|1/f(p)|.$$

If $\sum_{n=1}^{\infty}\min\frac{\nu_n^{(i)}}{M_n^{(i)}(g_j)M_n^{(i)}(1/f)}=\infty$ *for all* j, *then* f *belongs to* $\mathscr{M}'$.

Proof. let $I_n=[\sum_{i=1}^{n-1}\nu_i, \sum_{i=1}^{n}\nu_i]$ and for $r\in I_n$ we denote by $\gamma_r^{(i)}$ the level curve $\{p\in R;\ u(p)=r\}$ contained in $D_n^{(i)}$. In $\bigcup_n D_n$ we can put $\omega=adu+bdv$ and $df=f_u du+f_v dv$, hence if we set $L(r)=\sum_{i=1}^{m_n}\int_{\gamma_r^{(i)}}|dw_j|\int_{\gamma_r^{(i)}}|d\log f|$ and $L_n=\min_{r\in I_n}L(r)$, then

$$\begin{aligned}L_n&\leqq\sum_{i=1}^{m_n}\left(\int_{\gamma_r^{(i)}}|g_j|^2dv\int_{\gamma_r^{(i)}}|1/f|^2dv\right)^{1/2}\left(\int_{\gamma_r^{(i)}}|b|^2dv\int_{\gamma_r^{(i)}}|f_v|^2dv\right)^{1/2}\\&\leqq\sum_{i=1}^{m_n}\left(\int_{\gamma_r^{(i)}}|g_j|^2dv\int_{\gamma_r^{(i)}}|1/f|^2dv\right)^{1/2}\sum_{i=1}^{m_n}\left(\int_{\gamma_r^{(i)}}|b|^2dv\int_{\gamma_r^{(i)}}|f_v|^2dv\right)^{1/2}\\&\leqq 2\pi\nu_n\max_i\frac{M_n^{(i)}(g_j)M_n^{(i)}(1/f)}{\nu_n^{(i)}}\left(\int_0^{2\pi}|b|^2dv\int_0^{2\pi}|f_v|^2dv\right)^{1/2}.\end{aligned}$$

By integration with respect to $r\in I_n$ we have

$$L_n\left[2\pi\max\frac{M_n^{(i)}(g_j)M_n^{(i)}(1/f)}{\nu_n^{(i)}}\right]^{-1}\leqq\|\omega\|_{D_n}\|df\|_{D_n},$$

hence it follows from the assumption that $\varliminf_{n\to\infty}L_n=0$. Consequently there exists a subsequence $\{n'\}$ such that

$$\lim_{n'\to\infty}\left|\int_{\partial\Omega_{n'}}w_j d\log f\right|\leq\lim_{n'\to\infty}L_{n'}=0,\qquad \text{q.e.d.}$$

With a slight modification we can prove

Proposition 8. *Let* f *be a meromorphic function such that* $\|d\log f\|_{\cup D_n}<+\infty$ *and* f *satisfies* (1). *If* $\sum_{n=1}^{\infty}\min_i\frac{\nu_n^{(i)}}{M_n^{(i)}(g_j)}=\infty$ *for all* j, *then* f *belongs to* $\mathscr{M}'$.

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