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On the existence and uniqueness of diffusion processes with Wentzell's boundary conditions

By

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1. Introduction.

It is well-known that a diffusion process on a domain D in $\mathbf{R}^d$ (or more generally, a d-dimensional manifold) with smooth boundary ∂D is determined by a pair of analytical data (A, L) where A is a second order differential operator of elliptic type (possiblyde generate) and L is a Wentzell's boundary condition. The problem of constructing diffusion process for given (A, L) has been discussed by many authors: Cf. e. g. Sato–Ueno [6], Bony–Courrège–Priouret [3] and Taira [8] for analytical construction, Ikeda–Watanabe [5] and El Karoui [4] for construction by the method of stochastic differential equations (SDE's), and Stroock–Varadhan [7] and Anderson [1], [2] for construction by the method of martingale problems. Also, a direct construction of path functions was discussed in Watanabe [10] and Ikeda-Watanabe [5] by using the notion of Poisson point process of Brownian excursions. This method can cover more general cases than the method of SDE but the problem to show that it is actually an (A, L)-diffusion, more specifically, it is actually a unique solution of SDE, remains unanswered.

The purpose of the present paper is to answer this problem. For this, we start with any solution of SDE and then show that it coincides with the process constructed by the method of Poisson point process of Brownian excursions just mentioned. This shows that any solution of SDE is given as a well-determined functional of a Poisson point process of Brownian excursions and some auxiliary Brownian motions: Consequently, we can conclude that the solution of SDE is unique and hence (A, L)-diffusion is unique. Also, we can conclude at the same time that the process constructed by the method of Poisson point process of Brownian excursions is actually this unique (A, L)-diffusion process. In this way, we can show the unique existence of (A, L)-diffusions in the case when the coefficient of the reflection may degenerate on some part of the boundary: So far in the probabilistic construction by the methods of SDE and the martingale problem, this coefficient is usually assumed to be positive everywhere on the boundary.

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2. (A, L)-diffusions as solutions of stochastic differential equations (SDE's).

Let $D=(\mathbf{R}^d)^+=\{x=(x^1,\dots,x^d);x^d\geq 0\}$, $\mathring{D}=\{x\in D;x^d>0\}$ and $\partial D=\{x\in D;x^d=0\}$. Suppose we are given a second-order differential operator A on D:

$$Af(x)=\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(x)\frac{\partial^2 f}{\partial x^i\partial x^j}(x)+\sum_{i=1}^{d}b^i(x)\frac{\partial f}{\partial x^i}(x)$$

and Wentzell's boundary condition:

$$Lf(x)=\frac{1}{2}\sum_{i,j=1}^{d-1}\alpha^{ij}(x)\frac{\partial^2 f}{\partial x^i\partial x^j}(x)+\sum_{i=1}^{d-1}\beta^i(x)\frac{\partial f}{\partial x^i}(x)+\mu(x)\frac{\partial f}{\partial x^d}(x)-\rho(x)\cdot Af|_{\partial D}(x)$$

where $a^{ij}(x)$ and $b^i(x)$ $i,j=1,\dots,d$ are bounded Borel-measurable functions on D such that $a^{ij}(x)=a^{ji}(x)$, $\sum_{i,j=1}^{d}a^{ij}(x)\xi^i\xi^j\geq 0$ for all $\xi\in\mathbf{R}^d$ and $\alpha^{ij}(x)$, $\beta^i(x)$ $i,j=1,\dots,d-1$, $\mu(x)$ and $\rho(x)$ are bounded Borel-measurable functions on ∂D such that $\alpha^{ij}(x)=\alpha^{ji}(x)$, $\sum_{i,j=1}^{d-1}\alpha^{ij}(x)\xi^i\xi^j\geq 0$ for all $\xi\in\mathbf{R}^{d-1}$, $\mu(x)\geq 0$ and $\rho(x)\geq 0$.

Definition 2.1. By an (A, L)-*process* starting at $x\in D$, we mean a D-valued continuous $(\mathscr{F}_t)$-adapted process $X(t)$ defined on a probability space $(\Omega,\mathscr{F},P)$ with a filtration $(\mathscr{F}_t)$ such that $X(0)=x$ and, for every $f\in\mathbf{C}_b^2(D)$ (:=the class of twice continuously differentiable functions on D with bounded derivatives), it holds that

$$f(X(t))-f(x)=an\ (\mathscr{F}_t)\text{-}martingale+\int_0^t(Af)(X(s))ds+\int_0^t(Lf)(X(s))d\varphi(s)$$

where $\varphi(t)$ is an $(\mathscr{F}_t)$-adapted continuous non-decreasing process with $\varphi(0)=0$ such that

$$\int_0^t I_{\partial D}(X(s))d\varphi(s)=\varphi(t)\qquad\text{for all } t\geq 0,\ \text{a.s.}\tag{2.1}$$

and

$$\int_0^t I_{\partial D}(X(s))ds=\int_0^t\rho(X(s))d\varphi(s)\qquad\text{for all } t\geq 0,\ \text{a.s.}\tag{2.2}$$

