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On some extension property for BMO functions on Riemann surfaces

By

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Introduction.

In previous papers [6] and [7] we investigated two BMO spaces $BMO(R, m)$ and $BMO(R, \lambda)$ on Riemann surface R with universal covering $D=\{|z|<1\}$, with respect to Lebesgue measure dm=dxdy on the unit disk D and the hyperbolic measure $d\lambda=dxdy/(1-|z|^2)^2$ on D. These spaces are defined by using the universal covering map. On the other hand, in case Ω is a plane domain, we can consider another BMO space $\widetilde{BMO}(\Omega, m)$ with respect to Lebesgue measure dm on Ω, which seems to be more natural than $BMO(\Omega, m)$. Reimann [11] and Jones [8] proved the quasi-conformal invariance of the space $\widetilde{BMO}(\Omega, m)$, which shows that this space depends only on the conformal structure of Ω. From such an observation we shall define in this paper a new space $\widehat{BMO}(R, m)$ on an arbitrary Riemann surface R and investigate its fundamental property.

In §1 we shall study about the relation on the spaces $BMO(\Omega, m)$ and $\widetilde{BMO}(\Omega, m)$ for a plane domain Ω and especially we show some necessary and sufficient conditions for which these two spaces coincide each other. In §2 we define newly a space $\widehat{BMO}(R, m)$ on an arbitrary Riemann surface R and show that many results obtained for $\widetilde{BMO}(\Omega, m)$ are valid also for $\widehat{BMO}(R, m)$. The next §3 is concerned with some extension property for the functions of $BMO(R, m)$ and $\widehat{BMO}(R, m)$. It is well known that BMO functions on quasi-disks have extension property. Here we show that the similar result is also valid for compact bordered Riemann surfaces.

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§1 BMO spaces on plane domains.

Let Ω be a plane domain and dm the 2-dimensional Lebesgue measure. We can define the following BMO space on Ω naturally.

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Definition 1. $\widetilde{BMO}(\Omega, m)=\left\{f\in L^1_{loc}(\Omega) : \|f\|_{\widetilde{BMO}(\Omega,m)}=\sup m(B)^{-1}\int_B|f-f(B, m)|dm<+\infty\right\}$, where the supremum is taken for every disk B in Ω and $f(B, m)=m(B)^{-1}\int_B f\, dm$. $\widetilde{BMO}H(\Omega, m)=\widetilde{BMO}(\Omega, m)\cap H(\Omega)$, $\widetilde{BMO}A(\Omega, m)=\widetilde{BMO}(\Omega, m)\cap A(\Omega)$.

The following two propositions play the fundamental roles throughtout this paper.

Proposition 1. ([8], [12]) *Let $\alpha>0$ and f be a function of $L^1_{loc}(\Omega)$ such that $\sup m(B)^{-1}\int_B|f-f(B, m)|dm$ $(\equiv M)<+\infty$, where the supremum is taken for every disk B in Ω with radius $r(B)<\alpha d(B, \partial\Omega)$ and $d(B, \partial\Omega)$ being the distance between B and the boundary $\partial\Omega$ of Ω. Then f belongs to $\widetilde{BMO}(\Omega, m)$ and $\|f\|_{\widetilde{BMO}(\Omega,m)}\leqslant C_1(\alpha)M$, where $C_1(\alpha)\geqslant 1$ is a constant depending only on the constant α.*

Proposition 2. ([8], [11]) *Let Ω, Ω' be plane domains, f a quasi-conformal map from Ω onto Ω' having the maximal delatation K and g a function of $\widetilde{BMO}(\Omega', m)$, then $g\circ f$ also belongs to $\widetilde{BMO}(\Omega, m)$ and $C_2(K)^{-1}\|g\|_{\widetilde{BMO}(\Omega',m)}\leqslant\|g\circ f\|_{\widetilde{BMO}(\Omega,m)}\leqslant C_2(K)\|g\|_{\widetilde{BMO}(\Omega',m)}$, where $C_2(K)\geqslant 1$ is a constant depending only on K.*

It is surprising that the constant $C_2(K)$ above is independent of the choice of Ω, Ω' and f. Next we define the following BMO space on Riemann surfaces. Let R be a Riemann surface having the universal covering $D=\{|z|<1\}$ and $\pi : D \longrightarrow R$ its universal covering map.

Definition 2. ([9]) $BMO(R, m)=\{f\in L^1_{loc}(R) : f\circ\pi\in BMO(D, m)\}$, $BMOH(R, m)=BMO(R, m)\cap H(R)$, $BMOA(R, m)=BMO(R, m)\cap A(R)$.

The space $BMO(R, m)$ is determined independently of the choice of universal covering map, since Proposition 2 implies that $C_2(1)^{-1}\|f\circ\pi\|_{\widetilde{BMO}(D,m)}\leqslant\|f\circ\pi'\|_{\widetilde{BMO}(D,m)}\leqslant C_2(1)\|f\circ\pi\|_{\widetilde{BMO}(D\, m)}$ for a function f on R and another universal covering map π'. Hence we can define the norm of $f\in BMO(R, m)$ by $\|f\|_{BMO(B,m)}=\sup\|f\circ\pi\|_{\widetilde{BMO}(D,m)}$, where the supremum is taken for every universal covering map π. Note that the norm $\|\cdot\|_{BMO(R,m)}$ is conformally invariant.

For harmonic functions on plane domains the following characterizations are known.

Proposition 3. ([3]) (1) *A harmonic function h on a plane domain Ω belongs to $\widetilde{BMO}H(\Omega, m)$ if and only if $\sup_{z\in\Omega} d(z, \partial\Omega)|\nabla h(z)|<+\infty$, and there exists universal constants A, $A'>0$ such that $A\sup_{z\in\Omega} d(z, \partial\Omega)|\nabla h(z)|\leqslant\|h\|_{\widetilde{BMO}H(\Omega,m)}\leqslant A'\sup_{z\in\Omega} d(z, \partial\Omega)|\nabla h(z)|$.*

(2) *A harmonic function h on a plane domain Ω having universal covering D belongs to $BMOH(\Omega, m)$ if and only if $\sup_{z\in\Omega}\rho_\Omega(z)^{-1}|\nabla h(z)|<+\infty$, where $\rho_\Omega(z)|dz|$ denotes the hyperbolic metric on Ω, and there exists universal constants A, $A'>0$ such that $A\sup_{z\in\Omega}$*

$\rho_\Omega(z)^{-1}|\nabla h(z)|\leqslant\|h\|_{BMOH(\Omega,m)}\leqslant A'\sup_{z\in\Omega}\rho_\Omega(z)^{-1}|\nabla h(z)|$.

