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On the L^2-boundedness of pseudo-differential operators

By

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§1. Introduction.

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. Let m, ρ and δ be real numbers with $0 \leq \rho, \delta \leq 1$. If a smooth function $p(x, \xi)$ on $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ satisfies

$$(1.1) \qquad |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for any multi-indices α and β, then we say that $p(x, \xi)$ belongs to Hörmander's class $S_{\rho,\delta}^m$ (see, for example, [5]). For $p(x, \xi)$ in $S_{\rho,\delta}^m$ we define the pseudo-differential operator $p(X, D_x)$ by

$$(1.2) \qquad p(X, D_x)u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)d\xi,$$

where $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, that is, $\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx$ and we denote $p(X, D_x) \in L_{\rho,\delta}^m$. The function $p(x, \xi)$ is called the symbol of the operator $p(X, D_x)$. In [4], Hörmander proved that if all of the operators in $L_{\rho,\delta}^m$ are $L^2(\mathbf{R}^n)$-bounded then $m \leq \min\left\{0, \frac{n}{2}(\rho-\delta)\right\}$, by giving counter examples. When $\delta < 1$, Calderón and Vaillancourt in [1] showed that $m \leq \min\left\{0, \frac{n}{2}(\rho-\delta)\right\}$ implies the $L^2(\mathbf{R}^n)$-boundedness of the operators in $L_{\rho,\delta}^m$. Moreover, when $\rho=\delta<1$, there are many generalized theorems to the case of non-regular symbols (see, for example, [3], [7] and [12]). On the other hand, when $\delta=1$, Chin-Hung-Ching in [2] proved that $S_{1,1}^0$ does not always define $L^2(\mathbf{R}^n)$-bounded operators, and Rodino in [11] proved that the operator in $L_{\rho,1}^{-n(1-\rho)/2}$ is not always $L^2(\mathbf{R}^n)$-bounded, by constructing the counter examples.

In the present paper we give also an example of symbols which is in $S_{\rho,1}^{-n(1-\rho)/2}$ but define operators unbounded in $L^2(\mathbf{R}^n)$, and we show that the decreasing order $n(1-\rho)/2$ of symbols is critical in a sense (see [6]). Our example is similar to the example constructed by Chin-Hung-Ching in the case $\rho=1$, and therefore a little different from the one of Rodino in [11].

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In Section 2 we give an $L^2(\mathbf{R}^n)$-boundedness theorem and in Section 3 we construct an example of symbols to show that the theorem in Section 2 is critical.

§2. L^2-boundedness theorem.

Let $\|f\|_{L^2}$ denote the norm of $L^2(\mathbf{R}^n)$. We use $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. Let S denote the set of Schwartz rapidly decreasing functions in $\mathbf{R}^n$. We assume that the symbols in the present paper are at least measurable in $\mathbf{R}_x^n\times\mathbf{R}_\xi^n$. Then the following lemma is shown in [10].

Lemma 2.1. *We assume that a symbol $p(x, \xi)$ satisfies that the support of $p(x, \xi)$ is contained in $\{\xi;\ |\xi|\leq R\}$ and*

$$|\partial_\xi^\alpha p(x, \xi)|\leq N \qquad \text{for } |\alpha|\leq\kappa=\left[\frac{n}{2}\right]+1. \tag{2.1}$$

Then the operator $p(X, D_x)$ is $L^2(\mathbf{R}^n)$-bounded and we have

$$\|p(X, D_x)u\|_{L^2}\leq CN\|u\|_{L^2} \qquad \text{for } u \text{ in } S, \tag{2.2}$$

where the constant C is independent of $p(x, \xi)$.

