J. Math. Kyoto Univ. (JMKYAZ)
28-1 (1988) 1-12

Micro-local energy method of Mizohata and hypoellipticity

By

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§0. Introduction

In the present paper, we are mainly concerned with partial hypoellipticity (with respect to the x-variables) of the equation (in $\mathbf{R}^{d+1}$)

$$\frac{\partial u}{\partial t}+a(x, D_x)u=f, \tag{0.1}$$

where $a(x, D_x)$ is a second order partial differential operator (in $\mathbf{R}^d$ with coefficients of class C^∞) satisfying the following conditions.

(A.1) *There exists a constant C such that*

$$\sum_{|\nu|=1}\|a^{(\nu)}(x, D_x)v\|^2+\sum_{|\mu|=1}\|a_{(\mu)}(x, D_x)v\|_{-1}^2$$
$$\leqq C(\mathrm{Re}(a(x, D_x)v, v)+\|v\|^2), \ \forall v\in\mathcal{S}(\boldsymbol{R}^d).$$

(A.2) *For any $\varepsilon>0$, there exists a constant C_ε such that*

$$\|\log\langle D_x\rangle v\|^2\leqq\varepsilon\ \mathrm{Re}(a(x, D_x)v, v)+C_\varepsilon\|v\|^2, \ \forall v\in\mathcal{S}(\boldsymbol{R}^d).$$

Here we use standard notations: $a^{(\nu)}_{(\mu)}(x, \xi)=\partial_\xi^\nu(-i\partial_x)^\mu a(x, \xi)$, and $\log\langle D_x\rangle=\log(2-\Delta_x)^{1/2}$. $\|\ \|$ and $\|\ \|_{-1}$ stand for the norms in $L^2(\boldsymbol{R}^d)$ and $H^{-1}(\boldsymbol{R}^d)$ (the Sobolev space of order -1) respectively.

Our main purpose is:

Theorem 1. *Assume that second order differential operator $a(x, D_x)$ satisfies the conditions* (A.1) *and* (A.2). *Suppose that $u\in C^1([0, \delta]; \mathcal{D}'_x(U))$ and $f\in C([0, \delta]; \mathcal{D}'_x(U))$ satisfy the eguation* (0.1) *in $(0, \delta)\times U$, where $\delta>0$ and U is an open set of $\boldsymbol{R}^d$. Let (x_0, ξ^0) be a point of $U\times(\boldsymbol{R}^d\setminus 0)$. Then, if $(x_0, \xi^0)\notin WF(f(\cdot, t))$ for $0\leqq t\leqq\delta$, it follows that $(x_0, \xi^0)\notin WF(u(\cdot, t))$ for $0<t\leqq\delta$.*

Since $au(x)=f(x)$ implies $\partial u/\partial t+au=f$ (where u and f are independent of t), Theorem 1 yields the result due to Y. Morimoto [7]:

Corollary. *Provided second order differential operator $a(x, D_x)$ satisfies the conditions* (A.1) *and* (A.2), *then it follows that $a(x, D_x)$ is hypoelliptic in $\boldsymbol{R}^d$.*

Our argument can be applied without modification to the proof of the follow-

Received August, 22 1986

ing Morimoto's theorem:

Theorem 2. (Y. Morimoto [8]). *Let $a(x, D_x)$ be a differential operator of order m with coefficients of class $C^\infty(\boldsymbol{R}^d)$. Assume that for any $\varepsilon>0$, there exists a constant C_ε such that*

$$\text{(A.3)} \qquad \|(\log\langle D_x\rangle)^m v\|^2+\sum_{0<|\nu+\mu|<m}\|(\log\langle D_x\rangle)^{|\nu+\mu|}a_{(\mu)}^{(\nu)}(x, D_x)v\|^2_{-|\mu|}$$

$$\leqq\varepsilon\|a(x, D_x)v\|^2+C_\varepsilon\|v\|^2,\ \forall v\in\mathcal{S}(\boldsymbol{R}^d).$$

Then, $a(x, D_x)$ is hypoelliptic. Moreover we have $WF(au)=WF(u)$ for all $u\in\mathcal{D}'(\boldsymbol{R}^d)$.

Let us now explain the conditions (A.1) and (A.2). To the operators of the form $a(x, D_x)=\sum_{j=1}^k X_j^* X_j (or=\sum_{j=1}^k X_j^2)$, where X_j, $j=1,\dots, k$ are C^∞ real vector fields generating a Lie algebra of rank d at every point, Theorem 1 is applicable. In fact, the condition (A.1) is satisfied by the operators of the form $\sum_{j=1}^k X_j^2+X_0 (X_j, j=0,\dots, k$ are real vector fields of class C^∞), and, provided $X_1,\dots, X_k$ satisfy the condition on Lie algebra, the operators of the form $a(x, D_x)=\sum_{j=1}^k X_j^* X_j$ $(or=\sum_{j=1}^k X_j^2)$ have

$$\text{(A}'\text{.2)} \qquad \|\langle D\rangle^\sigma v\|^2\leqq const.\ (\mathrm{Re}(a(x, D_x)v, v)+\|v\|^2),\ \forall v\in\mathcal{S}(\boldsymbol{R}^d)$$

for some σ, $0<\sigma\leqq 1$. (See L. Hörmander [1] chap. 22.2.) Theorem 1 is also applicable to the infinitely degenerate elliptic operators (they are our main targets). For example, the operator

$$a_1(x, D_x)=D_{x_1}^2+x_1^{2k}D_{x_2}^2+\exp(-1/|x_2|^\gamma)D_{x_3}^2$$

satisfies (A.1) and (A.2), if k is a non-negative integer and $0<\gamma<\frac{1}{k+1}$. (See Y. Morimoto [7] proposition 4.)

