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Supplements to my previous papers; a refinement and applications

By

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Introduction.

This article is supplements to my previous papers [2] and [3]. In §1, we give a refinement (Theorem 1) of a fundamental variational formula ([3, Theorem 1]), which leads us more precise formulas than those in [3, Theorems 2–4] and also gives another proof of [2, Theorem 2]. (See §2.) As direct applications of Theorem 1, we show in §3 a variational formula for the modified canonical injection, and in §4 formulas under variation by connecting boundary arcs.

§1. A refinement of the variational formula.

In this paper, we use the same notations as in [3], and show the following refinement of our previous formula in [3, Theorem 1].

Theorem 1. *Under the same notations and assumptions as in* [3, Theorem 1] (*also cf. Remark at the end of* [3, §1]), *it holds that*

$$\iint_{R_0'}\omega_{t,s}=t\cdot\iint_{R_0'}\varphi_{0,0}\cdot\mu\wedge{}^*\psi+2\pi\cdot\sum_{j=1}^{n}\eta_j\cdot s_j^2(a_{j,1}(0)\cdot b_{j,2}(0)+a_{j,2}(0)\cdot b_{j,1}(0))$$
$$+o(t+\sum_{j=1}^{n}s_j^2),$$

where $\varphi_{0,0}=a_{j,k}(z_{j,k})dz_{j,k}$ *and* $\psi=b_{j,k}(z_{j,k})dz_{j,k}$ *on* $\bar{U}_{j,k}=\{|z_{j,k}|<1\}$ *for every* j *and* k.

Proof. In the proof of [3, Theorem 1], we have shown that

$$\iint_{R_0'}\omega_{t,s}=t\cdot\iint_{K}\varphi_{0,0}\cdot\mu\wedge{}^*\psi$$
$$+\sum_{j=1}^{n}\iint_{V_j}a_{t,s}(F_{t,s}(z))\cdot(-s_j/2)\cdot\left(\frac{z}{|z|}\right)^2\cdot(1/|z|)d\bar{z}\wedge{}^*\psi+o(t),$$

where $V_j=\{0<|z_{j,1}|<1/2\}\cup\{0<|z_{j,2}|<1/2\}$ for every j.

Set $a_{t,s}(z_{j,k,t,s})=\sum_{n=-\infty}^{\infty}a_{n,j,k,t,s}\cdot z_{j,k,t,s}{}^n$ on $\{s_j<|z_{j,k,t,s}|<1\}$ and $b_{j,k}(z_{j,k})=\sum_{n=0}^{\infty}b_{n,j,k}\cdot z_{j,k}{}^n$ on $\bar{U}_{j,k}$ for every j and k. Fix j and k, then

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$$I_{j,k}=\iint_{\{0<|z_{j,k}|<1/2\}}a_{t,s}(F_{t,s}(z))(-s_j/2)\left(\frac{z}{|z|}\right)^2(1/|z|)d\bar{z}\wedge{}^*\phi$$

$$=\int_0^{1/2}\int_0^{2\pi}a_{t,s}(F_{t,s}(re^{i\theta}))\cdot(-s_j/2)(e^{i2\theta}/r)(2b_{j,k}(z))rdrd\theta$$

$$=-s_j\cdot\int_0^{1/2}\int_0^{2\pi}\sum_{n=-\infty}^{\infty}a_{n,j,k,t,s}\cdot(F_{t,s}(r)\cdot e^{i\theta})^n\cdot\sum_{m=0}^{\infty}b_{m,j,k}\cdot r^m e^{im\theta}\cdot e^{i2\theta}drd\theta$$

$$=-2\pi s_j\cdot\sum_{n=0}^{\infty}b_{n,j,k}\cdot a_{-n-2,j,k,t,s}\cdot\int_0^{1/2}(r^n/F_{t,s}(r)^{n+2})dr.$$