Theorem 2.2. *Let $0\leq\rho\leq 1$. Suppose that a symbol $p(x, \xi)$ satisfies that*

$$|\partial_\xi^\alpha p(x, \xi)|\leq N\langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|}\omega(\langle\xi\rangle^{-1}) \qquad \text{for } |\alpha|\leq\kappa, \tag{2.3}$$

where $\omega(t)$ is a non-negative and non-decreasing function on $[0, \infty)$ and satisfies

$$\int_0^1\frac{\omega(t)^2}{t}dt=M_2^2<\infty. \tag{2.4}$$

Then the operator $p(X, D_x)$ is $L^2(\mathbf{R}^n)$-bounded and we have

$$\|p(X, D_x)u\|_{L^2}\leq C(1+M_2)N\|u\|_{L^2} \qquad \text{for } u \text{ in } S, \tag{2.5}$$

where the constant C is independent of $p(x, \xi)$.

Proof. By the assumptions and Lemma 2.1 we may assume that the symbol $p(x, \xi)$ has the support in $\{\xi:\ |\xi|\geq 2\}$ and satisfies

$$|\partial_\xi^\alpha p(x, \xi)|\leq N|\xi|^{-n(1-\rho)/2-\rho|\alpha|}\omega(|\xi|)^{-1} \qquad \text{for } |\alpha|\leq\kappa. \tag{2.3'}$$

We take a smooth function $f(t)$ on $\mathbf{R}^1$ such that the support is contained in the interval $\left[\frac{1}{2}, 1\right]$ and $\int_0^\infty\frac{f(t)^2}{t}dt=1$. Then since $\int_0^\infty\frac{f(t|\xi|)^2}{t}dt=1$ for $\xi\neq 0$, we have

$$p(X, D_x)u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi$$

$$=\int_0^\infty\frac{dt}{t}\int K_t(x, z)G_t(x-tz)dz,$$

where

$$K_t(x, z)=(2\pi)^{-n}\int e^{iz\cdot\xi}p\left(x, \frac{\xi}{t}\right)f(|\xi|)d\xi,$$

and

$$G_t(x)=f(t|D_x|)u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}f(t|\xi|)\hat{u}(\xi)d\xi.$$

Since the support of $p\left(x,\ \frac{\xi}{t}\right)f(|\xi|)$ is contained in $\left\{\xi;\ \frac{1}{2}\leq|\xi|\leq 1,\ |\xi|\geq 2t\right\}$, we can write

$$\begin{aligned}(2.6)\qquad p(X,\ D_x)u(x)&=\int_0^{1/2}\frac{dt}{t}\int K_t(x,\ z)G_t(x-tz)dz\\&=\int_0^{1/2}\frac{dt}{t}\int_{|z|\leq t^{\rho-1}}K_t(x,\ z)G_t(x-tz)dz\\&\quad+\int_0^{1/2}\frac{dt}{t}\int_{|z|\geq t^{\rho-1}}K_t(x,\ z)G_t(x-tz)dz\\&=Iu(x)+IIu(x).\end{aligned}$$

Using the Schwarz ineuality we have

$$|Iu(x)|^2\leq\left(\int_0^{1/2}t^m\frac{dt}{t}\int|K_t(x,\ z)|^2dz\right)\left(\int_0^{1/2}\frac{dt}{t}\int_{|z|\leq t^{\rho-1}}t^{-m}|G_t(x-tz)|^2dz\right),$$

where $m=-n(1-\rho)$. By assumptions and the Parseval equality we have

$$\begin{aligned}\int_0^{1/2}t^m\frac{dt}{t}\int|K_t(x,\ z)|^2dz&=(2\pi)^{-n}\int_0^{1/2}t^m\frac{dt}{t}\int\left|p\left(x,\ \frac{\xi}{t}\right)f(|\xi|)\right|^2d\xi\\&\leq C^2N^2\int_0^{1/2}t^m\frac{dt}{t}\int_{1/2\leq|\xi|\leq 1}\left(\frac{|\xi|}{t}\right)^m\omega\left(\left(\frac{|\xi|}{t}\right)^{-1}\right)^2d\xi\\&\leq C^2N^2\int_0^{1/2}\omega(2t)^2\frac{dt}{t}=C^2N^2M_2^2.\end{aligned}$$