On the other hand, it is remarkable that the condition (A.2) is necessary for hypoellipticity of the operators of the form $a(x, D_x)=D_{x_1}^2+b(x', D_{x'})$, where $b(x', D_{x'})$ is a formally selfadjoint differential operator of second order satisfying $(b(x', D_{x'})v, v)\geqq -const.\ \|v\|^2$, $\forall v\in\mathcal{S}(\boldsymbol{R}^{d-1})$. (From this fact, we can show that the operator $a_1(x, D_x)$ with $k=0$ is not hypoelliptic in any neighborhood of the origin $0\in\boldsymbol{R}^d$, *if* $\gamma\geqq 1$. See Y. Morimoto [7] or T. Hoshiro [2].)

We rely heavily on the works [4] and [5] by S. Mizohata, who initiated the micro-local energy method for the characterizations of the analytic and the Gevrey wave front sets. We remark that our method is quite elementary. In fact, we make use of basic calculus of pseudo-differential operators.

Hypoellipticity of $a(x, D_x)$ under the conditions (A.1) and (A.2) (or under the condition (A.3)) was recently studied by Y. Morimoto, related to the work by S. Kusuoka - D. Strook [3]. He has obtained the corollary to Theorem 1 and also Theorem 2 in [7] and [8] respectively. Essentially, our results are not new. However, we believe that our method is one of the most direct way to arrive at the results.

The plan of this paper is as follows: In §1, we explain our method. The proof of Theorem 1 will be given in §2. In §3, we will prove the propositions stated in §1. Finally in §4, we will prove Theorem 2.

Acknowledgments. The author is deeply indebted to Professors S. Mizohata, W. Mastumoto and N. Shimakura for helpful encouragements.

§1. Preliminaries.

In this section, we introduce some notations and techniques which are necessary for the proofs of Theorems. Let us first define a sequence of cut-off functions (*micro-localizers*) $\{\alpha_n(\xi),\ \beta_n(x)\}_{n\geqq 1}$, of size $r_0>0$. It is well known that there exists a sequence $\phi_N\in C_0^\infty(\boldsymbol{R}^d)$, $N=1, 2,\ldots$, having the following properties:

(C-1) *ϕ_N has its support in $\{y;\ |y|\leqq r_0\}$, and is equal to* 1 *in $\{y;\ |y|\leqq r_0/2\}$.*

(C-2) *For any positive integer K_0,*

$$|\partial_y^{p+\nu}\phi_N|\leqq C_{K_0}(CN)^{|p|},$$

$$|p|\leqq N \text{ and } |\nu|\leqq K_0,$$

where C and C_{K_0} are constants independent of N.
(For the construction of ϕ_N, see S. Mizohata [5].)

Now we define

$$N_n=[\log n]+1,\ \alpha_n(\xi)=\phi_{N_n}\Big(\frac{\xi}{n}-\xi^0\Big) \text{ and } \beta_n(x)=\phi_{N_n}(x-x_0). \tag{1.1}$$

Note: Let us notice that the support of $\alpha_n(\xi)$ is contained in $\{\xi;\ |\xi-n\xi^0|\leqq nr_0\}$, in other words, since r_0 is small, the support of $\alpha_n(\xi)$ is contained in a small conic neighborhood of ξ^0. We remark that, in the following arguments, we are mainly concerned with the investigations when n is large. So we may regard N_n as $\log n$. Also we remark that, in our method, (C-2) is very important because we have $|\alpha_n^{(p+\nu)}(\xi)|\leqq C_{K_0}(CN_n)^{|p|}n^{-|p+\nu|}$ for $|p|\leqq N_n$, $|\nu|\leqq K_0$ and $|\beta_{n(p+\mu)}(x)|\leqq C_{K_0}(CN_n)^{|p|}$ for $|q|\leqq N_n$, $|\mu|\leqq K_0$, which enable us to do *the quantitative analysis.* ($\alpha_n^{(p+\nu)}=\partial_\xi^{p+\nu}\alpha_n$, $\beta_{n(q+\mu)}=(-i\partial_x)^{p+\mu}\beta_n$.)

Let us put, for $u\in\mathscr{D}'(\mathbf{R}^d)$,

$$\begin{cases} S_n^M(u)=\displaystyle\sum_{|p+q|\leqq N_n}\|c_{p,q}^n\alpha_n^{(p)}(D)\beta_{n(q)}(x)u\|^2 \\ \text{with } c_{p,q}^n=M^{-|p+q|}n^{|p|}(\log n)^{-|p+q|}. \end{cases} \tag{1.2}$$

$S_n^M(u)$ could be called a *micro-local energy* of u in a neighborhood of $(x_0,\ n\xi^0)$. Now, we have the following propositions on the relation between the wave front sets and behavior of $S_n^M(u)$ as $n\to\infty$, whose proofs will be given in §3.

Proposition 1. *Let u be a distribution and suppose $(x_0,\ \xi^0)\notin WF(u)$. Then we have, if r_0 is small:*

$$(1.3)\qquad \begin{cases} \textit{For any positive number } s, \textit{ there exists a } M>0 \textit{ such that} \\ \qquad S_n^M(u)=O(n^{-2s}) \ \textit{as}\ n\to\infty. \end{cases}$$

Note: In general, $\phi_n=O(n^{-k})$ means that there exists a constant B such that $|\phi_n|\leqq Bn^{-k}$, when n is large.