Remark 1. Above proposition shows that in case of the unit disk D, the space $\widetilde{BMO}A(D, m)=BMOA(D, m)$ coincides with Bloch space $\mathscr{B}(D)=\{f\in A(D):\|f\|_{\mathscr{B}}=\sup_{z\in D}(1-|z|^2)|f'(z)|<+\infty\}$, (cf.[2]).

Note that (2) in above proposition is a direct consequence of (1) since in case of the unit disk D we have $2^{-2}d(z, \partial D)^{-1}\leqslant\rho_D(z)\leqslant d(z, \partial D)^{-1}$. Since $\rho_\Omega(z)\leqslant d(z, \partial\Omega)^{-1}$ for every plane domain Ω having the universal covering D, we obtain $BMOH(\Omega, m)\subset\widetilde{BMO}H(\Omega, m)$. Further every Dirichlet function on Ω belongs to $\widetilde{BMO}(\Omega, m)$ (see [7]), especially it holds that $HD(\Omega)\subset\widetilde{BMO}H(\Omega, m)$. As for $BMO(\Omega, m)$, Metzger's result (see [10]) implies $AD(\Omega)\subset BMOA(\Omega, m)$, nevertheless $HD(\Omega)$ is not contained in $BMOH(\Omega, m)$ in general (see[6]).

Further we need the following result.

Proposition 4. ([7]) (1) *Let Ω be a hyperbolic plane domain and $g_\Omega(z, \zeta)$ its Green function with pole $\zeta\in\Omega$, then $A\leqslant\|g_\Omega(\cdot, \zeta)\|_{\widetilde{BMO}(\Omega,m)}\leqslant A'$, where A, $A'>0$ are universal constants.*

(2) *Let μ be a positive measure on D such that its Green potential $f=\int_D g(\cdot, \zeta)d\mu(\zeta)$ belongs to $\widetilde{BMO}(D, m)$, then $\mu(B)\leqslant A\|f\|_{\widetilde{BMO}(D,m)}$ for every disk B in D whose hyperbolic radius is equal to* 1, *where $A>0$ is a universal constant.*

In above proposition, the constant 1 has no special meaning. Now we can prove the following.

Theorem 1. *Let Ω be a plane domain with universal covering map $\pi: D\longrightarrow\Omega$, then $BMO(\Omega, m)\subset\widetilde{BMO}(\Omega, m)$. Further the following conditions are equivalent*;

(1) $BMO(\Omega, m)=\widetilde{BMO}(\Omega, m)$,

(2) $BMOH(\Omega, m)=\widetilde{BMO}H(\Omega, m)$,

(3) *There exists a constant $L>0$ such that $d(z, \partial\Omega)^{-1}\leqslant L\rho_\Omega(z)$, $z\in\Omega$,*

(4) *There exists a constant $M>0$ such that for every $\zeta\in\Omega$, the domain $\{z\in\Omega:\rho_\Omega(z, \zeta)<M\}$ is simply connected, where $\rho_\Omega(z, \zeta)$ is the hyperbolic distance between z and ζ,*

(5) $\log\pi'(z)\in\widetilde{BMO}A(D, m)$ $(=\mathscr{B}(D))$,

(6) $\log\rho_\Omega(z)\in BMO(\Omega, m)$,

further if Ω is hyperbolic, the next condition is also equivalent to above conditions,

(7) $\sup\|g_\Omega(\cdot, \zeta)\|_{BMO(\Omega,m)}<+\infty$.

Proof. Let f be a function of $BMO(\Omega, m)$ and B a disk in Ω and B' one of the connected components of $\pi^{-1}(B)$, then π is conformal on B' and so $\|f\circ\pi\|_{\widetilde{BMO}(B',m)}\leqslant\|f\|_{BMO(\Omega,m)}$ by definition. Further we have $\|f\|_{\widetilde{BMO}(B,m)}\leqslant C_2(1)\|f\circ\pi\|_{\widetilde{BMO}(B',m)}$ by Proposition 2, it follows that $\|f\|_{\widetilde{BMO}(\Omega,m)}\leqslant C_2(1)\|f\|_{BMO(\Omega,m)}$, which implies $BMO(\Omega, m)\subset\widetilde{BMO}(\Omega, m)$. "(1)$\rightarrow$(2)" is trivial. "(4)$\rightarrow$(1)" is a consequence of Proposition 1.

((2)$\rightarrow$(3)) Suppose Ω does not satisfy the condition (3), then there exists two sequences $\{z_n\}_{n=1}^{\infty}$ on Ω and $\{\zeta_n\}_{n=1}^{\infty}$ on $\partial\Omega$ such that $|z_n-\zeta_n|^{-1}\geqslant n\rho_\Omega(z_n)$. We set $u_n(z)=\log|z-\zeta_n|$. Since $\log|z|$ belongs to $\widetilde{BMO}(\mathbf{C}, m)$, the $\widetilde{BMO}(\Omega, m)$ norms of $u(n=1, 2,\cdots)$ are bounded above. On the other hand, by Proposition 3, $\|u_n\|_{BMO(\Omega, m)}\geqslant A\rho_\Omega(z_n)|\nabla u_n(z_n)|\geqslant An\longrightarrow+\infty$, hence $BMOH(\Omega, m)$ does not concide with $\widetilde{BMOH}(\Omega, m)$ by open mapping theorem.

((3)→(4)) We show that if we choose a constant M so that $M<\pi L^{-1}$ then (4) is valid with this constant M. If it were not so, there exists a point $z_0\in\Omega$ such that the domain $\Omega_0=\{z\in\Omega : \rho_\Omega(z, z_0)<M\}$ is not simply connected, then there exists a simple closed curve α in Ω_0 such that $\int_\alpha\rho_\Omega(\zeta)|d\zeta|<2M$ and α surrounds some point $\xi_0\in\partial\Omega$. Hence $\int_\alpha d(\zeta, \partial\Omega)^{-1}|d\zeta|\leqslant\int_\alpha L\rho_\Omega(\zeta)|d\zeta|<2\pi\leqslant\int_\alpha r^{-1}|rd\theta|\leqslant\int_\alpha|\zeta-\zeta_0|^{-1}|d\zeta|\leqslant\int_\alpha d(\zeta, \partial\Omega)^{-1}|d\zeta|$, where $\zeta-\zeta_0=re^{i\theta}$. This is a contradiction.

