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On the contraction of the Teichmüller metrics

Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

By

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Introduction and main results

The universal Teichmüller space $T(1)$ can be represented as a quotient space of QS by the Möbius group $PSL(2,\mathbf{R})$, where QS is the group of all quasi-symmetric homeomorphisms of a circle. But QS contains another topological subgroup, which is much larger than $PSL(2,\mathbf{R})$, the subgroup S of symmetric homeomorphisms. S can be defined as the closure with respect to the quasi-symmetric topology of the group of real analytic homeomorphisms of the circle. Recently, Gardiner-Sullivan showed that $QS\mathrm{mod}S$ also have a natural complex Banach manifold structure and a natural quotient metric $\bar{d}$, which we also call the Teichmüller metric on $QS\mathrm{mod}S$, coming from the Teichmüller metric d on $T(1)$.

Since the manifold $QS\mathrm{mod}S$ is also universal in a sense (cf. [3], and also see [4]), it is important to investigate where and how extent the quotient map π contracts the metrics.

We recall some definitions. First, in $T(1)$, the Teichmüller metric can be described by using extremal quasiconformal mappings. Fix a normalized quasiconformal mapping f of the unit disk D onto itself. And denote by μ_f the complex dilatation of f. Set

$$k_f=\|\mu_f\|_\infty=\operatorname*{ess.sup}_{z\in D}|\mu_f(z)|$$

and

$$k_0(f)=\inf_g k_g\ ,$$

where g moves all quasiconformal mappings of D with the same boundary value as f.

We say that f is extremal (in $T(1)$-sense) if $k_f=k_0(f)$. Recall that the Teichmüller distance $d([f],[g])$, from a point $[g]$ to another point $[f]$ in $T(1)$, is equal to

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$$\frac{1}{2}\log\frac{1+k_0(g\circ f^{-1})}{1-k_0(g\circ f^{-1})}\ .$$

Similarly, denote

$$\bar{k}_f=\inf_U \operatorname{ess.sup}_{z\in U} |\mu_f(z)|\ ,$$

where U moves all neighborhoods of ∂D in D. (Thus $\bar{k}_f$ is called the boundary dilatation of f.) And set

$$\bar{k}_0(f)=\inf_g \bar{k}_g$$

where g moves all quasiconformal mappings of D with the same boundary value as f.

We say that f is extremal in QSmod$S$-sense if $\bar{k}_0(f)=\bar{k}_f$. Recall that the Teichmüller distance $\bar{d}(\pi[f],\pi[g])$, from a point $\pi[f]$ to another point $\pi[g]$ in QSmod$S$, is equal to

$$\frac{1}{2}\log\frac{1+\bar{k}_0(g\circ f^{-1})}{1-\bar{k}_0(g\circ f^{-1})}\ .$$

Now the principle of Teichmüller contraction ([2]) concerns a curve $C_\mu=\{[f^t]\mid |t|<1\}$ or $\pi C_\mu=\{\pi[f^t]\mid |t|<1\}$, where $\mu_{f^t}=t\mu/\|\mu\|_\infty$ with a given Beltrami coefficient μ. Such curves are called Beltrami lines. It is known that such a curve is a geodesic if μ is extremal [11]. Moreover, for extremal μ in $T(1)$- or QSmod$S$-sense, the natural mapping I_μ from the open interval $(-1, 1)$ with the Poincaré metric onto C_μ or πC_μ with the Teichmüller metric is an isometry.

Teichmüller contraction says that, if the mapping I_μ fails to preserve distance between two points, then it is strictly contracting at all pairs of points on the same Beltrami line and within a specified distance from the two given points. See the next section. This property of the mapping I_μ is called a coiling property by Sullivan [17].

Relating to these phenomena, it is interesting to discuss the following

Problem. For what kind of points $[f]\in T(1)$, does the distance 0 to $[f]$ really contract under the projection π?