The converse is also true. We can state in the following form:

Proposition 2. *Put:*

$$\tilde{S}_n^M(u)=\sum_{|p|\leqq N_n}\|c_{p,0}^n\alpha_n^{(p)}(D)\beta_n(x)u\|^2.$$

If $\tilde{S}_n^M(u)$ *satisfies* (1.3), *then* $(x_0, \xi^0)\notin WF(u)$.

Note: It follows from Proposition 1 and 2 that $(x_0, \xi^0)\notin WF(u)$ is equivalent to (1.3) because $\tilde{S}_n^M(u)\leqq S_n^M(u)$. Also notice that $S_n^M(u)\leqq S_n^{M'}(u)$ if $M\geqq M'$.

Let us notice the following fact. In the definition (1.2), if we take M sufficiently large, the values of $\|c_{p,q}^n\alpha_n^{(p)}\beta_{n(q)}u\|$, (p, q) satisfying $|p+q|=N_n$, are so small that we can neglect them, when n is large. More precisely,

Lemma 1. *Let u be a distribution. For any positive number s, if we choose M sufficiently large, then*

$$\mathring{S}_n^M(u)=\sum_{|p+q|=N_n}\|c_{p,q}^n\alpha_n^{(p)}(D)\beta_{n(q)}(x)u\|^2$$

$$=O(n^{-2s}) \ \textit{as}\ n\to\infty.$$

In the sequel, we call a term (or sum of terms) *negligible* if it satisfies the same property as $\mathring{S}_n^M(u)$ in Lemma 1. By the same argument as in Proof of Lemma 1, we can show that the sum of $\|c_{p,q}^n\alpha_n^{(p)}\beta_{n(q)}u\|^2$ with respect to (p, q) satisfying $N_n-n_0\leqq|p+q|\leqq N_n$, n_0 being any fixed positive integer independent of n, is also negligible.

Proof of Lemma 1. First, we take a $\zeta\in C_0^\infty$ so that $\zeta(x)=1$ in $\{x;\ |x-x_0|\leqq r_0\}$. (Therefore $\beta_n\subset\subset\zeta$.) Then, for some k, we have $\zeta u\in H^{-k}$. Let us now recall that, for $c(x, \xi)\in S_{1,0}^{-k}$, the semi-norm of $c(x, \xi)$ in $S_{1,0}^{-k}$ is defined in such a way that

$$|c|_l^{(-k)}=\max_{|\nu+\mu|\leqq l}\sup|c_{(\nu)}^{(\mu)}(x, \xi)|/\langle\xi\rangle^{-k-|\mu|}.$$

From the condition (C-2) and fact that $K^{-1}n\leqq\sqrt{1+|\xi|^2}\leqq K\cdot n$ for $\xi\in\operatorname{supp}[\alpha_n^{(p)}]$, it follows that the semi-norms (in $S_{1,0}^{-k}$) of symbols $\sigma(c_{p,q}^n\alpha_n^{(p)}\beta_{n(q)})$, (p, q) satisfying $|p+q|\leqq N_n$, are estimated by $n^k\cdot C'(2C/M)^{|p+q|}$, where C (resp. M) is the same constant as in (C-2) (resp. (1.2)). Therefore, the values of

$$\|c_{p,q}^n\alpha_n^{(p)}(D)\beta_{n(q)}(x)\langle D_x\rangle^k\|_{L^2\to L^2},$$

(p, q) satisfying $|p+q|\leqq N_n$, are also estimated by $n^k\cdot C''(2C/M)^{|p+q|}$.

To see these estimates, observe that

$$(1.4)\qquad |\sigma(\alpha_n^{(p)}\beta_{n(q)})|_l^{(-k)} \leqq const.\ |\alpha_n^{(p)}|_{l_1}^{(-k)}|\beta_{n(q)}|_{l_1}^{(0)}$$

$$\leqq const.\ n^{k-|p|}(CN_n)^{|p|}(CN_n)^{|q|},$$

where *const.*s are independent of n, p and q (we take $K_0=l_1$ in (C-2)). We choose $l=d+2$ in order to apply (1.4) to the values of $\|c_{p,q}^n\alpha_n^{(p)}\beta_{n(q)}\langle D_x\rangle^k\|_{L^2\to L^2}$, (p, q) satisfying $|p+q|\leqq N_n$.

On the other hand, the number of terms with (p, q) satisfying $|p+q|=N_n$ cannot exceed $(2d)^{N_n}$. Therefore we have

$$\begin{aligned}\mathring{S}_n^M(u) &= \sum_{|p+q|=N_n}\|c_{p,q}^n\alpha_n^{(p)}\beta_{n(q)}\langle D_x\rangle^k\langle D_x\rangle^{-k}\zeta u\|^2\\ &\leqq C_1\sum_{|p+q|=N_n}\|c_{p,q}^n\alpha_n^{(p)}\beta_{n(q)}\langle D_x\rangle^k\|_{L^2\to L^2}^2\\ &\leqq C_2\cdot n^{2k}\cdot(2C/M)^{2N_n}\cdot(2d)^{N_n},\end{aligned}$$

where C_j, $j=1, 2$, are constants independent of n.

Now, for the given positive number s, we choose M in such a way that $(2C/M)\cdot\sqrt{2d}\leqq e^{-s-k}$. Then the last inequality is estimated by $const.\cdot n^{-2s}$ when n is large (recall that $N_n\fallingdotseq\log n$).

Note: There are several choices of micro-local energy forms related to the criterion of the wave front sets (see S. Mizohara [4]). The reader might feel that our form $S_n^M(u)$ looks like strange because, in spite of the analysis in C^∞-class, the sum on $|p+q|$ is taken up to $N_n\fallingdotseq\log n$ (which becomes large with n). However in our arguments, the choice, together with the choice of $c_{p,q}^n$ and the condition (C-2), is essential in view of the assumption (A.2).