((5)→(4)) For an analytic function f on D, its Schwarzian derivative is defined by $S_f(z)=(f''/f')'-2^{-1}(f''/f')^2$, then it is known that if $|S_f(z)|\leqslant2(1-|z|^2)^{-2}$, $z\in D$, then f is conformal on D (cf. [4]). Let π satisfy the condition (5). Using $\mathscr{B}(D)$-norm instead of $\widetilde{BMOA}(D, m)$-*norm*, we have $|(\pi''/\pi')|\leqslant C(1-|z|^2)^{-1}$ on D with some constant $C>0$ and a simple calculation shows $|S_\pi(z)|\leqslant C'(1-|z|^2)^{-2}$. Let $0<a<1$, γ a Möbius transformation of D and $g(z)=\pi(a\gamma(z))$, then $S_g(z)=a^2 S_\pi(a\gamma(z))(\gamma'(z))^2$, hence we have $|S_g(z)|\leqslant a^2C'(1-|a\gamma(z)|^2)^{-2}|\gamma'(z)|^2\leqslant a^2C'(1-|z|^2)^{-2}$. Therefore if we choose a so that $a^2C'\leqslant2$, the map g becomes conformal for every Möbius transformation γ of D, which implies the condition (4).

The similar argument shows "(4)→(5)", and since $-\log(1-|z|^2)=\log\rho_\Omega(\pi(z))+\log|\pi'(z)|$, "(5)↔(6)" follows from the fact $\log(1-|z|^2)\in BMO(D, m)$. "(1)→(7)" follows from Proposition 4 (1) and the closed graph theorem. Finally "(7)→(4)" follows from Proposition 4 (2). Q.E.D.

§2. BMO spaces on Riemann surfaces.

Let R be an arbitrary Riemann surface. We define the following new BMO space on R which reduces to $\widetilde{BMO}(R, m)$ when R is a plane domain.

Definition 3. $BMO(R,m)=\{f\in L^1_{loc}(R) : \|f\|_{\widehat{BMO}(R,m)}=\sup(1/\pi)\int_D|f\circ\phi-f\circ\phi(D, m)|dm<+\infty\}$, where the supremum is taken for every conformal map ϕ of D into R.

$$\widehat{BMOH}(R, m)=\widehat{BMO}(R, m)\cap H(R),\ \widehat{BMOA}(R, m)=\widehat{BMO}(R, m)\cap A(R).$$

Note that $\|f\|_{\widehat{BMO}(R,m)}=\sup_\phi\|f\circ\phi\|_{\widetilde{BMO}(D,m)}$.

The metric $d(z, \partial\Omega)^{-1}|dz|$ on a plane domain Ω called quasi-hyperbolic metric (cf. [5]) is conformally invariant. Indeed, Koebe's one-quarter theorem shows that for a conformal map f of Ω onto Ω', $4^{-1}(d(z, \partial\Omega)^{-1}|dz|)\leqslant d(f(z), \partial\Omega')^{-1}|df(z)|\leqslant4(d(z, \partial\Omega)^{-1}|dz|)$ (also see [5]). Now we define the corresponding metric $\hat\rho_R(z)|dz|$ on an arbitrary Riemann surface R by $\hat\rho_R(z)=\inf\rho_S(z)$ $(\geqslant\rho_S(z))$, where the infimum is taken for every simply connected domain S on R containing z. Equivalently, $\hat{\rho}_R(z)=\inf|\phi'(0)|^{-1}$, where the infimum is taken for every conformal map ϕ of D into R such that $\phi(0)=z$. When Ω is a plane domain, the second expression and Koebe's one-quarter theorem imply $4(d^{-1}(z, \partial\Omega)^{-1}|dz|)\leqslant\hat{\rho}_\Omega(z)|dz|\leqslant d(z, \partial\Omega)^{-1}|dz|$. Thus the metric $\hat{\rho}_R(z)|dz|$ is considered as a generalization of $d(z, \partial\Omega)^{-1}|dz|$. We call $\hat{\rho}_R(z)|dz|$ the generalized quasi-hyperbolic metric. Now we investigate the relation between $\hat{\rho}_R(z)$ and the injective radius with center z. Let R be a Riemann surface with universal covering D. We define the injective radius $r_R(z)$ with center $z\in R$ by $r_R(z)=\sup\ \{r>0$: the domain$\{\zeta\in R : \rho_R(z, \zeta)<r\}$ is simply connected$\}$, then we have

Lemma 1. *Let R be a Riemann surface with universal covering D, then*

$$4^{-1}l(r_R(z))^{-1}\rho_R(z)|dz|\leqslant\hat{\rho}_R(z)|dz|\leqslant l(r(z))^{-1}\rho_R(z)\,|dz|,$$

where $l(r)$ denotes the Euclidean radius of the disk in D with center the origin and hyperbolic radius r.

Proof. Let $\pi : D \longrightarrow R$ be the universal covering map such that $\pi(0)=z$. Set $\phi_0(\zeta)=\pi(l(r_R(z))\zeta)$, then ϕ_0 is a conformal map of D into R such that $\phi_0(0)=z$, hence $\hat{\rho}_R(z)\leqslant|\phi_0'(0)|^{-1}=l(r_R(z))^{-1}|\pi'(0)|^{-1}=l(r_R(z))^{-1}\rho_R(z)$. Next, let ϕ be an arbitrary conformal map of D into R such that $\phi(0)=z$. Let Ω be the component of $\pi^{-1}(\phi(D))$ containing the origin. Then $g=\pi^{-1}\circ\phi : D \longrightarrow \Omega$ is conformal and so Koebe's one-quarter theorem shows that $4^{-1}l(r_R(z))^{-1}\rho_R(z)\leqslant 4^{-1}d(0, \partial\Omega)^{-1}|\pi'(0)|^{-1}\leqslant|g'(0)|^{-1}|\pi'(0)|^{-1}=|\phi'(0)|^{-1}$, hence the assertion follows.