Here by [3, Lemma 2-i)], $a_{t,s}(z)$ is uniformly bounded on $\{s_j<|z_{j,k,t,s}|<1/2\}$, we can find, by Cauchy's integral formula, a constant M such that $|a_{n,j,k,t,s}|\leq M/s_j^n$ for every negative n. And since $r<F_{t,s}(r)$ and $\int_0^{1/2}(s_j/F_{t,s}(r)^2)dr=1$, we have

$$I_{j,k}=-2\pi b_{0,j,k}\cdot a_{-2,j,k,t,s}+O(\sum_{n=1}^{\infty}s_j^{n+2})$$

$$=-2\pi b_{j,k}(0)\cdot a_{-2,j,k,t,s}+o(s_j^2).$$

Now since $z_{j,1,t,s}=\eta_j\cdot s_j^2/z_{j,2,t,s}$ on $\{s_j^2<|z_{j,k,t,s}|<1\}$, where $a_{t,s}(z)$ is well-defined and holomorphic, we can see that

$$a_{-2,j,k,t,s}=-\eta_j\cdot s_j^2\cdot a_{0,j,3-k,t,s}.$$

And since $a_{0,j,k,t,s}$ converges to $a_{j,k}(0)$ by [3, Lemma 2-ii)], we conclude that

$$I_{j,k}=2\pi\eta_j\cdot s_j^2\cdot b_{j,k}(0)\cdot a_{j,3-k}(0)+o(s_j^2).$$

Summing these $I_{j,k}$ up, we have the desired formula. q.e.d.

§2. Remarks on the formulas of Schiffer-Spencer's type.

First, using Theorem 1 instead of [3, Theorem 1] in the proofs of [3, Theorems 2–4], we can see the following

Remark 1. In [3, Theorems 2, 3 and 4], respectively, we can replace $o(\|(t, s)\|)$ by the following more precise quantities

(Th2) $2\pi\cdot\mathrm{Re}[\sum_{h\in X}\eta_h s_h^2(a_{d,h,1}(0)\cdot a_{d',h,2}(0)+a_{d,h,2}(0)\cdot a_{d',h,1}(0))]+o(\|(t, s)\|')$,

(Th3) $2\pi\cdot\mathrm{Im}[\sum_{h\in X}\eta_h s_h^2(b_{q,h,1}(0)\cdot a_{d,h,2}(0)+b_{q,h,2}(0)\cdot a_{d,h,1}(0))]+o(\|(t, s)\|')$, and

(Th4) $\mathrm{Re}[\sum_{h\in X}\eta_h s_h^2(b_{q,h,1}(0)\cdot b_{q',h,2}(0)+b_{q,h,2}(0)\cdot b_{q',h,1}(0))]+o(\|(t, s)\|')$.

Here we set $\theta(d^{(\prime)}, R_0)=a_{d^{(\prime)},h,k}(z_{h,k})dz_{h,k}$ and $\phi(q^{(\prime)}, R_0)=b_{q^{(\prime)},h,k}(z_{h,k})dz_{h,k}$ on $\bar{U}_{h,k}$ for every h and k,

$$X=\{h\in[1, n]: C_h \text{ is essentially trivial}\}$$

(which we have assumed to be coincident with $\{m+1, ..., n\}$ in [3, §2]), and finally

$$\|(t, s)\|' = t + \sum_{j=1}^{H} \frac{1}{\log(1/s(j))} + \sum_{h \in X} s_h^2.$$

Now note that $\phi(q, R_0)$ in [3] is identical with $-i \cdot \phi(q; R_0)$ in [2], and that the classical Schiffer-Spencer's variation in [2] is a special case of pinching deformation, where $\mu_t \equiv 0$, $n=1$, and U and the parameter $s=s_1$ are chosen in a special manner. In particular, we see that

$$\|\sigma(C_j, R_{t,s})\|^2_{R'_{t,s}} = \pi/\log(1/s).$$

Using this fact instead of [3, Theorem 5], we can realize the following

Remark 2. The formulas in [2, Theorem 2] can be shown by Theorem 1 and the same argument as in the proofs of [3, Theorems 2–4].

§3. A formula for the modified canonical injection.