§2. Proof of Theorem 1.

Let us admit Proposition 1 and 2 for the moment. Our assumption for $f(x, t)$ is:

$$(2.1)\quad\begin{cases}\textit{Given any positive number } s, \textit{ there exist positive constants } M=M_s \textit{ and } A=A\\ \textit{such that}\\ \qquad\qquad S_n^M(f(\cdot, t))\leqq An^{-2s}\\ \textit{for } t;\ 0\leqq t\leqq\delta \textit{ and } n;\ \textit{large.}\end{cases}$$

Under this assumption (for the given s), we are going to prove:

$$(2.2)\quad\begin{cases}\textit{For any } \eta(0<\eta<\delta), \textit{ there exists a } B=B(\eta, s) \textit{ such that}\\ \qquad\qquad S_n^M(u(\cdot, t))\leqq Bn^{-2s}\\ \textit{for } t;\ \eta\leqq t\leqq\delta \textit{ and } n;\ \textit{large.}\end{cases}$$

We show this by the method of S. Mizohata [5]. Since the proof is long, we devide it into three steps.

(Step 1.) First, we operate $\alpha_n^{(p)}(D)\beta_{n(q)}(x)$, (p, q) satisfying $|p+q|\leqq N_n$, to the both sides of the equation (0.1). Then

$$\partial_t u_{p,q}=-\alpha_n^{(p)}\beta_{n(q)}a(x, D_x)u+f_{p,q},$$

where we denote $u_{p,q}=\alpha_n^{(p)}\beta_{n(q)}u$ and $f_{p,q}=\alpha_n^{(p)}\beta_{n(q)}f$.
Therefore we have

$$\begin{aligned}
(2.3)\qquad \frac{d}{dt}\|u_{p,q}\|^2 &= 2\mathrm{Re}(-\alpha_n^{(p)}\beta_{n(q)}a(x, D_x)u+f_{p,q}, u_{p,q})= \\
&= 2\mathrm{Re}\{-(a(x, D_x)u_{p,q}, u_{p,q}) \\
&\quad + \sum_{|\nu|=1}(a^{(\nu)}(x, D_x)u_{p,q+\nu}, u_{p,q})-\sum_{|\mu|=1}(a_{(\mu)}(x, D_x)u_{p+\mu,q}, u_{p,q}) \\
&\quad - \sum_{2\leqq|\nu+\mu|\leqq N_0}\frac{(-1)^{|\nu|}}{\nu!\mu!}(a_{(\mu)}^{(\nu)}(x, D_x)u_{p+\mu,q+\nu}, u_{p,q}) \\
&\quad -(r_{p,q}(x, D_x)u, u_{p,q})+(f_{p,q}, u_{p,q})\} \\
&= 2\mathrm{Re}\sum_{j=1}^{6}I_j,
\end{aligned}$$

where N_0 is a large integer whose definition will be given later (see (2.4)).

(Step 2.) Now, we have the following:

Lemma 2. (i) *The terms $r_{p,q}(x, D_x)u$, (p, q) satisfying $|p+q|\leqq N_n$, are negligible, that is, if we take N_0 sufficiently large, then*

$$\sum_{|p+q|\leqq N_n}\|c_{p,q}^n r_{p,q}(x, D_x)u\|^2=0(n^{-2s}) \quad as\ n\to\infty.$$

Here, $\phi_n(t)=0(n^{-k})$ means that there exists a constant B such that $|\phi_n(t)|\leqq Bn^{-k}$, for t; $0\leqq t\leqq\delta$ and n; large.
(ii) *For every positive number L,*

$$\begin{aligned}
|I_2|\leqq L^{-1}(\log n)^{-2}\ const. &\sum_{|\nu|=1}\{\mathrm{Re}(a(x, D_x)u_{p,q+\nu}, u_{p,q+\nu}) \\
&+\|u_{p,q+\nu}\|^2\}+L(\log n)^2\|u_{p,q}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
|I_3|\leqq L^{-1}n^2(\log n)^{-2}\ const. &\sum_{|\mu|=1}\{\mathrm{Re}(a(x, D_x)u_{p+\mu,q}, u_{p+\mu,q}) \\
&+\|u_{p+\mu,q}\|^2\}+L(\log n)^2\|u_{p,q}\|^2,
\end{aligned}$$

where const.s are independent of L, n, p and q.
(iii) *For every positive number L,*

$$\begin{aligned}
|I_4|\leqq L^{-1}\ const. &\sum_{2\leqq|\nu+\mu|\leqq N_0} n^{2(2-|\nu|)}(\log n)^{-4}\times \\
&\times\{\mathrm{Re}(a(x, D_x)u_{p+\mu,q+\nu}, u_{p+\mu,q+\nu})+\|u_{p+\mu,q+\nu}\|^2\} \\
&+L(\log n)^2\|u_{p,q}\|,
\end{aligned}$$

where const. is independent of L, n, p and q.

Proof of Lemma 2. (i) Take a function $\zeta \in C_0^\infty(\boldsymbol{R}^d)$ satisfying $\zeta(x)=1$ in $\{x;\ |x-x_0|\leqq r_0\}$. Then, $\beta_n \subset\subset \zeta$ and there exists a positive number k such that $\zeta u(\cdot,\ t)\in H^{-k}(\boldsymbol{R}^d)$ for $t;\ 0\leqq t\leqq \delta$.