The following theorem shows that in the definition of $\widehat{BMO}$ it is enough to take the supremum over some family of "tame" conformal maps. Let $B_z=\{\zeta\in R$: $\rho_R(z, \zeta)<r_R(z)\}$ and $\phi_z : D \longrightarrow B_z$ the conformal map such that $\phi_z(0)=z$.

Theorem 2. *Let R be a Riemann surface with universal covering D, then for every function f of $L^1_{loc}(R)$ we have*

$$\sup_{z\in R}\|f\circ\phi_z\|_{\widetilde{BMO}(D,m)}\leqslant\|f\|_{\widehat{BMO}(R,m)}\leqslant A\sup_{z\in R}\|f\circ\phi_z\|_{\widetilde{BMO}(D,m)},$$

where $A\geqslant 1$ is a universal constant. In other words f belongs to $\widehat{BMO}(R, m)$ if and only if $\sup\ m(B)^{-1}\int_B|f\circ\pi-f\circ\pi(B,\ m)|dm(\equiv M)<+\infty$, and $M\leqslant\|f\|_{\widehat{BMO}(R,m)}\leqslant A'M$, where $\pi : D \longrightarrow R$ is a universal covering map and the supremum is taken for every disk B in D such that no two points of B is equivalent and $A'\geqslant 1$ is a universal constant.

Lemma 2. *For a Riemann surface R with universal covering D, we have*

$$\{\zeta\in R : \hat{\rho}_R(z, \zeta)<12^{-1}\}\subset B_z,\ \ z\in R,$$

where $\hat{\rho}_R(z, \zeta)$ denotes the generalized quasi-hyperbolic distance between z and ζ.

Proof. Let $B_z^0=\{\zeta\in R : \rho_R(z, \zeta)<2^{-1}r_R(z)\}$ and $z'\in B_z^0$ then $r_R(z')\leqslant r_R(z)+2^{-1}r_R(z)=(3/2)r_R(z)$ hence $l(r_R(z'))\leqslant l((3/2)r_R(z))\leqslant(3/2)l(r_R(z))$ and so by Lemma 1 we have $\hat{\rho}(z')\geqslant 4^{-1}l(r_R(z'))^{-1}\rho_R(z')\geqslant 6^{-1}l(r_R(z))^{-1}\rho_R(z')$. It follows that $\hat{\rho}_R(z, \partial B_z^0)\geqslant$

$6^{-1}l(r_R(z))^{-1}(2^{-1}r_R(z))=12^{-1}r_R(z)l(r_R(z))^{-1}\geqslant 12^{-1}$, which implies the assertion.

Proof of Theorem 2. The first inequality is trivial. Next, let ϕ be a conformal map of D into R. By Proposition 1 it suffices to estimate the mean oscillation of $f\circ\phi$ on every disk B in D whose hyperbolic radius is less than 12^{-1}. Then the radius of $\phi(B)$ with respect to the generalized quasi-hyperbolic metric is less than 12^{-1} and so $\phi(B)\subset B_z$ for some $z\in R$ by above lemma. Hence the assertion follows from Proposition 2.

Next we give some characterization for BMO functions.

Theorem 3. (1) *$BMO(R, m)\subset\widehat{BMO}(R, m)$ for every Riemann surface R with universal covering D.*
(2) *Let Ω be a plane domain, then for every function f on Ω,*

$$\|f\|_{\widetilde{BMO}(\Omega, m)}\leqslant\|f\|_{\widehat{BMO}(\Omega, m)}\leqslant A\|f\|_{\widetilde{BMO}(\Omega, m)},$$

where $A\geqslant 1$ is a universal constant, especially $\widehat{BMO}(\Omega, m)=\widetilde{BMO}(\Omega, m)$.
(3) *A harmonic function h on an arbitrary Riemann surface R belongs to $\widehat{BMOH}(R, m)$ if and only if there exists a constant $M\geqslant 0$ such that $|\nabla h(z)|\leqslant M\hat{\rho}_R(z)$, $z\in R$.*
(4) *An analytic function f on an arbitrary Riemann surface R having universal covering D belongs to $\widehat{BMOA}(R, m)$ if and only if the Riemann surface of the inverse function of f does not contain arbitrary large schlicht disk, especially it holds that $\widehat{BMOA}(R, m)=BMOA(R, m)$.*

Proof. Let R be a Riemann surface having universal covering map $\pi: D\longrightarrow R$ and $f\in BMO(R, m)$. Let ϕ be a conformal map of D into R and Ω_0 one of the components of $\pi^{-1}(\phi(D))$. Since $\pi^{-1}\circ\phi: D\longrightarrow\Omega_0$ is conformal, we obtain $\|f\circ\phi\|_{\widetilde{BMO}(D, m)}=\|(f\circ\pi)\circ(\pi^{-1}\circ\phi)\|_{\widetilde{BMO}(D, m)}\leqslant C_2(1)\|f\circ\pi\|_{\widetilde{BMO}(\Omega_0, m)}\leqslant C_2(1)\|f\|_{BMO(R, m)}$ by Proposition 2. It follows that $\|f\|_{\widehat{BMO}(R, m)}\leqslant C_2(1)\|f\|_{BMO(R, m)}$, hence $BMO(R, m)\subset\widehat{BMO}(R, m)$. In case Ω a plane domain, the first inequality in (2) is trivial and the second one is a consequence of Proposition 2. Note that the condition in (3) is equivalent to the condition $\sup_{\phi}\{\sup_{z\in D}(1-|z|^2)|\triangle(h\circ\pi)(z)|\}\leqslant M$, where $\sup_{\phi}$ is taken for every conformal map ϕ of D into R, hence the assertion (3) follows from Proposition 3 and the definition of $\widehat{BMO}$. Finally, the assertion (4) follows fom the fact that an analytic function g on D belongs to Bloch space $\mathscr{B}(D)$ $(=\widetilde{BMOA}(D, m))$ if and only if the Riemann surface of the inverse function of g does not contain arbitrary large schlicht disk (see [2]).

Q.E.D.

Here we show a removability property.