Thus, an (A, L)-process $X(t)$ is always accompanied by an auxiliary increasing process $\varphi(t)$. (A, L)-processes can be given equivalently as solutions of the following stochastic differential equations (SDE's). To formulate SDE, we first choose arbitrarily but fix some bounded Borel-measurable functions $\sigma(x)=(\sigma_k^i(x))_{\substack{i=1,\dots,d\\k=1,\dots,r}}$ on D and $\tau(x)=(\tau_k^i(x))_{\substack{i=1,\dots,d-1\\k=1,\dots,q}}$ on ∂D such that

$$\sum_{k=1}^{r}\sigma_k^i(x)\sigma_k^j(x)=a^{ij}(x)\qquad i,j=1,\dots,d,\ x\in D\tag{2.3}$$

and

$$\sum_{k=1}^{q}\tau_k^i(x)\tau_k^j(x)=\alpha^{ij}(x)\qquad i,j=1,\dots,d-1,\ x\in\partial D.\tag{2.4}$$

We consider the following SDE to be satisfied by a D-valued continuous $(\mathscr{F}_t)$-semimartingale $X(t)=(X^1(t), \ldots, X^d(t))$ together with r-dimensional continuous $(\mathscr{F}_t)$-martingale $B(t)=(B^1(t), \ldots, B^r(t))$, q-dimensional continuous $(\mathscr{F}_t)$-martingale $M(t)=(M^1(t), \ldots, M^q(t))$ and $(\mathscr{F}_t)$-adapted continuous non-decreasing process $\varphi(t)$ with $\varphi(0)=0$ a.s.,

$$
(2.5) \qquad \begin{cases} dX^i(t)=\sum_{k=1}^{r}\sigma_k^i(X(t))I_{\mathring{D}}(X(t))dB^k(t)+b^i(X(t))I_{\mathring{D}}(X(t))dt \\ \qquad +\sum_{k=1}^{q}\tau_k^i(X(t))I_{\partial D}(X(t))dM^k(t)+\beta^i(X(t))I_{\partial D}(X(t))d\varphi(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad i=1, \ldots, d-1 \\ dX^d(t)=\sum_{k=1}^{r}\sigma_k^d(X(t))I_{\mathring{D}}(X(t))dB^k(t)+b^d(X(t))I_{\mathring{D}}(X(t))dt \\ \qquad +\mu(X(t))I_{\partial D}(X(t))d\varphi(t) \\ X(0)=x \end{cases}
$$

with subsidiary conditions: (2.1), (2.2) and

$$
(2.6) \qquad \langle B^k, B^l\rangle_t=\delta_{kl}t,\ \langle B^k, M^l\rangle_t=0 \text{ and } \langle M^k, M^l\rangle_t=\delta_{kl}\varphi(t).
$$

Thus, $\varphi(t)$ is an increasing process, called the *local time of* $X(t)$ *on the boundary*, which increases only when the process $X(t)$ is on the boundary. $B(t)$ is an r-dimensional $(\mathscr{F}_t)$-Brownian motion and $M(t)$ is a q-dimensional Brownian motion if the time is measured by the local time $\varphi(t)$ on the boundary. More precisely, we give the following

Definition 2.2. By a *solution* $X(t)$ *of* (2.5), we mean a D-valued continuous $(\mathscr{F}_t)$-semimartingale defined on a probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)$ such that $X(0)=x$ a.s. and the following is satisfied: There exist an r-dimensional continuous $(\mathscr{F}_t)$-martingale $B(t)$, a q-dimensional continuous $(\mathscr{F}_t)$-martingale $M(t)$ and a continuous $(\mathscr{F}_t)$-adapted non-decreasing process $\varphi(t)$ satisfying (2.1), (2.2) and (2.6), and stochastic differentials of this system of semimartingales $(X(t), B(t), M(t), \varphi(t))$ satisfy the relation (2.5).