Therefore we have

$$\begin{aligned}(2.7)\qquad \|Iu\|_{L^2}^2&\leq C^2N^2M_2^2\int\left(\int_0^{1/2}t^{-m}\frac{dt}{t}\int_{|z|\leq t^{\rho-1}}|G_t(x-tz)|^2dz\right)dx\\&=C_2N^2M_2^2\int_0^{1/2}t^{-m}\left(\int_{|z|\leq t^{\rho-1}}\left(\int|G_t(x)|^2dx\right)dz\right)\frac{dt}{t}\\&\leq C^2N^2M_2^2\int_0^{\infty}\left(\int|f(t|\xi|)\hat{u}(\xi)|^2d\xi\right)\frac{dt}{t}\\&=C^2N^2M_2^2\|u\|_{L^2}^2.\end{aligned}$$

Here and hereafter the constants C are not always the same at each occurence. In a similar way we have

$$\begin{aligned}|IIu(x)|^2&\leq\left(\int_0^{1/2}t^m\frac{dt}{t}\int|z|^{2\kappa}|K_t(x,\ z)|^2dz\right)\\&\quad\times\left(\int_0^{1/2}t^{-m}\frac{dt}{t}\int_{|z|\geq t^{\rho-1}}|z|^{-2\kappa}|G_t(x-tz)|dz\right)\end{aligned}$$

where $m=(2\kappa-n)(1-\rho)$. By assumptions we have

$$\int_0^{1/2} t^m \frac{dt}{t} \int |z|^{2\kappa} |K_t(x, z)|^2 dz = c_n \int_0^{1/2} t^m \frac{dt}{t} \int \left| \nabla_\xi^\kappa \left\{ p\left(x, \frac{\xi}{t}\right) f(|\xi|) \right\} \right|^2 d\xi$$

$$\leq C^2 N^2 \int_0^{1/2} t^m \frac{dt}{t} \int_{1/2 \leq |\xi| \leq 1} \left| \frac{\xi}{t} \right|^{-n(1-\rho)-2\kappa\rho} t^{-2\kappa} \omega\left(\left| \frac{\xi}{t} \right|^{-1} \right)^2 d\xi$$

$$\leq C^2 N^2 \int_0^{1/2} \omega(2t)^2 \frac{dt}{t} = C^2 N^2 M_2^2.$$

Therefore we have

$$\|IIu\|_{L^2}^2 \leq C^2 N^2 M_2^2 \int \left(\int_0^{1/2} t^{-m} \frac{dt}{t} \int_{|z| \geq t^{\rho-1}} |z|^{-2\kappa} |G_t(x-tz)|^2 dz \right) dx \tag{2.8}$$

$$= C^2 N^2 M_2^2 \int_0^{1/2} t^{-m} \left(\int_{|z| \geq t^{\rho-1}} |z|^{-2\kappa} \left(\int |G_t(x)|^2 dx \right) dz \right) \frac{dt}{t}$$

$$\leq C^2 N^2 M_2^2 \int_0^\infty \left(\int |f(t|\xi|) \hat{u}(\xi)|^2 d\xi \right) \frac{dt}{t}$$

$$= C^2 N^2 M_2^2 \|u\|_{L^2}^2.$$

From (2.6), (2.7) and (2.8) we obtain the estimate (2.5).

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Remark. (i) We note that when $\rho=1$ the estimate (2.5) has already been proved in [8].

(ii) If we replace the condition (2.4) by

$$\int_0^1 \frac{\omega(t)}{t} dt = M_1 < \infty, \tag{2.4'}$$

then we can prove the L^p-boundedness of the operator for $2 \leq p \leq \infty$ (see [9] and [10]). We can see easily that the condition (2.4)' is stronger than (2.4).

§3. An example of pseudo-differential operators unbounded in $L^2(\mathbf{R}^n)$.