Let us observe the followings. From the condition (C-2) and the fact that $K^{-1}n\leqq\sqrt{1+|\xi|^2}\leqq K\cdot n$ for $\xi\in\operatorname{supp}[\alpha_n^{(p)}]$, it follows that the semi-norms of $c_{p,q}^n r_{p,q}(x,\ D_x)$ in $S_{1,0}^{s+1-N_0}$, $(p,\ q)$ satisfying $|p+q|\leqq N_n$, are estimated by $n^{-s}\ C'(2C/M)^{|p+q|}$. Therefore the values of

$$\|c_{p,q}^n r_{p,q}(x,\ D_x)\langle D_x\rangle^k\|_{L^2\to L^2},$$

$(p,\ q)$ satisfying $|p+q|\leqq N_n$, are also estimated by $n^{-s}C''(2C/M)^{|p+q|}$, provided

$$s+1-N_0\leqq -k. \tag{2.4}$$

To see these estimates, observe that (cf. S. Mozohata [4] page 58)

$$\begin{aligned} &|r_{p,q}|_l^{(s+1-N_0)}\\ &\leqq const. \sum_{|\nu+\mu|=N_0+1}|\alpha_n^{(p+\mu)}|_{l_1}^{(s-|\mu|)}|\beta_{n(q+\nu)}|_{l_1}^{(0)}|a_{(\mu)}^{(\nu)}|_{l_1}^{(2-|\nu|)}\\ &\leqq const.\ n^{-s-|p|}(CN_n)^{|p+q|}, \end{aligned}$$

where *const.*s are constants independent of n, p and q.

So, let us take N_0 in such a way that the inequality (2.4) holds. Then we have (notice that, in (2.3), u may be replaced by ζu, because $a(x,\ D_x)$ is a differential operator and $\beta_n\subset\subset\zeta$)

$$\begin{aligned} &\sum_{|p+q|\leqq N_n}\|c_{p,q}^n r_{p,q}(x,\ D_x)u\|^2\\ &=\sum_{|p+q|\leqq N_n}\|c_{p,q}^n r_{p,q}(x,\ D_x)\langle D_x\rangle^k\langle D_x\rangle^{-k}\zeta u\|^2\\ &\leqq C_1\sum_{|p+q|\leqq N_n}\|c_{p,q}^n r_{p,q}(x,\ D_x)\langle D_x\rangle^k\|_{L^2\to L^2}^2\\ &\leqq C_2\sum_{p,q}(2C/M)^{2|p+q|}n^{-2s}\\ &\leqq C_2 n^{-2s}, \end{aligned}$$

provided M satisfies $2d(2C/M)^2<1/2$. (Observe that the number of the terms with $(p,\ q)$ satisfying $|p+q|=j$ cannot exceed $(2d)^j$.)

(ii) Let us show the second inequality, because the proof of the first one is similar. First, it is clear that

$$|I_3|\leqq L^{-1}d(\log n)^{-2}\sum_{|\mu|=1}\|a_{(\mu)}u_{p+\mu,q}\|^2+L(\log n)^2\|u_{p,q}\|^2.$$

Let us take a function $\phi\in C_0^\infty$ so that $\phi(y)=1$ in $\{y;\ |y|\leqq r_0\}$, and define $\tilde{\alpha}_n(\xi)=\phi\left(\frac{\xi}{n}-\xi^0\right)$. Then, since $\alpha_n\subset\subset\tilde{\alpha}_n$ and the commutator $[\tilde{\alpha}_n,\ a_{(\mu)}]$ is bounded in $OPS_{1,0}^1$, we have

$$\begin{aligned}\|a_{(\mu)}u_{p+\mu,q}\|^2 &= \|a_{(\mu)}\alpha(D_x)u_{p+\mu,q}\|^2\\ &\leqq 2\|\tilde{\alpha}(D_x)a_{(\mu)}u_{p+\mu,q}\|^2 + C_1\|\langle D_x\rangle u_{p+\mu,q}\|^2\\ &\leqq C_2 n^2\{\|\langle D_x\rangle^{-1}a_{(\mu)}u_{p+\mu,q}\|^2 + \|u_{p+\mu,q}\|^2\},\end{aligned}$$

where C_j, $j=1, 2$, are constants independent of n, p and q.

Now, in view of (A.1), the latter inequality of (ii) is proved.

(iii) It follows from $a_{(\mu)}^{(\nu)} \in S_{1,0}^{2-|\nu|}$ that

$$\begin{aligned}|I_4| &\leqq L^{-1}(\log n)^{-2}\ const. \sum_{2\leqq|\nu+\mu|\leqq N_0} \|\langle D_x\rangle^{2-|\nu|}\ u_{p+\mu,q+\nu}\|^2 + L(\log n)^2\|u_{p,q}\|^2\\ &\leqq L^{-1}(\log n)^{-2}\ const. \sum_{2\leqq|\nu+\mu|\leqq N_0} n^{2(2-|\nu|)}\|u_{p+\mu,q+\nu}\|^2 + L(\log n)^2\|u_{p,q}\|^2,\end{aligned}$$

where *const.*s are independent of L, n, p and q.

Now, in view of (A.2), the assertion of (iii) is proved. (Observe that $K^{-1}n \leqq \sqrt{1+|\xi|^2} \leqq K\cdot n$ for $\xi \in \operatorname{supp}[\alpha_n^{(p)}]$.)

