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Some remarks on the positivity of fundamental solutions for certain parabolic equations with constant coefficients

By

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§1. Introduction and results.

It is well known that the fundamental solution $(4\pi t)^{-n/2}\exp(-|x|^2/4t)$ of the heat operator $\frac{\partial}{\partial t}-\triangle$ is nonnegative. In this note we will show that this property does never hold for the parabolic equations of higher order with respect to the space variables. Here we will restrict ourselves to the case of single equations with constant coefficients. The general case of parabolic systems with variable coefficients will be treated in the other paper by the latter author.

Let us consider the Cauchy problem

$$\begin{cases} (1.1) \quad Lu\equiv\frac{\partial}{\partial t}u(t, x)-\sum_{|\alpha|\leqq 2m}a_\alpha\left(\frac{\partial}{\partial x}\right)^\alpha u(t, x)=0 \qquad 0<t\leqq T(<\infty),\ x\in \boldsymbol{R}^n, \\ (1.2) \quad u(0, x)=u_0(x). \end{cases}$$

We assume that:

(*i*) *m is a positive integer and a_α are real constants,*

(*ii*) *L is parabolic, i.e. there exists a positive constant δ such that for any $\xi\in\mathbf{R}^n$ we have*

$$A_{2m}(i\xi)\equiv\sum_{|\alpha|=2m}a_\alpha(i\xi)^\alpha\leqq-\delta|\xi|^{2m}.$$

We say that $E(t, x)$ is a fundamental solution of L if it satisfies

$$LE(t, x)=0 \qquad 0<t\leqq T,\ x\in\mathbf{R}^n,$$

and $\quad \lim_{t\to+0}E(t, x)=\delta(x),$

where $\delta(x)$ is Dirac's delta. Since the coefficients of L are constant, one of the fundamental solutions is explicitly given by means of the Fourier transform:

$$E_0(t, x)=\int\exp\{ix\xi+t\{A_{2m}(i\xi)+A'(i\xi)\}\}d\xi,$$

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where $\quad A'(i\xi)=\sum_{|\alpha|<2m} a_\alpha(i\xi)^\alpha,$

$d\!\!\!{}^-\xi=(2\pi)^{-n}d\xi$ and the integration is extended over the whole space $\mathbf{R}^n$.

There are many works on the uniqueness of the solution of the problem (1.1)–(1.2). Here we only mention the following theorems.

Theorem 1.1 (Täcklind. See [1] Chap. 3 Sec. 2.). *Let $u(t, x)$ be the solution of* (1.1) *with initial data $u_0(x)\equiv 0$, and satisfy the inequality*

$$\sup_{0\leqq t\leqq T} |u(t, x)|\leqq \exp(|x|h(|x|))$$

where $h(r)$ is a continuous function such that the integral $\int_0^\infty h(r)^{1-2m}dr$ diverges. Then $u(t, x)$ is identically equal to zero.

It follows from this theorem that for each initial data $u_0(x)$ in $C_0^\infty(\mathbf{R}^n)$, there exists a unique solution of (1.1)–(1.2) which is bounded in $[0, T]\times\mathbf{R}^n$.

As for the distribution solution, we can easily show the following theorem.

Theorem 1.2 (See [2] Chap. 5 Sec. 2). *For each $u_0(x)$ in $\mathcal{S}'$, there exists a unique solution $u(t, x)$ of* (1.1)–(1.2) *which belongs to $C^1([0, T]; \mathcal{S}')$. Here $\mathcal{S}'$ is the space of temperate distributions.*

In each case, the unique solution is given by

$$u(t, x)=E_0(t, x) * u(x)\equiv\int E_0(t, x-y)u_0(y)dy.$$

So, $E_0(t, x)$ is the unique fundamental solution which gives the unique solution in each space mentioned above. And it is easily shown that $E_0(t, x)$ is real-valued, because the coefficients of L are real constants.

Now our main result is the following.

Theorem 1.3. *If $m\geqq 2$, then $E_0(t, x)$ is not nonnegative. More precisely, there exist positive constants t_0 and c_0 such that for any t in $(0, t_0)$ there exists some $x\in\mathbf{R}^n$ which satisfies*

$$E_0(t, x)\leqq -c_0 t^{-n/2m}.$$

§2. Proof of Theorem 1.3.