This problem has been investigated implicitly by many authors. As Gardiner and Sullivan pointed in [2] and [3] that, Strebel's frame mapping theorem (Theorem A below) implies the following

Proposition 1. *Let $[f]\in T(1)$ and suppose that $\bar{d}(0,\pi([f]))<d(0,[f])$. Then $[f]$ contains a Teichmüller mapping of finite type.*

On the other hand, even in the case that the point corresponds to a Teichmüller mapping of finite type, whether the distance contracts or not is a very delicate problem, and remains unsettled. At least, we know the following

Reich's example (cf. [12], [14]). In $\Omega=\{w=u+iv \mid 0<v<u^{\alpha},\ 0<u<A\}$, where $\alpha>1$, $0<A<\infty$. Suppose that $w=h(z)$ maps D conformally onto Ω, and define μ on D, by

$$\mu(z)\overline{dz}/dz=t\overline{dw}/dw$$

with a fixed positive $t<1$. Then f with the complex dilatation μ is a Teichmüller mapping of finite type. On the other hand, $\bar{d}(0,\pi([f]))=d(0,[f])$.

In fact, set $g_n(z)=n^{\alpha+1}e^{-nz}$ on Ω, and define φ_n by

$$\varphi_n(z)dz^2=g_n(w)dw^2\ ,$$

Then we can see that $\{\varphi_n\}$ is a degenerating Hamilton sequence for μ. Hence by Theorem B below, we conclude the assetion.

In this paper we will give a condition under which the projection is really a contraction. And using Gardiner's results (Principle of Teichmüller contraction), we also give an estimate of contraction.

For this purpose, let B be the set of all functions φ that are holomorphic on D and satisfy that

$$\|\varphi\|_1=\iint_D|\varphi|dxdy<\infty\ ,$$

and let $C(B)$ denote the infimum of the set of all $C\in(0,\infty]$ such that

$$\iint_D|\varphi|dxdy\leq C\iint_D|\mathrm{Re}\varphi|dxdy\ , \tag{1}$$

for every $\varphi\in B$ with $\mathrm{Im}\varphi(0)=0$. Clearly, $C(B)\geq 1$.

Remark. G. H. Hardy and J. E. Littlewood [6] proved that $C(B)<\infty$, and M. Ortel and W. Smith [13] gave a simple proof that $C(B)<20\sqrt{2}$. The better estimate due to S. Axler [1] is that $C(B)\leq 7$.

Next for every θ such that $\frac{\pi}{2}<\theta<\frac{\pi}{2}+\arcsin\left(\frac{1}{2C(B)-1}\right)\ (<\pi)$, $\sum_{\theta}$ denotes the subset of $T(1)$ consisting of elements $[f]$ which correspond to Teichmüller mappings of finite type whose complex dilatations $\mu=\mu_f$ satisfy the following condition:

There is a positive $\rho<1$ such that $\mu(z)=0$ or

$$\frac{\mu(z)}{|\mu(z)|}\in\{e^{it}\mid |t|\leq\theta\}$$

for every z with $\rho<|z|<1$.

Theorem 1. *For every element $[f]$ belonging to $\sum_{\theta}$ with $\frac{\pi}{2}<\theta<\frac{\pi}{2}+\arcsin\left(\frac{1}{2C(B)-1}\right)$, it follows that*

$$\bar{d}(0,\pi([f])) < d(0,[f]) .$$

Corollary (on contraction). *Under the same circumstance as in Theorem 1, we set* $\lambda = \bar{d}(0,\pi([f]))/d(0,[f])$ *and* $k = \|\mu_f\|_\infty$. *Fix* $k' < 1$ *and let* f^t *be the quasiconformal mapping of* D *onto itself such that* $\mu_{f^t} = (t/k)\mu_f$ *for every* $t \in [0,k')$. *Then* ($\lambda < 1$ *and*) *and there exists* $\lambda' < 1$ *depending only on* k, k', *and* λ *such that*

$$\bar{d}(0,\pi([f^t])) \leq \lambda' d_P(0,t) ,$$

for every t *with* $0 \leq t \leq k'$, *where* d_P *denotes the Poincaré metric on the unit disk.*

Finally, it is very interesting to solve the following problem.

Problem. Can the equivalence class of an arbitrary extremal quasiconformal mapping in QSmod$S$-sense contain an extremal mapping of Teichmüller type? And if so, what kind of order condition does the corresponding quadratic differential satisfy? (Also, see [9].)