By standard representation theorems of martingales, it is not difficult to show the following (cf. [5])

Proposition 2.1. *If* $X(t)$ *is a soltion to the SDE* (2.5), *then* $X(t)$ *is an* (A,L)-*process. Conversely, if* $X(t)$ *is an* (A, L)-*process, then* (*enlarging* $(\Omega, \mathscr{F}, P)$ *and* $(\mathscr{F}_t)$ *in the sense of* [5], *Def.* II-7.1 *if necessary*) $X(t)$ *is a solution to the above SDE* (2.5).

We say that the *uniqueness holds for* (A,L)-*processes or solutions of SDE* (2.5) if, for each fixed $x\in D$, the law on $\mathbf{C}([0, \infty)\to D)$ of any (A, L)-process $X(t)$ or any solution $X(t)$ of (2.5) such that $X(0)=x$ a.s. is unquie. It is well-known that if, for every $x\in D$, an (A, L)-process starting at x exists and furthermore, the uniqueness holds, then this system of (A, L)-processes defines a diffusion process (i.e. strong Markov continuous process) on D. It is called the (A,L)-*diffusion* *process*. Thus, to show the unique existence of (A, L)-diffusion process is equivalent to show the unique existence of solutions of SDE (2.5).

3. Existence and uniqueness of SDE (2.5).

It is proved in [5] that, if $\sigma_k^i(x)$, $b^i(x)$ are Lipschitz continuous on D and $\tau_k^i(x)$, $\beta^i(x)$, $\mu(x)$ are Lipschitz continuous on ∂D and if

$$a^{dd}(x)=\sum_{k=1}^{r}\sigma_k^d(x)\sigma_k^d(x)>0 \qquad \text{for all } x\in\partial D \tag{3.1}$$

and

$$\inf_{x\in\partial D}\mu(x)>0, \tag{3.2}$$

then a solution $X(t)$ to SDE (2.5) exists for every initial point $x\in\partial D$ and furthermore, the uniqueness of solutions holds. The main result of this paper is to weaken the condition (3.2) to the following one:

$$\inf_{x\in\partial D}[\mu(x)+\rho(x)]>0. \tag{3.3}$$

Namely, we have

Theorem 3.1. *If $\sigma_k^i(x)$ and $b^i(x)$ are bounded Lipschitz continuous on D and $\tau_k^i(x)$, $\beta^i(x)$, $\mu(x)$ are bounded Lipschitz continuous on ∂D and if, furthermore,* (3.1) *and* (3.3) *are satisfied, then the existence and uniqueness of solutions to SDE* (2.5) *hold.*

Corollary. *For a given (A,L), if we can choose $\sigma_k^i(x)$ and $\tau_k^i(x)$ such that* (2.3), (2.4) *and all the conditions in Theorem* 3.1 *are satisfied, then the (A,L)-diffusion exists uniquely.*

Remark 3.1. The condition (3.3) is known as a *transversality condition.* This implies that everywhere on the boundary, there occurs either reflection or sojourn.

The rest of this paper is devoted to the proof of Theorem 3.1.

The existence of solutions can be verified in several ways: By the transformations of solutions (cf. [5], Chap IV §7), we can reduce the problem to the case $\sigma_k^d(x)\equiv\delta_{1k}$, $k=1, \ldots, r$, and $b^d(x)\equiv 0$. If we set $\mu_n(x)=\mu(x)+\frac{1}{n}$, $x\in\partial D$, $n=1,2, \ldots$, then we know that solution $X^n(t)$ exists for $(\sigma, b, \tau, \beta, \mu_n, \rho)$ together with auxiliary processes $(B^n(t), M^n(t), \varphi^n(t))$. We can easily show that $\mathfrak{X}^n(t)=(X^n(t), B^n(t), M^n(t), \varphi^n(t))$, $n=1, 2, \cdots$ are tight and a limiting process $\mathfrak{X}(t)=(X(t), B(t), M(t), \varphi(t))$ in the sense of probability law satisfies (2.1), (2.2), (2.5) and (2.6).

In [5] and [10], we gave a direct construction of D-valued continuous process from a Poisson point process of Brownian excursions. If we apply the results of [12] (cf. [11]), we can show that this process is a solution to SDE (2.5). Thus, this is another way to obtain the existence of solutions. We recall this construction here because it play an important role in the proof of uniqueness given below.