(Step 3.) Let us now observe that $c_{p,q}^n(\log n)^{-1} = M\ c_{p,q+\nu}^n$ for $|\nu|=1$, $c_{p,q}^n n(\log n)^{-1} = M\ c_{p+\mu,q}^n$ for $|\mu|=1$, and $c_{p,q}^n(\log n)^{-2}n^{2-|\nu|} \leqq M^{|\nu+\mu|}c_{p+\mu,q+\nu}^n$ for $|\nu+\mu| \geqq 2$ (when n is large). Therefore, it follows from (2.3) and Lemma 2 that

$$\begin{aligned}(2.5)\qquad &\frac{d}{dt}\|c_{p,q}^n u_{p,q}\|^2\\ &\leqq 2\{-\operatorname{Re}(ac_{p,q}^n u_{p,q},\ c_{p,q}^n u_{p,q}) +\\ &+ \{2+3L(\log n)^2\}\|c_{p,q}^n u_{p,q}\|^2 +\\ &+ L^{-1}\ const.\ M^{2N_0} \sum_{1\leqq|\nu+\mu|\leqq N_0} (\operatorname{Re}(ac_{p+\mu,q+\nu}^n u_{p+\mu,q+\nu},\ c_{p+\mu,q+\nu}^n u_{p+\mu,q+\nu})\\ &+ \|c_{p+\mu,q+\nu}^n u_{p+\mu,q+\nu}\|^2)\} + \textit{negligible terms}.\end{aligned}$$

Let us take L sufficiently large and sum up the inequalities (2.5) with respect to (p, q) satisfying $|p+q| \leqq N_n - N_0$. Then, the third terms on the right hand side of (2.5) will be absorbed into the first ones, that is, we have (also observe that the terms $\frac{d}{dt}\|c_{p,q}^n u_{p,q}\|^2$ and $\operatorname{Re}(ac_{p,q}^n u_{p,q},\ c_{p,q}^n u_{p,q})$, (p, q) satisfying $N_n - N_0 \leqq |p+q| \leqq N_n$, are negligible)

$$\begin{aligned}(2.6)\qquad \frac{d}{dt}S_n^M(u(\cdot, t)) \leqq &-\sum_{|p+q|\leqq N_n} \operatorname{Re}(ac_{p,q}^n u_{p,q},\ c_{p,q}^n u_{p,q})\\ &+ 2\{2+3L(\log n)^2 + L^{-1}\ const.\ M^{2N_0}\}\ S_n^M(u(\cdot, t)) + 0(n^{-2s}),\end{aligned}$$

where *const.* is independent of M, L and n.

Furthermore, it follows from the assumption (A.2) that

$$(2.7)\qquad \frac{d}{dt}S_n^M(u(\cdot, t)) \leqq -\frac{1}{\varepsilon}(\log n)^2 S_n^M(u(\cdot, t)) + 0(n^{-2s})$$

$$\leqq -\frac{1}{\varepsilon}(\log n) S_n^M(u(\cdot,\ t)) + 0(n^{-2s}),$$

where $\varepsilon > 0$ can be chosen arbitararily small. (For any positive number ε, the inequality (2.7) holds when n is sufficiently large.)

Therefore we have

$$S_n^M(u(\cdot,\ t)) \leqq \exp(-(\log n)t/\varepsilon) S_n^M(u(\cdot,\ 0)) + 0(n^{-2s})$$

$$= n^{-t/\varepsilon} S_n^M(u(\cdot,\ 0)) + 0(n^{-2s}).$$

Now, the assertion of Theorem 1 is proved, because $S_n^M(u(\cdot,\ 0)) = 0(n^k)$ for some k. q.e.d.

§3. Proofs of Propositions.

Proof of Proposition 1. Here, we consider the values of $\|c_{p,q}^n \alpha_n^{(p)} \beta_{n(q)} u\|$, (p, q) satisfying $|p+q| \leqq N_n$, when $(x_0,\ \xi_0) \notin \mathrm{WF}(u)$ and r_0 (size of micro-localizers) is small.

Take a function $\zeta \in C_0^\infty$ so that $\zeta(x) = 1$ in $\{x;\ |x - x_0| \leqq r_0\}$. Then we have $\beta_n \subset\subset \zeta$ (i.e. $\zeta = 1$ in the support of β_n) and $\zeta u \in H^{-k}$ for some k. Let us write

$$c_{p,q}^n \alpha_n^{(p)} \beta_{n(q)} u = c_{p,q}^n \alpha_n^{(p)} \beta_{n(q)} (\zeta u)$$

$$= \sum_{|\nu| \leqq N_0} \nu !^{-1} c_{p,q}^n \beta_{n(q+\nu)} \alpha^{(p+\nu)} (\zeta u) + c_{p,q}^n r'_{p,q}(x,\ D_x)(\zeta u),$$

where N_0 is a large integer whose definition will be given later (see (3.1)).

On the last terms of the right hand side, we have the following: By the same argument as in the proof of Lemma 2 (i), the values of

$$\|c_{p,q}^n r'_{p,q}(x,\ D_x) \langle D_x \rangle_x^k \|_{L^2 \to L^2},$$

$(p,\ q)$ satisfying $|p+q| \leqq N_n$, are estimated by $n^{-s} \mathit{const.} (2C/M)^{|p+q|}$ (*const.* is independent of n, p and q), provided

$$k - N_0 - 1 \leqq -s. \tag{3.1}$$

Therefore, if we take M so that $(2d)(2C/M)^2 < 1/2$, then

$$\sum_{|p+q| \leqq N_n} \|c_{p,q}^n r'_{p,q}(x,\ D_x)(\zeta u)\|^2$$

$$= \sum_{|p+q| \leqq N_n} \|c_{p,q}^n r'_{p,q}(x,\ D_x) \langle D_x \rangle^k \langle D_x \rangle^{-k} (\zeta u)\|^2$$

$$\leqq \mathit{const.} \sum_{|p+q| \leqq N_n} \|c_{p,q}^n r'_{p,q}(x,\ D_x) \langle D_x \rangle^k \|^2_{L^2 \to L^2}$$

$$\leqq \mathit{const.} \sum_{q,p} (2C/M)^{2|p+q|} n^{-2s}$$

$$\leqq \mathit{const.}\ n^{-2s},$$

where *const.*s are independent of n.