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1. Preliminaries and known results

We start with the following frame mapping theorem due to K. Strebel [16]. Let h be an orientation-preserving homeomorphism of ∂D onto itself which admits a quasiconformal extension $\bar{h}$ into an interior neighborhood. Such a mapping $\bar{h}$ is called a frame mapping or more accurately an interior frame mapping associated with h. The infimum of the maximal dilatations of all frame mappings associated with h is called the (interior) dilatation of the homeomorphism h.

Theorem A (Frame mapping theorem). *Let* h *be an orientation preserving homeomorphism of* ∂D *onto itself which admits a quasiconformal extension into* D. *Suppose that the dilatation of* h *is smaller than the maximal dilatation* K_0 *of an extremal mapping* f_0 *for the boundary value* h.

Then every Hamilton sequence $\{\varphi_n\}$ *for the complex dilatation* μ_{f_0} *of* f_0 *converges in* L^1*-norm to a uniquely determined holomorphic differential* φ_0 *with* $\|\varphi_0\|_1 = 1$. *Consequently, the complex dilatation* μ_{f_0} *has the form* $k_0\bar{\varphi}_0/|\varphi_0|$ *with* $k_0 = (K_0 - 1)/(K_0 + 1)$.

Here note that, for every $[f] \in T(1)$, the dilatation of $f|_{\partial D}$ is nothing but $\bar{k}_0(f)$. Hence Theorem A implies Proposition 1.

Next the following Theorem is due to F. P. Gardiner [2], which plays a fundamental role in this paper.

Theorem B. (The Hamilton-Reich-Strebel condition for extremality in $QS \bmod S$). *For every* $[f] \in T(1)$, $\bar{k}_f = \bar{k}_0(f)$ *if and only if*

$$\sup_{\{\varphi_n\}} \limsup_{n\to\infty} \left| \mathrm{Re} \int\int \varphi_n \mu_f dxdy \right| = \bar{k}_f \ ,$$

where the supremum is taken over all degenerating sequences $\{\varphi_n\}$ *for* μ_f *in* B_1.

Here $B_1 = \{\varphi \in B \mid \|\varphi\|_1 = 1\}$ and a degenerating sequence means that it converges to zero uniformly on compact subsets of D.

Now recall that, to characterize the complex dilatation of an extremal quasiconformal mapping, the following fundamental theorem due to R. Hamilton [5], E. Reich, and K. Strebel [15] is very useful.

Theorem C. *A Beltrami coefficient* μ *is extremal if and only if one of the following statements holds*:

1) *There exist* $\varphi \in B_1$ *and* $k \in [0,1)$ *such that* $\mu = k\bar{\varphi}/|\varphi|$ *for almost everywhere on* D.
2) *There is a degenerating sequence* $\{\varphi_n\}$ *in* B_1 *such that*

$$\lim_{n\to\infty} \left| \int\int_D \varphi_n \mu \, dxdy \right| = \|\mu\|_\infty \ .$$

Finally we cite the following Principle of Teichmüller contraction due F. P. Gardiner [2], it gives the motivation of this research.

Principle of Teichmüller contraction. *Assume* $\|\mu\|_\infty = 1$, $0 < k_1 < k_2 < 1$, *and* $d(0, [f^{k_1}]) \leq \lambda_1 d_p(0,k_1)$ *or* $\bar{d}(0, \pi([f^{k_1}])) \leq \lambda_1 d_p(0,k_1)$ *with some* $\lambda_1 < 1$, *where and in the sequel,* f^k *is the quasiconformal mapping of* D *onto itself such that* $\mu_f = k\mu$ *for every positive* $k < 1$. *Then there exists a* $\lambda_2 < 1$ *depending only on* k_1, k_2, *and* λ_1 *such that*

$$d(0,[f^k]) \leq \lambda_2 d_p(0,k) \quad \text{or} \quad \bar{d}(0,\pi([f^k])) \leq \lambda_2 d_p(0,k)$$

respectively, for all k *with* $0 \leq k \leq k_2$.

2. Proofs of main results.

Our proof of Theorem 1 also give a general criterion for extremality of quasiconformal mappings. Such a criterion is interesting in itself, and has been investigated by many mathematicians. For example, see the works made by M. Ortel and W. Smith [13], X. Z. Huang [8], K. Strebel [16], Z. Li [10], etc.