Theorem 4. *Let $\{z_n\}_{n=1}^{\infty}$ be a hyperbolically separated sequence on D, that is, there exists a constant $a>0$ such that $\rho_D(z_i, z_j)\geqslant a$ $(i\neq j)$. Let $D'=D\setminus\bigcup_n\{z_n\}$ and f a function of $\widetilde{BMO}(D', m)$, then $f\in\widetilde{BMO}(D, m)$ and $\|f\|_{\widetilde{BMO}(D, m)}\leqslant C(a)\|f\|_{\widetilde{BMO}(D', m)}$ where $C(a)\geqslant 1$ is a constant depending only on a.*

Remark 2. This result was shown in [3] when f is harmonic on D.

Remark 3. Since D is a uniform domain, above theorem implies that D' is a uniform domain (cf. [5]).

Proof. By Proposition 1, it suffices to show that if f belongs to $\widetilde{BMO}(D\smallsetminus\{0\}, m)$, then $f\in\widetilde{BMO}(D, m)$ and $\|f\|_{\widetilde{BMO}(D,m)}\leqslant A\|f\|_{\widetilde{BMO}(D\smallsetminus\{0\},m)}$ with some universal constant $A>0$. Note that it is known that if g belongs to $\widetilde{BMO}(\mathbf{C}\smallsetminus\{0\}, m)$ then $g\in\widetilde{BMO}(\mathbf{C}, m)$ and $\|g\|_{\widetilde{BMO}(\mathbf{C},m)}\leqslant A'\|g\|_{\widetilde{BMO}(\mathbf{C}\smallsetminus\{0\},m)}$ with a universal constant $A'>0$ (see [12] 5p), and the same argument is valid here.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence on Riemann surface R such that $\hat{\rho}_R(z_i, z_j)\geqslant a>0$ $(i\neq j)$, and ϕ aconformal map of D into R, then the sequence $\phi^{-1}(\bigcup_n\{z_n\})$ is a hyperbolically separated sequence on D having the same constant a, hence we have

Corollary 1. *Let R be an arbitrary Riemann surface, $\{z_n\}_{n=1}^{\infty}$ a generalized quasi-hyperbolically separated sequence on R such that $\hat{\rho}_R(z_i, z_j)\geqslant a>0$ $(i\neq j)$, and $R'=R\setminus\bigcup_n\{z_n\}$. Let f be a function of $\widehat{BMO}(R', m)$, then $f\in\widehat{BMO}(R, m)$ and $\|f\|_{\widehat{BMO}(R,m)}\leqslant C(a)\|f\|_{\widehat{BMO}(R,m)}$, where $C(a)\geqslant 1$ is a constant depending only on a.*

Th e next corollary is a generalization of Proposition 4 (1).

Corollary 2. *Let R be a hyperbolic Riemann surface, then*

$$A\leqslant\|g_R(\cdot, \xi)\|_{\widehat{BMO}(R,m)}\leqslant A', \ \xi\in R$$

where A, $A'>0$ are universal constants.

Proof. We can show the existence of the constant A by considering the mean oscillation of $g_R(\cdot, \zeta)$ on a sufficiently small local disk containing ζ. Next, let ϕ be a conformal map of D into $R\setminus\{\zeta\}$ and f the analytic function on D such that $Re(f(z))=g_R(\phi(z), \zeta)$, then the Riemann surface of the inverse function of f does not contain a schlicht disk whose radius is larger than π. Therefore, by Theorem 3 (4), we have $\|g_R(\cdot, \zeta)\|_{\widehat{BMO}(R\smallsetminus\{\zeta\},m)}\leqslant A''$ for some universal constant $A''>0$, and the assertion follows from Corollary 1.

Finally we need the following lemma to prove our main theorem below.

Lemma 3. *Let R be a Riemann surface having universal covering D, B_0 a local disk on R and set $R_0=R\smallsetminus\bar{B}_0$, then we have*

(1) *For any $f\in\widehat{BMO}(R_0, m)$, there exists a function g of $\widehat{BMO}(R, m)$ such that $g|R_0=f$.*

(2) *For any $g\in BMO(R, m)$, the function $f=g|R_0$ belongs to $BMO(R_0, m)$.*

Proof. Let B_1 be a local disk on R such that $\bar{B}_0\subset B_1$ and set $2a=\rho_R(\partial B_0, \partial B_1)$. Let $q: B_1\longrightarrow D$ be a conformal map such that $q(B_0)=\{|z|<r_0\}$ $(0<r_0<1)$. Let f be in $\widehat{BMO}(R_0, m)$, then $f\circ q^{-1}\in\widetilde{BMO}(\{r_0<|z|<1\}, m)$. Since $\{r_0<|z|<1\}$ is a uniform domain (see [5]), there exists a function $k\in\widetilde{BMO}(D, m)$ such that $k|\{r_0<|z|<1\}=f\circ q^{-1}$. Let g be a function on R such that $g=k\circ q$ on $\bar{B}_0$, $g=f$ on R_0. We show that g belongs to $\widehat{BMO}(R, m)$. Let ϕ be a conformal map of D into R and B a disk in D whose hyperbolic radius is less than a. First we assume $\phi(B)\cap B_0\neq\phi$. Then, since $\phi(B)\subset B_1$ and $q\circ\phi$ is conformal, we have $\|g\circ\phi\|_{\widetilde{BMO}(B,m)}\leqslant C_2(1)\|k\|_{\widetilde{BMO}(D,m)}$ by Proposition 2. Next we assume $\phi(B)\cap B_0=\phi$. Then $\phi(B)\subset R_0$ and so $\|g\circ\phi\|_{\widetilde{BMO}(D,m)}\leqslant\|f\|_{\widehat{BMO}(R_0,m)}$ by definition of BMO. It follows by Proposition 1 that $\|g\circ\phi\|_{\widetilde{BMO}(D,m)}\leqslant C_1'(a)\max\{C_2(1)\|k\|_{\widetilde{BMO}(D,m)}, \|f\|_{\widehat{BMO}(R_0,m)}\}$, hence g belongs to $\widehat{BMO}(R, m)$.