In this section, we use also the notations in [1]. Fix (t, s) arbitrarily, and let $H_{t,s}(\omega)$ be the projection of $H_{f_{t,s}}(\omega)$ to the orthogonal complement $\Gamma_{t,s}$ of $\Gamma_N(R_{t,s}, R_0)$ in $\Gamma_h(R_{t,s}, R_0)$, and set

$$A_{t,s}(\omega) = H_{t,s}(\omega) + i^* H_{t,s}(\omega)$$

for every $\omega \in \Gamma_h(R_0)$. Also denote by $\Gamma_a(R_{t,s}, R_0)$ the *real* Hilbert space consisting of all square integrable holomorphic differentials φ on $R_{t,s}$ such that $\int_c \varphi = 0$ for every $c \in L(R_{t,s}, R_0)$. Then we have the following

Proposition. *The linear map $A_{t,s}$ is a (real) isomorphism between $\Gamma_h(R_0)$ and $\Gamma_a(R_{t,s}, R_0)$ for every (t, s). Moreover, it holds that*

$$\int_d \mathrm{Re}\ A_{t,s}(\omega) = \int_d \omega$$

for every $\omega \in \Gamma_h(R_0)$ and 1-cycle d on R'_0.

Proof. It is clear that $\Gamma_{t,s}$ is isomorphic to $\Gamma_h(R_{t,s}, R_0)/\Gamma_N(R_{t,s}, R_0)$. Hence by [1, Theorem 1-i)], $H_{t,s}$ is an isomorphism between $\Gamma_{t,s}$ and $\Gamma_h(R_0)$.

Next $\alpha \in \Gamma_{t,s}$ if and only if $\alpha \in \Gamma_h(R_{t,s}, R_0)$ and $-({}^*\alpha, \sigma(c))_{R'_{t,s}} = (\alpha, {}^*\sigma(c))_{R'_{t,s}} = 0$ for every $c \in L(R_{t,s}, R_0)$ by definition, which is equivalent to the condition that $\int_c {}^*\alpha = 0$ for every $c \in L(R_{t,s}, R_0)$. Hence $\alpha \in \Gamma_{t,s}$ if and only if $\alpha + i^*\alpha \in \Gamma_a(R_{t,s}, R_0)$, which shows the first assertion.

Finally, since $\int_d {}^*\sigma(c) = 0$ for every $c \in L(R_{t,s}, R_0)$ and 1-cycle d on R'_0, we have the second assertion by [1, Lemma 5]. q.e.d.

Now by Theorem 1, we can show the following

Theorem 2. *For every 1-cycle d on R'_0 and $\omega \in \Gamma_h(R_0)$, it holds that*

$$\int_d \mathrm{Im}\ A_{t,s}(\omega) - \int_d \mathrm{Im}\ A_{0,0}(\omega) = t \cdot \mathrm{Im} \iint_{R'} A_{0,0}(\omega) \cdot \mu \wedge {}^*\theta(d, R_0)$$

$$+\ 2\pi\cdot\mathrm{Im}[\sum_{j=1}^{n}\eta_j s_j^2(a_{j,1}(0)\cdot a_{d,j,2}(0)+a_{j,2}(0)\cdot a_{d,j,1}(0))]\ +\ o(t+\sum_{j=1}^{n}s_j^2),$$

where $A_{0,0}(\omega)=\omega+i^*\omega=a_{j,k}(z_{j,k})dz_{j,k}$ *on* $\bar{U}_{j,k}$ *for every* j *and* k.

Proof. Fix a 1-cycle d and $\omega\in\Gamma_h(R_0)$. We set $\theta_{t,s}={}^*H_{f_{t,s}}(\omega)-i\cdot H_{f_{t,s}}(\omega)$ and $\varphi_{t,s}=-i\cdot A_{t,s}(\omega)$. Then, letting $(a_{j;t,s})_{j=1}^{H}$ be the unique solution (cf. [3, §4]) of the equations

$$\int_{C_j}{}^*H_{f_{t,s}}(\omega)=\sum_{k=1}^{H}a_{k;t,s}\cdot\int_{C_j}\phi(C_k,\ R_{t,s})\qquad(j=1,\ \dots,\ H),$$

it holds that

$$\varphi_{t,s}=\theta_{t,s}-\sum_{j=1}^{H}a_{j;t,s}\cdot\phi(C_j,\ R_{t,s}).$$

Here recall that $\theta_{t,s}$ converges to $\varphi_{0,0}$ strongly metrically as $|(t,\ s)|$ tends to 0 (cf. the proof of [1, Theorem 4]). In particular, every $\int_{C_j}{}^*H_{f_{t,s}}(\omega)$ converges to $\int_{C_j}{}^*\omega=0$, hence so does every $a_{j;t,s}$ as $|(t,\ s)|$ tends to 0. Hence by [3, Theorem 6], $\varphi_{t,s}$ converges to $\varphi_{0,0}$ strongly metrically as $|(t,\ s)|$ tends to 0. And since

$$\|A_{t,s}(\omega)\|_{R'_{t,s}}\leqq 2\|H_{t,s}(\omega)\|_{R'_{t,s}}\leqq 2\|H_{f_{t,s}}(\omega)\|_{R'_{t,s}},$$

$\{\|A_{t,s}(\omega)\|_{R'_{t,s}}\}$ is bounded by [1, Lemma 3]. Thus we conclude that $\varphi_{t,s}$ satisfies the conditions 1)–3) in [3, Theorem 1].