As we remarked above, we may and do assume that

$$\sigma_k^d(x) \equiv \delta_{1k}, \quad k=1, \ldots, r, \text{ and } b^d(x) \equiv 0. \tag{3.4}$$

Let

$$\mathscr{W}_+^r = \{w \in \mathbf{C}([0, \infty) \to [0, \infty) \times \mathbf{R}^{r-1});\ w(0)=0 \text{ and } {}^\exists\sigma(w) \in (0, \infty) \text{ such}$$
$$\text{that } w^1(t)>0 \text{ if } t \in (0, \sigma(w)) \text{ and } w^1(t)=0 \text{ if } t \geq \sigma(w)\}$$

and n^+ be the r-dimensional Brownian positive excursion law, i.e., σ-finite measure on $(\mathscr{W}_+^r, \mathscr{B}(\mathscr{W}_+^r))$ given as in [5] p. 215. On the measure space $(\mathscr{W}_+^r, n^+)$, we consider the following "SDE" (cf. [9], [10])

$$\begin{cases} X_t^i = x^i + c\sum_{k=1}^r \int_0^{t\wedge\sigma(w)} \sigma_k^i(X_s)\,dw^k(s) + c^2\int_0^{t\wedge\sigma(w)} b^i(X_s)\,ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1, \ldots, d-1 \\ X_t^d = cw^1(t) \end{cases}$$

where $c>0$ is a given constant and $(x^1, \ldots, x^{d-1}, 0) \in \partial D$. The solution $X(t) = X^c(t, x, w)$, $t \geq 0$, $x=(x^1, \ldots, x^{d-1}, 0) \in \partial D$, $w \in \mathscr{W}_+^r$, is a well-determined n^+-measurable functional such that for all $x \in \partial D$ and almost all $w \in \mathscr{W}_+^r(n^+)$, $t \mapsto X(t) \in D$ is continuous, $X(0)=x$ and $X(t\wedge\sigma(w))=X(t)$. Define $\Phi^x(t)=\Phi(t, x, w)$, $t \geq 0$, $x \in \partial D$, $w \in \mathscr{W}_+^r$ by

$$\Phi(t, x, w) = \begin{cases} X^{\mu(x)}(t/\mu(x)^2, x, w) & \text{if } \mu(x)>0 \\ x & \text{if } \mu(x)=0. \end{cases} \tag{3.5}$$

Then $\Phi^x(t\wedge\hat{\sigma}^x(w))=\Phi^x(t)$ where $\hat{\sigma}^x(w)=\mu(x)^2\sigma(w)$. Now the construction of the D-valued continuous process X which *should* be the (A, L)-diffusion process will be carried out as follows: *Firstly,* we take a sufficiently big probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)$ on which we can realize the following objects:

i) A filtration $(\mathscr{G}_t)$ on Ω such thet $\mathscr{G}_t \subset \mathscr{F}_0$ for every $t \geq 0$ and an r-dimensional $(\mathscr{G}_t)$-Brownian motion $\hat{B}(t)$.

ii) A stationary $(\mathscr{F}_t)$-Poisson point process $p[t]$ on $\mathscr{W}_+^r$ with the characteristic measure n^+, i.e., a Poisson point process of r-dimensional positive Brownian excursions.

iii) A q-dimensional $(\mathscr{F}_t)$-Brownian motion $(\tilde{B}(t))$.

Let $x \in D$ be given and let $x(t)$ be the solution to the SDE

$$\begin{cases} dx^i(t) = \sum_{k=1}^r \sigma_k^i(x(t))\,d\hat{B}^k(t) + b^i(x(t))\,dt \qquad i=1, \ldots, d-1 \\ dx^d(t) = d\hat{B}^1(t) \\ x(0)=x. \end{cases} \tag{3.6}$$

Set $\sigma^0=\inf\{t: x^d(t)=0\}$ and $\xi_0=x(\sigma^0)$. Note that σ^0 and $\xi_0\in\partial D$ are $\mathscr{F}_0$-measurable random variables. *Next,* we consider the following SDE on ∂D of the jump-type

$$\begin{cases}\xi^i(t)=\xi_0^i+\sum_{k=1}^{q}\int_0^t\tau_k^i(\xi(s))d\tilde{B}^k(s)+\int_0^t\bar{\beta}^i(\xi(s))ds \\ \qquad +\int_0^{t+}\int_{\mathscr{W}_+^r}\phi^i(\xi(s-),\ w)I_{\{\sigma(w)<1\}}\tilde{N}_p(dsdw) \\ \qquad +\int_0^{t+}\int_{\mathscr{W}_+^r}\phi^i(\xi(s-),\ w)I_{\{\sigma(w)\geq 1\}}N_p(dsdw) \qquad i=1,\ \dots,\ d-1 \\ \xi^d(t)\equiv 0\end{cases}$$