Next we assume g be a function of $BMO(R, m)$. Let $\pi: D\longrightarrow R$ and $\pi_0: D\longrightarrow R_0$ be universal covering maps. Let $j: R_0\longrightarrow R$ an inclusion map and $\tilde{j}: D\longrightarrow D$ its lift. Then $\tilde{j}: D\longrightarrow D\smallsetminus\pi^{-1}(\bar{B}_0)$ is a universal covering map, and there exists a constant $b>0$ such that for every disk B in D whose hyperbolic radius is less than b, the map $\tilde{j}: B\longrightarrow\tilde{j}(B)$ is conformal. Hence by Proposition 2, $\|f\circ\pi_0\|_{\widetilde{BMO}(B,m)}\leqslant C_2(1)\|g\circ\pi\|_{\widetilde{BMO}(\tilde{j}(B),m)}\leqslant C_2(1)\|g\circ\pi\|_{\widetilde{BMO}(D,m)}\leqslant C_2(1)\|g\|_{BMO(R,m)}$. It follows by Proposition 1 that $\|f\|_{BMO(R_0,m)}\leqslant C_1'(b)C_2(1)\|g\|_{BMO(R,m)}$.

Q.E.D.

Now we can prove the following result.

Theorem 5. *For a Riemann surface R having universal covering D, the following conditions are equivalent;*

(1) $BMO(R, m)=\widehat{BMO}(R, m)$,

(2) $\inf_{z\in R} r_R(z)>0$, *that is, there exists a constant $M>0$ such that for every $\zeta\in R$, the domain $\{z\in R: \rho_R(z, \zeta)<M\}$ is simply connected,*

(3) *There exists a constant $L>0$ such that $\hat{\rho}_R(z)\leqslant L\rho_R(z)$, $z\in R$.*

Further if R is hyperbolic, the next condition is also equivalent;

(4) $\sup_{\zeta\in R}\|g_R(\cdot, \zeta)\|_{BMO(R,m)}<+\infty$.

Proof. "(2) $\longleftrightarrow$ (3)" is a consequence of Lemma 1 and "(2) $\longrightarrow$ (1) follows from Proposition 1. In case R is a hyperbolic surface, "(1) $\longrightarrow$ (4)" is a consequence of Corollary 2 and closed graph theorem and "(4) $\longrightarrow$ (2)" follows from Proposition 4 (2).

Next we prove "(1) $\longrightarrow$ (2)" in case that R is not a hyperbolic surface. Let B_0 be a local disk on R and set $R_0=R\smallsetminus\bar{B}_0$. Let $f\in\widehat{BMO}(R_0, m)$, then by Lemma 3 (1) there exists a function g of $\widehat{BMO}(R, m)$ such that $g|R_0=f$. Since $BMO(R, m)=\widehat{BMO}(R, m)$, it follows by Lemma 3 (2) that f belongs to $BMO(R_0, m)$. Therefore R_0 is a hyperbolic surface satisfying the condition (1) and so R_0 satisfies the condition (2). Here we remark that $\inf_{z\in R_0} r_{R_0}(z)=2^{-1}\inf\{l_{R_0}(\alpha): \alpha$ is a closed curve on R_0 which is not homotopic to a point$\}$, where $l_{R_0}(\alpha)$ denotes the hyperbolic length in R_0 of the curve α. And further by Schwarz's lemma, it holds that $l(a)\rho_{R_0}(z)\leqslant\rho_R(z)\leqslant\rho_{R_0}(z)$ on $\{\rho_R(z, B_0)>a\}$. Hence R also satisfies the condition (2).

Q.E.D.

For arbitrary Riemann surface R having universal covering D, the inclusion $BMO(R, \lambda)\subset BMO(R, m)$ holds, where $BMO(R, \lambda)$ is the BMO space on R with respect to the hyperbolic measure $d\lambda$ (cf. [6], [9], [12]). If R is compact, we have $BMO(R, \lambda)=BMO(R, m)$ (*see* [9]), hence

Corollary 3. *For arbitrary Riemann surface R with universal covering D, it holds that*

$$BMO(R, \lambda)\subset BMO(R, m)\subset \widehat{BMO}(R, m),$$

and if R is compact, we have

$$BMO(R, \lambda)=BMO(R, m)=\widehat{BMO}(R, m).$$

§3. Some extension property for BMO functions on Riemann surfaces.

Let Ω be a quasi-disk, that is, Ω is a domain in $\mathbf{C}$ surrounded by a simple closed curve α and there exists a quasi-conformal hemeomorphism f of $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$ which maps Ω onto $\hat{\mathbf{C}}\setminus\bar{\Omega}$ and keeps α pointwise fixed. For a function $g\in\widetilde{BMO}(\Omega, m)$ we define the function k on $\mathbf{C}$ by $k(z)=g(z)$ on Ω and $k(z)=g(f(z))$ on $\mathbf{C}\setminus\bar{\Omega}$. Since the 2-dim. measure of α vanishes, k is well defined as a function of $L^1_{loc}(\mathbf{C})$. Then k belongs to $\widetilde{BMO}(\mathbf{C}, m)$ and $\|k\|_{\widetilde{BMO}(\mathbf{C}, m)}\leqslant C(K)\|g\|_{\widetilde{BMO}(\Omega, m)}$, where $C(K)\geqslant 1$ is a constant depending only on the maximal delation K of f (see [8]). In this section we prove the corresponding result for compact Riemann surfaces. First we prove the following simple lemma.

Lemma 4. *Let Ω be a plane domain, $\Omega_+=\Omega\cap\{\mathrm{Im}\ z>0\}$ and $\Omega_-=\Omega\cap\{\mathrm{Im}\ z<0\}$. Let f be a function on Ω such that* (1)$\|f\|_{\widetilde{BMO}(\Omega_+, m)}$, $\|f\|_{\widetilde{BMO}(\Omega_-, m)}\leqslant M$, (2) *for every point $z\in\Omega$ such that $\bar{z}\in\Omega$, it holds that $f(\bar{z})=f(z)$. Then f belongs to $\widetilde{BMO}(\Omega, m)$ and $\|f\|_{\widetilde{BMO}(\Omega, m)}\leqslant AM$, where $A\geqslant 1$ is a universal constant.*

Proof. First we remark that in the definition of $\widetilde{BMO}$, we can replace the word "disk B" by "square Q whose sides are parallel to the coordinate axes" (see the proofs of Hilfssatz 2 in [12] and Lemma 2. 3 in [8]). Further by Proposition 1, it suffices to estimate the mean oscillation on the square Q in Ω such that $side(Q)\leqslant d(Q, \partial\Omega)$ and $Q\cap\mathbf{R}\neq\phi$, where $side(Q)$ denotes the length of the side of Q. Let (x_0, y_0) be the center of Q, and Q' the square with center $(x, side(Q)/2)$ and $side(Q')=side(Q)$. Then $Q'\subset\Omega_+$, hence $m(Q)^{-1}\int_Q|f-f(Q, m)|dm \leqslant 2m(Q)^{-1}\int_Q|f-f(Q', m)|dm=\int_{Q\cap\Omega_+}+\int_{Q\cap\Omega_-}\leqslant 4m(Q')^{-1}\int_{Q'}|f-f(Q', m)|dm$ and so the assertion follows.