Now set $\psi=\theta(d,\ R_0)$, then it is clear that ψ satisfies the conditions A) and B) in [3, Theorem 1]. And recalling that

$$I_{f_{t,s}}(\mathrm{Im}\ \varphi_{t,s})=-I_{f_{t,s}}(H_{f_{t,s}}(\omega))=-\omega=\mathrm{Im}\ \varphi_{0,0},$$

by [1, Lemmas 6 and 7], we can show similarly as in [3, §4] that

$$\int_d\varphi_{t,s}-\int_d\varphi_{0,0}=\int_d\mathrm{Im}\ A_{t,s}(\omega)-\int_d\mathrm{Im}\ A_{0,0}(\omega)=\mathrm{Re}\iint_{R'_0}\varphi_{t,s}\circ f_{t,s}^{-1}\wedge{}^*\psi.$$

Hence by Theorem 1, we conclude the desired formula. q.e.d.

§4. Variation by connecting boundary arcs.

Let $\{S_i\}_{i=1}^{I}$ be a finite set of Riemann surfaces with (not necessarily closed) boundary, and $P=\{p_{j,k}\}_{j=1,k=1}^{n,\ 2}$ be a finite set of mutually disjoint boundary points of them such that $S_i\cap P\neq\phi$ for every i. Fix a neighborhood $W_{j,k}$ of $p_{j,k}$ and a local coordinate $z_{j,k}$ on $W_{j,k}$ such that $z_{j,k}(W_{j,k})=W=\{|z|<1,\ \mathrm{Im}\ z\geqq 0\}$ and $z_{j,k}(p_{j,k})=0$ for every j and k, where we also assume that $\overline{W}_{j,k}$ are mutually disjoint.

Let μ_t be a Beltrami differential on $S_{0,0}-W$, where $W=\bigcup_{j,k}W_{j,k}$ and $S_{0,0}$ is the interior of $\bigcup_{i=1}^{I}S_i$, satisfying the conditions a) and b) in [3, §1] with $R_0=S_{0,0}$

and $U=W$. Let f_t be the quasiconformal mapping of $S_{0,0}$ onto another $S_{t,0}$ with the complex dilatation μ_t. Set $S_{t,s}$ be the union of Riemann surfaces with boundary obtained from $S_{t,0}$ by deleting $f_t \circ z_{j,k}^{-1}(\{|z|<s_j;\ \mathrm{Im}\ z \geqq 0\})$ from $S_{t,0}$ and identifying the newly resulting borders by the mapping $z_{j,2}^{-1}(-s_j^2/z_{j,1}(p))$ for every (t, s) with $t \geqq 0$ and $s_j \in [0, 1/2)$. Here we assume that $S_{t,s}$ is connected when every $s_j > 0$.

Now set $B_{j,k} = z_{j,k}^{-1}(\{|z|<1;\ \mathrm{Im}\ z=0\})$ for every j and k, and let $\overline{R_0'}$ be the double of $S_{0,0}$ with respect to $\bigcup_{j,k} B_{j,k}$, and R_0 be the Riemann surface with nodes resulting from $\overline{R_0'}$ by identifying $p_{j,1}$ with $p_{j,2}$ for every j. Then we see that above variation of $S_{0,0}$ by connecting boundary arcs is nothing but pinching deformation of R_0 in [3] (, where $U_{j,k}$ is the double of $W_{j,k}$ with respect to $B_{j,k} - \{p_{j,k}\}$ for every j and k, and μ_t and f_t are the natural symmetric extension of the above μ_t and f_t). Here, for every (t, s), $\theta(d, S_{t,s})$ and $\phi(q, S_{t,s})$ can be extended to meromorphic differentials on $R_{t,s}$, and it holds that

$$\theta(d, S_{t,s}) = \theta(d, R_{t,s}) + \theta(I_{t,s}(d), R_{t,s}), \text{ and}$$

$$\phi(q, S_{t,s}) = \phi(q, R_{t,s}) - \phi(I_{t,s}(q), R_{t,s})$$

for every 1-cycle d and point q on $S_{0,0}$, where $I_{t,s}$ is the canonical anti-conformal involution of $R_{t,s}$ onto itself fixing the border of $S_{t,s}$ pointwise. Hence we can see the following