where

$$\phi(x,\ w)=\Phi(\hat{\sigma}^x(w),\ x,\ w)-x=X^{\mu(x)}(\sigma(w),\ x,\ w)-x\in\partial D \tag{3.7}$$

and

$$\begin{cases}\bar{\beta}^i(x)=\beta_i(x)+\mu(x)^2\int_{\mathscr{W}_+^r}\left[\int_0^{\sigma(w)}b^i(X^{\mu(x)}(t,\ x,\ w))dt\right]I_{\{\sigma(w)<1\}}n^+(dw) \\ \qquad -\mu(x)\int_{\mathscr{W}_+^r}\left[\int_0^1\sigma_1^i(X^{\mu(x)}(t,\ x,\ w))\,dw^i(t)\right]I_{\{\sigma(w)\geq 1\}}n^+(dw).\end{cases} \tag{3.8}$$

We can show as in [9], [10] that $\bar{\beta}^i(x)$ is bounded, Lipschitz continuous and also that

$$\int_{\mathscr{W}_+^r\cap\{\sigma(w)<1\}}|\phi(x,\ w)-\phi(y,\ w)|^2n^+(dw)\leq\text{const. }|x-y|^2.$$

Hence the solution $\xi(t)$ exists uniquely as a well-determined functional of ξ_0, $\tilde{B}(t)$ and $p[t]$. *Thirdly,* define an $(\mathscr{F}_t)$-adapted right-continuous increasing process $A(t)$ by

$$A(t)=\sigma^0+\int_0^{t+}\int_{\mathscr{W}_+^r}\hat{\sigma}^{\xi(s-)}(w)N_p(dsdw)+\int_0^t\rho(\xi(s))ds.$$

By virtue of the assumption (3.3), we can show ([10]) that $t\longmapsto A(t)$ is strictly increasing and $\lim_{t\uparrow\infty}A(t)=\infty$ with probability one. For every $t\geq 0$, there exists a unique $s\geq 0$ such that $A(s-)\leq t<A(s)$. If $s=0$, i.e., $0\leq t\leq\sigma^0$, then we define $X(t)$ to be the solution $x(t)$ to SDE (3.6). If $s>0$ and $A(s-)<A(s)$, then this implies that $s\in\mathbf{D}_p$ and we set $X(t)=\Phi(t-A(s-),\ \xi(s-),\ p[s])$. If $s>0$ and $A(s-)=A(s)$, then $\xi(s)=\xi(s-)$ and we set $X(t)=\xi(s)=\xi(s-)$. In this way, a D-valued process $X(t)$ is constructed and it is not difficult to show (cf. [12]) that $t\longmapsto X(t)\in D$ is continuous a.s. And $X(t)$ is a well-determined functional of $\hat{B}(t)$, $\tilde{B}(t)$, $p[t]$:

$$X=\mathfrak{X}(\hat{B},\ \tilde{B},\ p). \tag{3.9}$$

We are now going to show the uniqueness of solutions of SDE (2.5). For this, we start with any solution X of (2.5) and then construct (by enlarging $(\Omega,\ \mathscr{F},\ P)$ if necessary) $\hat{B}$, $\tilde{B}$, p as above such that X is given by (3.9). Since the joint law of $(\hat{B}, \tilde{B}, p)$ is unique as mutually independent Brownian motions and a Poisson point process, this clearly implies the uniqueness of solutions.

So let $X(t)$ be a solution to SDE (2.5). By the assumption (3.4), we have

$$X^d(t)=x^d+\int_0^t I_{\mathring{D}}(X(s))dB^1(s)+\int_0^t \mu(X(s))d\varphi(s). \tag{3.10}$$

Let $e(t)=\int_0^t I_{\mathring{D}}(X(s))ds$ and $e^{-1}(t)$ be the right continuous inverse of $t \longmapsto e(t)$ defined on $(0, e(\infty))$. Let $\tilde{B}^i(t)=\int_0^{e^{-1}(t)} I_{\mathring{D}}(X(s))dB^i(s)$ $i=1, \dots, r$. Then it is well-known ([5]) that $\tilde{B}(t)=(\tilde{B}(t))_{0\leq t<e(\infty)}$ is a part of r-dimensional Brownian motion $(\tilde{B}(t))_{0\leq t<\infty}$. From (3.10), we have

$$\tilde{X}^d(t)=x^d+\tilde{B}^1(t)+\tilde{l}(t) \qquad t\in[0, e(\infty)) \tag{3.11}$$