Theorem 6. *Let R be a compact Riemann surface with universal covering D, $\alpha_j(1\leqslant j\leqslant n)$ disjoint simple closed curves on R such that $R\setminus\bigcup_{j=1}^{n}\alpha_j$ consists of two components R_1, R_2 and f a quasi-conformal homeomorphism of R with maximal dilatation K which* *maps* R_1 *onto* R_2 *and keeps* $\bigcup_{j=1}^{n}\alpha_j$ *pointwise fixed. Set* $b=\max\{l_R([\alpha_j]) : 1\leqslant j\leqslant n\}$, *where* $l_R([\alpha_j])$ *denotes the hyperbolic length of the geodesic* $[\alpha_j]$ *in the homotopy class of* α_j. *For a function* g *on* R_1, *we define the function* k *on* R *by* $k(z)=g(z)$ *on* R_1, $k(z)=g(f(z))$ *on* R_2, *then*

(1) *If* g *belongs to* $\widehat{BMO}(R_1, m)$, *then* $k\in\widehat{BMO}(R, m)$ *and* $\|k\|_{\widehat{BMO}(R,m)}\leqslant C(K)\|g\|_{\widehat{BMO}(R_1,m)}$, *where* $C(K)\geqslant 1$ *is a constant depending only on* K.

(2) *If* g *belongs to* $BMO(R_1, m)$, *then* $k\in BMO(R, m)$ *and* $\|k\|_{BMO(R,m)}\leqslant C(K, b)\|g\|_{BMO(R_1,m)}$, *where* $C(K, b)>0$ *is a constant depending only on* K *and* b.

Proof. Note that there exists a compact bordered Riemann surface R_1' and a conformal map s_1 of R_1' onto R_1. Then s_1 has a continuous extention to $\overline{R_1'}$. Let R' be the double of R_1'', j its anti-conformal involution and $R_2'=j(R_1')$. We define a map s on R' by $s(z)=s_1(z)$ on $\overline{R_1'}$, $s(z)=f(s_1(j(z))$ on R_2', then s is a quasi-conformal map having the maximal dilatation K. By Wolpert's theorem (cf. [1] 52p), we have $l_{R'}(s^{-1}(\alpha_j))=l_{R'}([s^{-1}(\alpha_j)])\leqslant Kl_R([\alpha_j])\leqslant Kb$. It follows by Proposition 2, we can assume from the beginning that R_1 is a compact bordered surface, R its double and f its anti-conformal involution.

Let g be a function of $\widehat{BMO}(R_1, m)$. Let H be the upper half plane, $\pi_1 : H \longrightarrow R_1$ the universal covering map and E the limit set of its covering transformation group. We can assume $\infty\in E$. Then the map π_1 induce the covering map $\pi_1 : \mathbf{C}\setminus E \longrightarrow R$ naturally. Let ϕ be a conformal map of D into R. Let Ω be one of the components of $\pi_1^{-1}(\phi(D))$, then $\pi_1 : \Omega \longrightarrow \phi(D)$ is a conformal map. Set $\Omega_+=\Omega\cap H$, $\Omega_-=\Omega\cap L$, where $L=\mathbf{C}\setminus H$, then by Proposition 2, $\|k\circ\pi_1\|_{\widetilde{BMO}(\Omega_+,m)}=\|(k\circ\phi)\circ(\phi^{-1}\circ\pi_1)\|_{\widetilde{BMO}(\Omega_+,m)}\leqslant C_2(1)\|k\circ\phi\|_{\widetilde{BMO}(\phi^{-1}(\pi_1(\Omega_+)),m)}\leqslant C_2(1)\|g\|_{\widehat{BMO}(R_1,m)}$. Since the same estimate holds on Ω_-, it follows by Lemma 4 that $\|k\circ\pi_1\|_{\widetilde{BMO}(\Omega,m)}\leqslant AC_2(1)\|g\|_{\widehat{BMO}(R_1,m)}$. Hence by Proposition 2, we have $\|k\circ\phi\|_{\widetilde{BMO}(D,m)}\leqslant C_2(1)\|k\circ\pi_1\|_{\widetilde{BMO}(\Omega,m)}\leqslant AC_2(1)^2\|g\|_{\widehat{BMO}(R_1,m)}$ and so $\|k\|_{\widehat{BMO}(R,m)}\leqslant AC_2(1)^2\|g\|_{\widehat{BMO}(R_1,m)}$.