In order to prove Theorem 1.3, let us introduce an auxiliary function

$$F(t, x)=t^{n/2m}E_0(t, t^{1/2m}x), \qquad 0<t\leqq T,\ x\in\mathbf{R}^n.$$

Then we have following propositions on this $F(t, x)$.

Proposition 2.1. *As t tends to 0, $F(t, x)$ converges uniformly to the function*

$$F_0(x)=\exp[ix\eta+A_{2m}(i\eta)]d\!\!\!{}^-\eta.$$

Proposition 2.2. $F_0(x)$ *satisfies*

$$\int F_0(x)dx=1,$$

and
$$\int x^\alpha F_0(x)dx=0 \textit{ if } 0<|\alpha|<2m.$$

Proof of proposition 2.1. Introducing new variables $\eta=t^{1/2m}\xi$, we have

$$F(t, x)=t^{n/2m}\int \exp[it^{1/2m}x\xi+t\{A_{2m}(i\xi)+A'(i\xi)\}]d\xi$$

$$=\int \exp[ix\eta+A_{2m}(i\eta)+tA'(it^{-1/2m}\eta)]d\eta.$$

Now, there exists some constant M such that

$$|tA'(it^{-1/2m}\eta)|\leqq Mt^{1/2m}(|\eta|^{2m-1}+1) \text{ if } 0<t<1.$$

So, using the estimate $|e^z-1|\leqq|z|e^{|z|}$ with $z\in\mathbf{C}$, we have

$$|F(t, x)-F_0(x)|\leqq\int|\exp(A_{2m}(i\eta))||\exp\{tA'(it^{-1/2m}\eta)\}-1|d\eta$$

$$\leqq Mt^{1/2m}\int(|\eta|^{2m-1}+1)\exp[-\delta|\eta|^{2m}+M\{|\eta|^{2m-1}+1\}]d\eta.$$

$$=M't^{1/2m},$$

where $0<t<1$ and M' does not depend on (t, x). Thus, $F(t, x)$ converges uniformly to $F_0(x)$, as t tends to 0.

Proof of proposition 2.2. By the formula of the Fourier transformation, we have

$$\int x^\alpha F_0(x)dx=\left(i\frac{\partial}{\partial\xi}\right)^\alpha \exp\{A_{2m}(i\xi)\}|_{\xi=0}. \tag{2.1}$$

On the other hand, we can write as

$$\left(\frac{\partial}{\partial\xi}\right)^\alpha \exp\{A_{2m}(i\xi)\}=p_\alpha(\xi)\exp\{A_{2m}(i\xi)\},$$

where $p_\alpha(\xi)$ is a polynomial in ξ of degree $(2m-1)|\alpha|$. It is easily shown by induction that $p_\alpha(\xi)$ is the linear combination of monomials of degree $2mj-|\alpha|$ with $j=1, 2,\ldots, |\alpha|$. So it does not contain the constant term unless $|\alpha|=2km$, $k=0, 1, 2,\cdots$. That is, the right hand side of (2.1) is equal to 0 if $0<|\alpha|<2m$, and equal to 1 if $|\alpha|=0$.

Proof of Theorem 1.3. From the Proposition 2.2, it follows that

$$\int F_0(x)dx=1$$

and
$$\int|x|^2F_0(x)dx=0$$

if $m\geqq2$. So $F_0(x)$ must change its sign, i.e. there exist two points $x^{(0)}$ and $x^{(1)}$ in $\mathbf{R}^n$ and a positive constant c_0 such that

$$F_0(x^{(0)}) \leqq -2c_0 \text{ and } F_0(x^{(1)}) \geqq 2c_0.$$

Since $F_0(x)$ is continuous and $F(t, x)$ converges to $F_0(x)$ uniformly, there exist some $\varepsilon>0$ and $t_0>0$ such that

$$F_0(t, x) \leqq -c_0 \text{ for } 0<t<t_0, \ |x-x^{(0)}|<\varepsilon.$$

Then

$$E_0(t, x)=t^{-n/2m}F(t, t^{-1/2m}x) \leqq -c_0 t^{-n/2m}$$

if $0<t<t_0$, $|x-t^{1/2m}x^{(0)}|<t^{1/2m}\varepsilon$. Thus our theorem is proved.

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