where $\tilde{X}^d(t)=X^d(e^{-1}(t))$ and $\tilde{l}(t)=\int_0^{e^{-1}(t)} \mu(X(s))d\varphi(s)$. Clearly, $\tilde{X}^d(t)$ is continuous in t and hence $\tilde{l}(t)$ is continuous. Also $\tilde{l}(t)$ increases only when $\tilde{X}(t)=0$. From this we can conclude that (3.11) is a part of *Skorohod equation*: $\tilde{X}^d(t)$ and $\tilde{l}(t)$ can be extended continuously on $[0, \infty)$, $\tilde{X}^d(t)\geq 0$, $\tilde{l}(t)$ is non-decreasing, $\int_0^t I_{\{\tilde{X}^d(s)=0\}} d\tilde{l}(s)=\tilde{l}(t)$ and (3.11) holds for every $t\in[0, \infty)$. It is known that $\tilde{l}(t)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t I_{(0,\varepsilon)}(\tilde{X}^d(s))ds$, and $\lim_{t\uparrow\infty}\tilde{l}(t)=\infty$ a. s. (cf. [5]). Let $\sigma^0=\inf\{t;\ X^d(t)=0\}$. If $\tilde{l}^{-1}(t)$ is the right-continuous inverse of $t \longmapsto \tilde{l}(t)$, then $\tilde{l}^{-1}(0)=\sigma^0$. Set

$$\mathbf{D}_p=\{t>0;\ \tilde{l}^{-1}(t)>\tilde{l}^{-1}(t-)\} \tag{3.12}$$

and

$$p[t](s)=\begin{cases}\tilde{B}(s+\tilde{l}^{-1}(t-))-\tilde{B}(\tilde{l}^{-1}(t-)) & 0\leq s\leq\tilde{l}^{-1}(t)-\tilde{l}^{-1}(t-)=:\sigma(p[t])\\ \tilde{B}(\tilde{l}^{-1}(t))-\tilde{B}(\tilde{l}^{-1}(t-)) & s\geq\sigma(p[t]).\end{cases} \tag{3.13}$$

Then $p[t]\in\mathscr{W}^r_+$, $t\in\mathbf{D}_p$ and it defines a stationary Poisson point process with characteristic measure n^+ (cf. [5]). We have $\tilde{l}(e(t))=\int_0^t \mu(X(s))d\varphi(s)$ and $\int_0^t I_{\partial D}(X(s))ds=t-e(t)=\int_0^t \rho(X(s))d\varphi(s)$. Hence it is easy to conclude from (3.3) that $\lim_{t\uparrow\infty}\varphi(t)=\infty$ a.s. Also we can easily deduce that $\sigma^0=\inf\{t;\ \varphi(t)>0\}$.

Let $A(t)$ be the right-continuous inverse of $t\longmapsto\varphi(t)$. Then, with probability one, $A(0+)=\sigma^0$, $t\longmapsto A(t)$ is strictly increasing and $\lim_{t\uparrow\infty}A(t)=\infty$. We now define point processes q_1 and q_2 on $\mathbf{C}([0, \infty)\to\mathbf{R}^r)$ and $\mathbf{C}([0, \infty)\to D)$, respectively, by

$$\mathbf{D}_{q_1}=\mathbf{D}_{q_2}=\{t\in(0, \infty);\ A(t-)<A(t)\}$$

and for $t\in\mathbf{D}_{q_1}=\mathbf{D}_{q_2}$,

$$q_1[t](s)=\begin{cases}B(A(t-)+s)-B(A(t-)) & 0\leq s\leq A(t)-A(t-)\\ B(A(t))-B(A(t-)) & s>A(t)-A(t-),\end{cases}$$

$$q_2[t](s)=\begin{cases} X(A(t-)+s) & 0\leq s\leq A(t)-A(t-) \\ X(A(t)) & s>A(t)-A(t-). \end{cases}$$

Let $\tilde{\mathscr{F}}_t=\mathscr{F}_{A(t)}$.

Lemma 3.1. (1) *$q_1[t]$ is an $(\tilde{\mathscr{F}}_t)$-adapted point process on $\mathscr{W}_+^r$ of the class (QL) (cf. [5]) with the compensator $\hat{N}_{q_1}(dsdw)=\mu(X(A(s)))dsn^+(dw)$.*