Next we prove the assertion (2). Let g be a function of $BMO(R_1, m)$, $\pi_1 : D \longrightarrow R_1$ and $\pi : D \longrightarrow R$ the universal covering maps, $j : R_1 \longrightarrow R$ the inclusion map and $\tilde{j} : D\longrightarrow D$ its lift. Note that by the collar lemma (cf. [1] 95p), there exists a constant $a>0$, which depends only on B, such that for each α_j the domain $U_j=\{z\in R : \rho_R(z, \alpha_j)<a\}$ becomes a collar neighborhood of α_j. Let $\pi^{-1}(\alpha_j)=\{\tilde{\alpha}_{j,t}\}_{t=1,2,\dots}$ be the decomposition into the component. Let B be a disk in D whose hyperbolic radius is less than a/2. First we assume that $B\cap\tilde{\alpha}_{j,t}\neq\phi$ for some $\tilde{\alpha}_{j,t}$. We can assume $\tilde{\alpha}_{j,t}$ is the interval $(-1. 1)$. Set $\Omega=\{z\in D : \rho_D(z, \tilde{\alpha}_{j,t})<a\}$, $\Omega_+=\Omega\cap H$, $\Omega_-=\Omega\cap L$, then $B\subset\Omega$. We can assume $\pi(\Omega_+)\subset R_1$. Set $\Omega_0=\tilde{j}^{-1}(\Omega_+)$. Since $\tilde{j}$ is a conformal map of D into D, the map $\tilde{j} : \Omega_0 \longrightarrow \Omega_+$ is conformal, hence by Proposition 2 we have $\|k\circ\pi\|_{\widetilde{BMO}(\Omega_+,m)}\leqslant C_2(1)\|k\circ\pi\circ\tilde{j}\|_{\widetilde{BMO}(\Omega_0,m)}=C_2(1)\|g\circ\pi_1\|_{\widetilde{BMO}(\Omega_0,m)}\leqslant C_2(1)\|g\|_{BMO(R_1,m)}$. Since the same estimate holds for Ω_-, it follows by Lemma 4 that $\|k\circ\pi\|_{\widetilde{BMO}(B,m)}\leqslant\|k\circ\pi\|_{\widetilde{BMO}(\Omega,m)}\leqslant AC_2(1)\|g\|_{BMO(R_1,m)}$. Next we assume that $B\cap\tilde{\alpha}_{j,t}=\phi$ for every $\tilde{\alpha}_{j,t}$, then we can assume $\pi(B)\subset R_1$. We set $\Omega_1=\tilde{j}^{-1}(B)$, then $j : \Omega_1 \longrightarrow B$ is a conformal map. Therefore by Proposition 2, we have $\|k\circ\pi\|_{\widetilde{MBO}(B,m)}\leqslant C_2(1)\|k\circ\pi\circ\tilde{j}\|_{\widetilde{BMO}(\Omega_1,m)}=C_2(1)\|g\circ\pi_1\|_{\widetilde{BMO}(\Omega_1,m)}\leqslant C_2(1)\|g\|_{BMO(R_1,m)}$. Summarizing

above, we obtain $\|k\circ\pi\|_{\widetilde{BMO}(B,m)}\leqslant AC_2(1)\|g\|_{BMO(R_1,m)}$ for every disk B in D whose hyperbolic radius is less than a/2. Hence the assertion (2) follows by Proposition 1. Q.E.D.

The following example shows that in the assertion (2) of above theorem, we can not replace the constant $C(K, b)$ by some constant "$C(K)$".

Example. Let $w=\gamma_0(z)$ be a Möbius transformation such that $(w-2)/(w+2)=t_0(z-2)/(z+2)$, where $t_0>0$ is asufficiently small constant, and $w=\gamma_t(z)$ a Möbius transformation such that $(w-1)/(w+1)=t(z-1)/(z+1)$, $0<t<10^{-1}$. Set $G=\langle\gamma_0, \gamma_t\rangle$, then the Riemann surface $R_1=H/G$ becomes a compact bordered surface having three boundary components. Let B_0, B_0' and B_t, B_t' be the disks surrounded by the isometric circles for γ_0 and γ_t respectively. The domain $N_0=H\smallsetminus(\bar{B}_0\cup\bar{B}_0'\cup\bar{B}_t\cup\bar{B}_t')$ is a fundamental domain for R_1. We define a function g on $\bar{N}_0$ by $g(z)=\log|z-1|$ on $\bar{N}_0\cap\{\mathrm{Re}\ z\geqslant 0\}$, $g(z)=\log|z+1|$ on $N_0\cap\{\mathrm{Re}\ z<0\}$. Then g belongs to $\widetilde{BMO}(N_0, m)$ and its $\widetilde{BMO}(N_0, m)$ norm is bounded above for $0<t<10^{-1}$. Since g takes the same value on every equivalent point on $\bar{N}_0$, g define a function on R. Set $\mathrm{I}=\{\gamma(N_0):\gamma\in G\}$. Let N_i, $N_j\in I$ have a common boundary arc in H, τ the reflection with respect to this arc. Then for every point $z\in N_i$ such that $\tau(z)\in N_j$, it holds that $g(z)=g(\tau(z))$. Further we remark that there exists a constant $a>0$ such that for every $t(0<t<10^{-1})$, each disk B in H whose hyperbolic radius is less than a intersects with at most two domains of I. Hence Proposition 1, 2 and Lemma 4 show that the $BMO(R_1, m)$ norm of $g(0<t<10^{-1})$ is bounded above. Let $\pi_1: H\longrightarrow R_1$ be the universal covering map, R the double of R_1 and k the function in Theorem 5. The map π_1 induces a covering map $\pi_1:\mathbf{C}\smallsetminus E\longrightarrow R$, where E denotes the limit set of G. Then N_0, $\{z\in\mathbf{C}:\bar{z}\in N_0\}$ and the free boundaries of N_0 makes a fundamental domain for this cvoering map, which we denote by $\tilde{N}_0$. For every $\varepsilon>0$ there exists a constant $t_1>0$ such that $\Omega_0=\{\varepsilon<|z-1|<2^{-1}\}\subset\tilde{N}_0$ for every $t(0<t<t_1)$. Let $\Omega_1=\{\log\varepsilon<\mathrm{Re}\ z<\log 2^{-1}\}$, and define the map $p:\Omega_1\longrightarrow\Omega_0$ by $p(z)=e^z+1$, then p is a universal covering map. Hence we can regard the domain Ω_1 as a subdomain of the universal covering D of R, and so it suffices to show that $\|k\circ p\|_{\widetilde{BMO}(\Omega_1,m)}\longrightarrow+\infty$ as $t\longrightarrow 0$. Let $Q=\{z=x+iy:\log\varepsilon<x<\log 2^{-1},\ 0<y<\log 2^{-1}\varepsilon^{-1}\}\subset\Omega_1$. Since $k\circ p=\mathrm{Re}z$, we have $m(Q)^{-1}\int_Q|k\circ p-k\circ p(Q, m)|dm=4^{-1}\log 2^{-1}\varepsilon^{-1}\longrightarrow+\infty$ as $\varepsilon\longrightarrow 0$ and so the assertion follows.

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Added in proof.

The equivalence of the condition (1), (3), (4), (5), (6), in Theorem 1 has been proved by B. G. Osgood [13]. Our proof is partially different from his and gives other new equivalent conditions.