To state the result, we set

$$S^k[\theta_1, \theta_2] = \{re^{it} \mid 0 \leq r \leq k, \ \theta_1 \leq t \leq \theta_2\} \ ,$$

where $0 < k \leq \infty$ and $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$. Then we have the following

Theorem 2. *Let κ be a bounded measurable function on D with $\|\kappa\|_\infty = k < 1$ and $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \arcsin\left(\frac{1}{2C(B)-1}\right)$. Suppose that there exists $k' < k$ and $0 < \rho < 1$, such that $\kappa(z) \in S^k[-\theta, \theta] \cup S^{k'}[\theta, 2\pi - \theta]$ for almost every $z \in D \setminus \{|z| \leq \rho\}$. Then κ is an extremal dilatation if and only if there exists $\varphi \in B_1$ such that*

$$\kappa = k\frac{\overline{\varphi}}{|\varphi|}$$

for almost everywhere on D.

Theorem 2 can be shown by the same way as in [8]. Namely, Theorem 2 follows by Theorem C and the following Lemma, which is also the key for the proof of Theorem 1.

Lemma. *Let κ be a bounded measurable function on D with $\|\kappa\|_\infty = 1$, and fix θ with $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \arcsin\left(\frac{1}{2C(B)-1}\right)$. Suppose that there exists $k' < 1$ and $0 < \rho < 1$ such that $\kappa(z) \in S^1[-\theta, \theta] \cup S^{k'}[\theta, 2\pi - \theta]$ for almost every $z \in D \setminus \{|z| \leq \rho\}$.*

Then for every degenerating sequence $\{\varphi_n\}$ it follows that

$$\limsup_{n\to\infty} \left| \iint_D \varphi_n \kappa \, dx dy \right| < 1$$

Proof of Lemma. We may assume that, for each n,

$$\varphi_n(0) = 0 \quad \text{and} \quad \iint_D \varphi_n \kappa \, dx dy > 0 \ .$$

Choose θ_1 so that

$$\pi/2 < \theta < \theta_1 < \pi/2 + \arcsin\left(\frac{1}{2C(B)-1}\right) .$$

Set

$$\Omega_n = \{z \in D \mid \varphi_n(z) \in S^\infty[\theta_1, 2\pi - \theta_1]\} \ ,$$

and

$$M = \{z \in D \mid \kappa(z) \in S^1[-\theta, \theta]\} \ .$$

Then, for every $z \in \Omega_n \cap M$, we have

$$\mathrm{Re}\,\varphi_n \kappa \leq |\varphi_n(z)\kappa(z)| \cos(\theta_1 - \theta) \leq |\varphi_n(z)| \cos(\theta_1 - \theta) \ .$$

Hence, for each n, we obtain

$$\iint_D \varphi_n \kappa \, dx dy$$

$$=\mathrm{Re}\iint_{D\setminus\Omega_n}\varphi_n\kappa dxdy+\mathrm{Re}\iint_{\Omega_n\cap M}\varphi_n\kappa dxdy+\mathrm{Re}\iint_{\Omega_n\setminus M}\varphi_n\kappa dxdy$$
$$\leq\iint_{D\setminus\Omega_n}|\varphi_n|dxdy+\cos(\theta_1-\theta)\iint_{\Omega_n\cap M}|\varphi_n|dxdy+k'\iint_{\Omega_n\setminus M}|\varphi_n|dxdy$$
$$\leq\iint_{D\setminus\Omega_n}|\varphi_n|dxdy+l\iint_{\Omega_n}|\varphi_n|dxdy\ ,$$

where $l=\max\{\cos(\theta_1-\theta),\ k'\}<1$.

Now suppose that

$$\limsup_{n\to\infty}\iint_D\varphi_n\kappa dxdy=\|\kappa\|_\infty=1$$

Then since $\|\varphi_n\|_1=1$, the above inequality gives that

$$\liminf_{n\to\infty}\iint_{\Omega_n}|\varphi_n|dxdy=0\ ,$$

which in turn gives a contradiction.