(2) *By enlarging $(\Omega, \mathscr{F}, P)$ and $(\tilde{\mathscr{F}}_t)$ if necessary, there exists an $(\tilde{\mathscr{F}}_t)$-stationary Poisson point process $\bar{p}$ on $\mathscr{W}_+^r$ with the characteristic measure n^+ such that $q_1[t]$ is given by*

$$N_{q_1}((0, t]\times E)=\int_0^{t+}\int_{\mathscr{W}_+^r} I_E(\tau_{\mu(X(A(s-)))}w)N_{\bar{p}}(dsdw) \tag{3.14}$$

$$E\in\mathscr{B}(\mathscr{W}_+^r)$$

and $q_2[t]$ is given by

$$N_{q_2}((0, t]\times E)=\int_0^{t+}\int_{\mathscr{W}_+^r} I_E(\Phi^{X(A(s-))}(w))N_{\bar{p}}(dsdw) \tag{3.15}$$

$$E\in\mathscr{B}(\mathbf{C}([0, \infty)\to D))$$

where $\Phi^x(w)=\Phi(\cdot, x, w)$ is defined by (3.5) and τ_c: $\mathscr{W}_+^r\to\mathscr{W}_+^r\cup\{\mathbf{0}\}$ is defined by

$$(\tau_c w)(\cdot)=\begin{cases} cw(\cdot/c^2) & \text{if } c>0 \\ \mathbf{0} & \text{if } c=0. \end{cases}$$

Here $\mathbf{0}$ denotes the path defined by $\mathbf{0}(t)=0\in\mathbf{R}^r$ for all $t\geq 0$.

Proof. As for the proof of (1), we consider the point process p defined by (3.12) and (3.13). We know that it is a stationary Poisson point process on $\mathscr{W}_+^r$ with the characteristic measure $n^+(dw)$, i.e., the compensator $\hat{N}_p(dsdw)=dsn^+(dw)$. If $A(s-)<A(s)$, then, on the interval $(A(s-), A(s))$, $\varphi(u)$ is flat and since $I_{\partial D}(X(u))du=\rho(X(u))d\varphi(u)$, $de(u)=du$ on this interval. Also, since $\tilde{l}(t)=\int_0^{e^{-1}(t)}\mu(X(s))d\varphi(s)$, $d\tilde{l}(u)=0$ on the interval $(e(A(s-)), e(A(s))=e(A(s-))+A(s)-A(s-))$. It is also clear that, if $\tilde{l}^{-1}(s-)<\tilde{l}^{-1}(s)$, then $d\varphi(e^{-1}(u))=0$ on the interval $(\tilde{l}^{-1}(s-), \tilde{l}^{-1}(s))$. From this we can conclude that $A(s-)<A(s)$ if and only if $\tilde{l}^{-1}(t-)<\tilde{l}^{-1}(t)$ where $t=\tilde{l}(e(A(s)))=\int_0^s\mu(X(A(\theta)))d\theta$ and that the intervals $(e(A(s-)), e(A(s)))$ and $(\tilde{l}^{-1}(t-), \tilde{l}^{-1}(t))$ coincide. This shows that $s\in\mathbf{D}_{q_1}$ if and only if $t=\int_0^s\mu(X(A(u)))du\in\mathbf{D}_p$ and $q_1[s]=p[t]$. In other words, q_1 is obtained from p by the time change $s\longmapsto t=\int_0^s\mu(X(A(\theta)))d\theta$. Now the assertion of (1) follows at once from the general theory of time change.

To prove (3.14), define a $\mathscr{W}_+^r$-valued point process q_3 by

$$N_{q_3}((0, t]\times E)=\int_0^{t+}\int_{\mathscr{W}_+^r} I_E(\tau_{\mu(X(A(s-)))^{-1}}w)I\{\mu(X(A(s-)))>0\}N_{q_1}(dsdw)$$

$$E \in \mathscr{B}(\mathscr{W}_+^r).$$

Then it is easy to see that its compensator is given by

$$\hat{N}_{q_3}(dtdw) = I_{\{\mu(X(A(t)))>0\}} dtn^+(dw).$$

Take an $(\bar{\mathscr{F}}_t)$-stationary Poisson point process q_4 on $\mathscr{W}_+^r$ with the compensator n^+ defined on a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ with a filtration $(\bar{\mathscr{F}}_t)$. We form the canonical extension

$$\hat{\Omega} = \Omega \times \bar{\Omega}, \quad \hat{\mathscr{F}} = \mathscr{F} \times \bar{\mathscr{F}}, \quad \hat{P} = P \times \bar{P}, \quad \hat{\mathscr{F}}_t = \tilde{\mathscr{F}}_t \times \bar{\mathscr{F}}_t$$

of $(\Omega, \mathscr{F}, P)$ and $(\tilde{\mathscr{F}}_t)$ and, on this extension, we define a Poisson point process $\bar{p}$ on $\mathscr{W}_+^r$ by

$$N_{\bar{p}}((0, t] \times E) = N_{q_3}((0, t] \times E) + \int_0^{t+} \int_{\mathscr{W}_+} I_{\{\mu(X(A(s-)))=0\}} I_E(w) N_{q_4}(dsdw).$$

It is easy to see ([5]) that $\bar{p}$ is $(\hat{\mathscr{F}}_t)$-stationary Poisson point process on $\mathscr{W}_+^r$ with the characteristic measure n^+ and (3.14) holds. Finally, (3.15) is intuitively obvious and can be proved rigorously as in [9]. Q.E.D.