On the other terms of the right hand side, we have the following: Let us observe that

$$\|\beta_{n(p+\nu)}\alpha_n^{(p+\nu)}(\zeta u)\| \leqq \sup|\beta_{n(p+\nu)}| \sup|\alpha_n^{(p+\nu)}| \times \{\int_{\Lambda_n}|\zeta u(\xi)|^2 d\xi\}^{1/2},$$

where $\Lambda_n=\{\xi;\ |\xi-n\xi_0|\leqq nr_0\}$. Therefore, it follows from the assumptions of Proposition 1 that, for any positive number s, there exists a constant C' (independent of n, p and q) such that the values of $\|\beta_{n(p+\nu)}\alpha^{(p+\nu)}(\zeta u)\|$ are estimated by $n^{-s-|p+\nu|}C'(CN_n)^{|p+q|}$.

Therefore, if we take M so that $(2d)(2C/M)^2<1/2$, then

$$\sum_{|p+q|\leqq N_n}\|c_{p,q}^n\beta_{n(p+\nu)}\alpha_n^{(p+\nu)}(\zeta u)\|^2$$
$$\leqq C_1\sum_{p,q}(2C/M)^{2|p+q|}n^{-2s-2|\nu|}\leqq 2C_1 n^{-2s-2|\nu|},$$

where C_1 is a constant independent of n.

Combining the above arguments, we have the assertion of Proposition 1.

Proof of Proposition 2. Let $\zeta\in C_0^\infty$ be a function satisfying $\zeta\subset\subset\beta_n$ (i.e. $\beta_n=1$ in the support of ζ). First, we shall show that $\|\alpha_n(D)\zeta u\|=0(n^{-s})$ as $n\to\infty$, when $S_n^M(u)=0(n^{-2s})$ holds. Let us consider

$$\|\alpha_n(D)\zeta u\|=\|\alpha_n(D)\zeta(\beta_n u)\|$$
$$\leqq\sum_{|\nu|\leqq N_0}\nu!^{-1}\sup|\zeta_{(\nu)}|\|\alpha_n^{(\nu)}\beta_n u\|+\|r''(x,\ D_x)\beta_n u\|.$$

In the above inequality, we have $\|\alpha_n^{(\nu)}\beta_n u\|=0(n^{-s-|\nu|}(\log n)^{|\nu|})$ from the hypothesis. So we must consider only the last term. Let us now take a function $\tilde{\zeta}\in C_0^\infty$ so that $\tilde{\zeta}=1$ in the support of β_n. Then, for some k, $\tilde{\zeta}u\in H^{-k}$. Therefore, by the same argument as in the proof of Lemma 2 (i), we can show that

$$\|r''(x,\ D_x)\beta_n u\|=\|r''(x,\ D_x)\beta_n\langle D_x\rangle^k\langle D_x\rangle^{-k}\tilde{\zeta}u\|$$
$$=0(n^{-s}),$$

provided N_0 satisfies the inequality $k-N_0-1\leqq -s$.

Now, we remark that, for the proof of Proposition 2, it suffices to show that, for any positive number s,

$$\sup\ |\alpha_n(\xi)\widehat{\zeta u}(\xi)|=0(n^{-s})\ \text{as}\ n\to\infty,$$

from the assumption of Proposition 2. To see this, let us observe that

$$|\alpha_n(\xi)\widehat{\zeta u}(\xi)|\leqq\|\alpha_n(D)\zeta u\|_{L^1}$$
$$\leqq const.\ \|(1+|x|^2)^l\alpha_n(D)\zeta u\|,$$

where l is a integer satisfying $l>d/2$. Furthermore,

$$(1+|x|^2)^l\alpha_n(D)(\zeta u)$$

$$= \sum_{|\kappa|\leqq 2l} (-1)^{|\kappa|} \kappa!^{-1} \alpha_n^{(\kappa)} [(-i\partial_x)^{\kappa}(1+|x|^2)^l](\zeta u)$$

$$= \sum_{|\kappa|\leqq 2l} \alpha_n^{(\kappa)} \zeta_\kappa u,$$

where we denote $\zeta_\kappa = (-1)^{|\kappa|}\kappa!^{-1}[(-i\partial_x)^{\kappa}(1+|x|^2)^l]\zeta$.

Thus we have

$$|\alpha_n(\xi)\widehat{\zeta u}(\xi)| \leqq const. \sum_{|\kappa|\leqq 2l} \|\alpha_n^{(\kappa)}\zeta_\kappa u\|.$$

In the last inequality, we can show that, for any positive number s, $\|\alpha_n \zeta_\kappa u\| = 0(n^{-s-|\kappa|}(\log n)^{|\kappa|})$ by the same argument as in the forepart of the proof, which completes the proof.

§4. Appendix (Proof of Theorem 2).

The proof of Theorem 2 is quite analogeous to that of Theorem 1. We shall show that, provided $a(x, D_x)u = f$ and $a(x, D_x)$ satisfies the condition (A.3), $S_n^M(f) = 0(n^{-2s})$ implies $S_n^M(u) = 0(n^{-2s})$ for any given positive number s.