Let l^∞ denote the set of bounded sequences $\{a_j\}_{j=1}^\infty$ and l^2 denote the set of sequences $\{a_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty |a_j|^2 < \infty$. We use these in the proof of the following

Theorem 3.1. *Let $0 \leq \rho < 1$ and let $\omega(t)$ be a non-negative and non-decreasing function on $\mathbf{R}^1$ which stisfies*

$$\int_0^1 \frac{\omega(t)^2}{t} dt = \infty. \tag{3.1}$$

Then we can construct a symbol $p(x, \xi)$, which satisfies

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{-n(1-\rho)/2-\rho|\alpha|+|\beta|} \omega(\langle \xi \rangle^{-1}) \tag{3.2}$$

for any α and β, so that $p(X, D_x)$ is not bounded in $L^2(\mathbf{R}^n)$.

Proof. We choose a sequence $\{\tilde{\eta}_{j,k}\}_{j=1,k=1}^{\infty, j^n-1}$ which has the following properties:

(3.3) $$|\tilde{\eta}_{j,k}|=j,$$

(3.4) $$|\tilde{\eta}_{j,k}-\tilde{\eta}_{j,k'}|>c_n \qquad \text{if } k\neq k',$$

where c_n is a constant independent of j with $0<c_n\leq 1$.

We take a $C_0^\infty(\mathbf{R}^n)$ function $\chi(\xi)$ with $\chi(\xi)=1$ for $|\xi|\leq 1/4$ and $\chi(\xi)=0$ for $|\xi|\geq 1/2$. We put $\eta_{j,k}=|\tilde{\eta}_{j,k}|^{\frac{\rho}{1-\rho}}\tilde{\eta}_{j,k}$, then we note that

(3.5) $$|\eta_{j,k}|=j^{\frac{1}{1-\rho}}$$

We define $p(x, \xi)$ by

(3.6) $$p(x, \xi)=\sum_{j=1,}^{\infty}\sum_{k=1}^{j^{n-1}}a_j|\eta_{j,k}|^{-\frac{n}{2}(1-\rho)}e^{-i\eta_{j,k}\cdot x}\chi((c_n|\eta_{j,k}|^\rho)^{-1}(\xi-\eta_{j,k})),$$

where $\{a_j\}_{j=1}^\infty$ is in l^∞. We can see that

$\{\chi((c_n|\eta_{j,k}|^\rho)^{-1}(\xi-\eta_{j,k}))\}_{j,k}$ have disjoint supports and that

(3.7) $$\frac{1}{2}|\eta_{j,k}|\leq|\xi|\leq\frac{3}{2}|\eta_{j,k}|$$

for any ξ in the support of $\chi((c_n|\eta_{j,k}|^\rho)^{-1}(\xi-\eta_{j,k}))$.
Hence $p(x, \xi)$ is well-defined as a C^∞-function on $\mathbf{R}_x^n\times\mathbf{R}_\xi^n$.
Moreover we have

(3.8) $$|\partial_\xi^\alpha\partial_x^\beta p(x, \xi)|=\sum_{j=1,}^{\infty}\sum_{k=1}^{j^{n-1}}a_j|\eta_{j,k}|^{-\frac{n}{2}(1-\rho)}(-i\eta_{j,k})^\beta(c_n|\eta_{j,k}|^\rho)^{-|\alpha|}$$
$$\times e^{-i\eta_{j,k}\cdot x}(\partial^\alpha\chi)((c_n|\eta_{j,k}|^\rho)^{-1}(\xi-\eta_{j,k})).$$

Hence we have

(3.9) $$|\partial_\xi^\alpha\partial_x^\beta p(x, \xi)|\leq\sum_{j=1,}^{\infty}\sum_{k=1}^{j^{n-1}}|a_j||\eta_{j,k}|^{-\frac{n}{2}(1-\rho)+|\beta|-\rho|\alpha|}$$
$$\times c_n^{-|\alpha|}|(\partial^\alpha\chi)((c_n|\eta_{j,k}|^\rho)^{-1}(\xi-\eta_{j,k}))|.$$

Here we set $t_j=(1+4j^{\frac{2}{1-\rho}})^{-1/2}$ and $a_j=\omega(t_j)$ $j=1, 2, \cdots\cdots$. Then we can see by (3.5) and (3.7) that $p(x, \xi)$ satisfies (3.2). We note that $\{a_j\}_{j=1}^\infty$ is bounded but $\{j^{-1/2}a_j\}_{j=1}^\infty$ is not in l^2.