Now it is easy to deduce from (2.2) that

$$A(t) = \sigma^0 + \int_0^{t+} \int_{\mathscr{W}_+^r} \mu(\xi(s-))^2 \sigma(w) N_{\bar{p}}(dsdw) + \int_0^t \rho(\xi(s)) ds$$

where $\xi(t) = X(A(t))$. Set $M(A(t)) = \tilde{B}(t)$. By (2.6), $\tilde{B}(t)$ is a q-dimensional $(\tilde{\mathscr{F}}_t)$-Brownian motion and finally, we can show, in exactly the same way as in [9], that

$$\begin{aligned} \xi^i(t) = \xi(0) &+ \sum_{k=1}^{q} \int_0^t \tau_k^i(\xi(s)) d\tilde{B}^k(s) + \int_0^t \bar{\beta}^i(\xi(s)) ds \\ &+ \int_0^{t+} \int_{\mathscr{W}_+^r} \phi^i(\xi(s-), w) I_{\{\sigma(w) \leq 1\}} \tilde{N}_{\bar{p}}(dsdw) \\ &+ \int_0^{t+} \int_{\mathscr{W}_+} \phi^i(\xi(s-), w) I_{\{\sigma(w) > 1\}} N_{\bar{p}}(dsdw) \qquad i = 1, \dots, d-1 \end{aligned}$$

where $\phi^i(x, w)$ and $\bar{\beta}^i(x)$ are the same as given by (3.7) and (3.8). Note that $\xi(0) = X(\sigma^0)$ is $\tilde{\mathscr{F}}_0$-measurable and hence is independent of $(\tilde{B}, \bar{p})$. Now we can conclude that X is obtained by (3.9) from $(\hat{B}, \tilde{B}, \bar{p})$ where $\hat{B}$ is any r-dimensional Brownian motion independent of $(\tilde{B}, \bar{p})$ such that $(\hat{B}(t))_{0 \leq t \leq \sigma^0}$ coincides with $(B(t))_{0 \leq t \leq \sigma^0}$. This completes the proof of the uniqueness of solutions of SDE (2.5).

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References

[1] R.F. Anderson, Diffusions with second order boundary conditions, Part I, Indiana Univ. Math. J., **25** (1976), 367–395.

[2] R.F. Anderson, Diffusions with second order boundary conditions, Part II, Indiana Univ. Math. J., **25** (1976), 403–441.

[3] J.M. Bony, Ph. Courrège and P. Priouret, Semi-groupe de Feller sur une varieté à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum, Ann. Inst. Fourier **18** (1968), 369–521.

[4] N. El Karoui, Processus de diffusion associé à un operateur élliptique dégénéré et à une condition frontière, Thèse d'Etat Paris VI, 1971.

[5] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland/Kodansha, 1981.

[6] K. Sato and T. Ueno, Multi-dimensional diffusion and the Markov process on the boundary, J. Math. Kyoto Univ., **4** (1965), 529–605.

[7] D.W. Stroock and S.R.S. Varadhan, Diffusion processes with boundary conditions, Comm. Pure Appl. Math., **24** (1971), 147–225.

[8] K. Taira, Difiusion processes and partial differential equations, a forthcoming book.

[9] S. Takanobu, On the uniqueness of solutions of stochastic differential equations with singular drifts, Publ, RIMS Kyoto Univ., **22** (1986), 813-848.

[10] S. Watanabe, Construction of diffusion processes with Wentzell's boundary conditions by means of Poisson point processes of Brownian excursions, Prob. Theory, Banach Center Publ. **5**, PWN-Polish Scientific Publishers, Warsaw, 1976, 255–271.

[11] S. Watanabe, Excursion point processes and diffusions, Proc. ICM Warsaw, PWN-Polish Scientific Publishers, 1984, 1117–1124.

[12] S. Watanabe, On the construction of a semimartingale from pieces by the method of excursion point processes, Ann. Inst. Henri Poincaré, Sup. au n°2, **23** (1987), 297-320.