Let us first operate $\alpha_n^{(p)}(D)\beta_{n(q)}(x)$, (p, q) satisfying $|p+q| \leqq N_n$, to the both sides of the equation $a(x, D_x)u = f$. Then

$$\alpha_n^{(p)}\beta_{n(q)} a u = \alpha_n^{(p)}\beta_{n(q)} f.$$

This implies

$$a u_{p,q} + \sum_{0<|\nu+\mu|\leqq N_0} \frac{(-1)^{|\nu|}}{\nu!\mu!} a_{(\mu)}^{(\nu)} u_{p+\mu, q+\nu} + r_{q,q} u = f_{p,q}.$$

Here we have used the same notations as in the proof of Theorem 1. Therefore we have

$$\begin{aligned} (4.1) \qquad & \|a u_{p,q}\|^2 \\ & \leqq const. \{ \sum_{0<|\nu+\mu|<m} \|a_{(\mu)}^{(\nu)} u_{p+\mu, q+\mu}\|^2 + \sum_{m\leqq|\nu+\mu|\leqq N_0} \|a_{(\mu)}^{(\nu)} u_{p+\mu, q+\nu}\|^2 \\ & \quad + \|r_{p,q} u\|^2 + \|f_{p,q}\|^2 \} \\ & = const. \ \{J_1 + J_2 + \|r_{p,q} u\|^2 + \|f_{p,q}\|^2\}, \end{aligned}$$

where *const.* means a constant independent of n, p and q.

Almost the same arguments as in the proof of Lemma 2 yield the following:

Lemma 3. (i) *For any positive number s, there exists a positive integer N_0 such that*

$$\sum_{|p+q|\leqq N_n} \|c_{p,q}^n r_{p,q} u\|^2 = 0(n^{-2s}) \ as \ n \to \infty.$$

(ii) *For any positive number ε, there exists a constant C_ε such that*

$$J_1 \leqq \sum_{0<|\nu+\mu|<m} \{(\log n)^{-|\nu+\mu|} n^{|\mu|}\}^2 \{\varepsilon \|a u_{p+\mu, q+\nu}\|^2 + C_\varepsilon \|u_{p+\mu, q+\nu}\|^2\},$$

and

$$J_2 \leqq \sum_{m \leqq |\nu+\mu| \leqq N_0} \{(\log n)^{-m} n^{m-|\nu|}\}^2 \{\varepsilon \| a u_{p+\mu,q+\nu} \|^2 + C_\varepsilon \| u_{p+\mu,q+\nu} \|^2\}.$$

Let us continue the proof of Theorem 2. Observe now that $(\log n)^{-|\nu+\mu|} n^{|\mu|} c^n_{p,q} = M^{|\nu+\mu|} c^n_{p+\mu,q+\nu}$ and, for (ν, μ) satisfying $|\nu+\mu| \geqq m$, $(\log n)^{-m} n^{m-|\nu|} c^n_{p,q} \leqq M^{|\nu+\mu|} c^n_{p+\mu,q+\nu}$ (when n is large). So, in view of (4.1) and Lemma 3, we have

(4.2) $\| a(c^n_{p,q} u_{p,q}) \|^2$

$$\leqq const.\ M^{2N_0} \{\varepsilon \sum_{0 < |\nu+\mu| \leqq N_0} \| a(c^n_{p+\mu,q+\nu} u_{p+\mu,q+\nu}) \|^2 + C_\varepsilon \| c^n_{p+\mu,q+\nu} u_{p+\mu,q+\nu} \|^2\}$$

$+$*negligible terms.*

Let us choose $\varepsilon > 0$ sufficiently small and sum up the both sides of (4.2) with respect to (p, q) satisfying $|p+q| \leqq N_n - N_0$. Then, the first terms on the right hand side of (4.2) will be absorbed into the left hand side, that is, we have (also observe that $\| a(c^n_{p,q} u_{p,q}) \|^2$ with (p, q) satisfying $N_n - N_0 \leqq |p+q| \leqq N_n$ are negligible)

$$\sum_{|p+q| \leqq N_n} \| a(c^n_{p,q} u_{p,q}) \|^2 \leqq const.\ S^M_n(u) + 0(n^{-2s}). \tag{4.3}$$

In view of the condition (A.3), it is clear that

$$(\log n)^{2m}\ S^M_n(u) \leqq const. \sum_{|p+q| \leqq N_n} \| a(c^n_{p,q} u_{p,q}) \|^2,$$

when n is large. Therefore, it follows from (4.3) that

$$S^M_n(u) = 0(n^{-2s}) \text{as } n \to \infty,$$

which completes the proof. q.e.d.

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References

[1] L. Hörmander, The analysis of linear partial differential operators III, Springer Verlag, 1985.

[2] T. Hoshiro, A property of operators characterized by iteration and a necessary condition for hypoellipticity, J. Math. Kyoto Univ **27** (1987) 401-416.

[3] S. Kusuoka and D. Strook, Applications of the Malliavin calculus, part II, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., **32** (1985), 1–76.

[4] S. Mizohata, On the Cauchy problem, Lectures delivered at the Wuhan University, Academic Press (Notes and Reports in Math. in Science and Engineering Vol. 3) 1986.

[5] S. Mizohata, On analytic regularities, Sèminaire de Vaillant à Paris VI, (1984).

[6] Y. Morimoto, Non-hypoellipticity for degenerate elliptic operators, Publ. RIMS Kyoto Univ., **22** (1986) 25–30.

[7] Y. Morimoto, A criterion for hypoellipticity of second order differential operators, preprint.

[8] Y. Morimoto, Criteria for hypoellipticity of differential operators, part II, Publ RIHS Kyoto Univ. **22** (1986) 1129-1154.