Next we take $\varphi(x)$ in S such that $\varphi(x)\neq 0$ and the support of $\hat{\varphi}(\xi)$ is contained in $\{\xi;\ |\xi|\leq c_n/4\}$, and we define $u_m(x)$ in S by

(3.10) $$\hat{u}_m(\xi)=\sum_{j=1,}^{m}\sum_{k=1}^{j^{n-1}}b_j\hat{\varphi}(\xi-\eta_{j,k}), \qquad m=1, 2, \cdots\cdots,$$

where $\{b_j\}_{j=1}^\infty$ is a sequence in l^∞. We note also that $\{\hat{\varphi}(\xi-\eta_{j,k})\}$ have disjoint supports. Therefore we have

(3.11) $$\|u_m\|_{L^2}^2=(2\pi)^{-n}\sum_{j=1,}^{m}\sum_{k=1}^{j^{n-1}}|b_j|^2\int|\hat{\varphi}(\xi-\eta_{j,k})|^2d\xi$$

$$=\Big(\sum_{j=1}^{m} j^{n-1}|b_j|^2\Big)\|\varphi\|_{L^2}^2.$$

On the other hand, by the definition, we have

$$p(X, D_x)u_m(x)=\sum_{j=1,k=1}^{m}\sum^{j^{n-1}} b_j(2\pi)^{-n}\int e^{ix\cdot\xi}p(x, \xi)\hat{\varphi}(\xi-\eta_{j,k})d\xi \tag{3.12}$$

$$=\sum_{j=1,k=1}^{m}\sum^{j^{n-1}} b_j\sum_{j'=1,k'=1}^{m}\sum^{j'^{n-1}} a_{j'}(2\pi)^{-n}\int|\eta_{j',k'}|^{-n(1-\rho)/2}$$

$$\times e^{i(\xi-\eta_{j',k'})\cdot x}\chi((c_n|\eta_{j',k'}|^{\rho})^{-1}(\xi-\eta_{j',k'}))\hat{\varphi}(\xi-\eta_{j,k})d\xi$$

$$=\sum_{j=1,k=1}^{m}\sum^{j^{n-1}} a_jb_j|\eta_{j,k}|^{-n(1-\rho)/2}(2\pi)^{-n}\int e^{i(\xi-\eta_{j,k})\cdot x}\hat{\varphi}(\xi-\eta_{j,k})d\xi$$

$$=\Big(\sum_{j=1}^{m} j^{n/2-1}a_jb_j\Big)\varphi(x).$$

Here we assume that $p(X, D_x)$ is L^2-bounded, that is,

$$\|p(X, D_x)u\|_{L^2}\leq C_0\|u\|_{L^2} \qquad \text{for any } u \text{ in } S.$$

Then, equalities (3.11) and (3.12) imply

$$|\sum_{j=1}^{m} j^{n/2-1}a_jb_j|^2\leq C_0^2\Big(\sum_{j=1}^{m}|b_j|^2j^{n-1}\Big) \qquad \text{for any } m. \tag{3.13}$$

Now by taking $b_j=j^{-n/2}a_j$, we obtain

$$\sum_{j=1}^{m} j^{-1}|a_j|^2\leq C_0^2 \qquad \text{for any } m.$$

This contradicts that $\{j^{-1/2}a_j\}_{j=1}^{\infty}$ does not belong to l^2.
Therefore the operator $p(X, D_x)$ is not $L^2(\mathbf{R}^n)$-bounded.

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