Theorem 3. i) *Let d and d' be 1-cycles on $S_{0,0}$, then it holds that*

$$\int_{d'} \sigma(d, S_{t,s}) - \int_{d'} \sigma(d, S_{0,0}) = 2t \cdot \mathrm{Re} \iint_{S_{0,0}} \theta(d, S_{0,0}) \cdot \mu \wedge {}^*\theta(d', S_{0,0})$$
$$-2\pi \cdot \mathrm{Re}\Big[\sum_{j=1}^{n} s_j^2 (a_{d,j,1}(0) \cdot a_{d',j,2}(0) + a_{d,j,2}(0) \cdot a_{d',j,1}(0))\Big] + o\Big(t + \sum_{j=1}^{n} s_j^2\Big).$$

ii) *Let q be a point on $S_{0,0} - W$. Let d be a 1-cycle on $S_{0,0} - \{q\}$ and assume that $\mu_t \equiv 0$ on some neighborhood of q on $S_{0,0}$. Then it holds that*

$$\int_d {}^*dg(\cdot\,, f_{t,s}^{-1}(q)) - \int_d {}^*dg(\cdot\,, q) = 2t \cdot \mathrm{Im} \iint_{S_{0,0}} \phi(q, S_{0,0}) \cdot \mu \wedge {}^*\theta(d, S_{0,0})$$
$$-2\pi \cdot \mathrm{Im}\Big[\sum_{j=1}^{n} s_j^2 (b_{q,j,1}(0) \cdot a_{d,j,2}(0) + b_{q,j,2}(0) \cdot a_{d,j,1}(0))\Big] + o\Big(t + \sum_{j=1}^{n} s_j^2\Big).$$

iii) *Let q and q' be distinct points on $S_{0,0} - W$, and assume that $\mu_t \equiv 0$ in some neighborhood of $\{q, q'\}$ on $S_{0,0}$. Then it holds that*

$$g(f_{t,s}^{-1}(q'), f_{t,s}^{-1}(q)) - g(q', q) = (t/\pi) \cdot \mathrm{Re} \iint_{S_{0,0}} \phi(q, S_{0,0}) \cdot \mu \wedge {}^*\phi(q', S_{0,0})$$
$$-\mathrm{Re}\Big[\sum_{j=1}^{n} s_j^2 (b_{q,j,1}(0) \cdot b_{q',j,2}(0) + b_{q,j,2}(0) \cdot b_{q',j,1}(0))\Big] + o\Big(t + \sum_{j=1}^{n} s_j^2\Big),$$

where $\theta(d^{(\prime)}, S_{0,0}) = a_{d^{(\prime)},j,k}(z_{j,k})dz_{j,k}$ and $\phi(q^{(\prime)}, S_{0,0}) = b_{q^{(\prime)},j,k}(z_{j,k})dz_{j,k}$ on $\bar{U}_{j,k}$ for every j an d k.

Proof. Since $I(C_j) = -C_j$ for every j and $\theta(d, R_{t,s}) \circ I = \overline{\theta(I(d), R_{t,s})}$ for every

(t, s) with $I=I_{t,s}$, we have

$$\int_{C_j}\theta(d, S_{t,s})=\int_{C_j}\sigma(d, R_{t,s})+\sigma(d, R_{t,s})\circ I=\int_{C_j+I(C_j)}\sigma(d, R_{t,s})=0 \qquad \text{for every } j.$$

Similarly, since $\phi(q, R_{t,s})\circ I=\overline{\phi(I(q), R_{t,s})}$, we have $\int_{C_j}\phi(q, S_{t,s})=0$ for every j.

Hence Theorem 1 and the same arguments as in the proofs of [3, Theorems 2–4] gives the desired formulas. q.e.d.

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