In fact, set $J=\{|z|\leq\rho\}$,

$$P_n=\{z\in D\mid \mathrm{Re}\varphi_n(z)<0\}\ ,$$

$Q_n=D\setminus P_n$, and $G_n=P_n\setminus(J\cup\Omega_n)$. Then, since $\varphi_n(0)=0$, we have

$$\iint_D|\mathrm{Re}\varphi_n|dxdy$$
$$=\iint_{P_n}|\mathrm{Re}\varphi_n|dxdy+\iint_{Q_n}|\mathrm{Re}\varphi_n|dxdy=2\iint_{P_n}|\mathrm{Re}\varphi_n|dxdy$$
$$=2\iint_{J\cap P_n}|\mathrm{Re}\varphi_n|dxdy+2\iint_{G_n}|\mathrm{Re}\varphi_n|dxdy+2\iint_{\Omega_n\setminus J}|\mathrm{Re}\varphi_n|dxdy$$
$$\leq2\iint_J|\varphi_n|dxdy+2\iint_{G_n}|\cos\theta_1\varphi_n|dxdy+2\iint_{\Omega_n}|\mathrm{Re}\varphi_n|dxdy$$
$$\leq2\iint_J|\varphi_n|dxdy+2|\cos\theta_1|\iint_{P_n}|\varphi_n|dxdy+2\iint_{\Omega_n}|\mathrm{Re}\varphi_n|dxdy$$
$$=2\iint_J|\varphi_n|dxdy+2|\cos\theta_1|\left[\iint_D|\varphi_n|dxdy-\iint_{D\setminus P_n}|\varphi_n|dxdy\right]$$
$$+2\iint_{\Omega_n}|\mathrm{Re}\varphi_n|dxdy\ .$$

Here since

$$\iint_{D\setminus P_n}|\varphi_n|dxdy\geq\iint_{Q_n}|\mathrm{Re}\varphi_n|dxdy=1/2\iint_D|\mathrm{Re}\varphi_n|dxdy\ ,$$

we have

$$(1+|\cos\theta_1|)\iint_D|\mathrm{Re}\varphi_n|dxdy$$
$$\leq2\iint_J|\varphi_n|dxdy+2|\cos\theta_1|\iint_D|\varphi_n|dxdy+2\iint_{\Omega_n}|\mathrm{Re}\varphi_n|dxdy\ .$$

Now by recalling the definition of $C(B)$ and that $|\cos\theta_1|=\sin(\theta_1-\pi/2)$ in this case, the above inequality implies that

$$\{1+(1-2C(B))\sin(\theta_1-\pi/2)\}\int\int_D|\mathrm{Re}\varphi_n|dxdy$$
$$\leq 2\int\int_J|\varphi_n|dxdy+2\int\int_{\Omega n}|\varphi_n|dxdy \ .$$

Here $\{1+(1-2C(B))\sin(\theta_1-\pi/2)\}>0$,

$$\liminf_{n\to\infty}\int\int_{\Omega n}|\varphi_n|dxdy=0 \ ,$$

and, since $\{\varphi_n\}$ is degenerating, $\lim_{n\to\infty}\int\int_J|\varphi_n|dxdy=0$. Hence we have

$$\liminf_{n\to\infty}\int\int_D|\mathrm{Re}\varphi_n|dxdy=0 \ .$$

So, by using inequality (1), we conclude that

$$\liminf_{n\to\infty}\int\int_D|\varphi_n|dxdy=0 \ ,$$

which contradicts the assumption that $\varphi_n \in B_1$ for every n.

Thus we have

$$\limsup_{n\to\infty}\int\int_D\varphi_n\kappa dxdy<1 \ ,$$

which implies the assertion.

Proof of Theorem 1. Suppose that $[f]\in\sum_\theta$. Then by the above Lemma and Theorem B, we conclude that $\bar{k}_0(f)<k_0(f)$.

In fact, suppose that $\bar{k}_0(f)=k_0(f)$. Then we have $\|\mu_f\|_\infty=k_0(f)=\bar{k}_0(f)$. Hence Theorem B implies that

$$\sup_{\{\varphi_n\}}\limsup_{n\to\infty}\left|\mathrm{Re}\int\int_D\varphi_n\mu_f dxdy\right|=\|\mu_f\|_\infty \ ,$$

where the supremum is taken over all degenerating sequences $\{\varphi_n\}$ in B_1. Then, by the diagonal argument, we can find a degenerating sequence in B_1 such that

$$\lim_{n\to\infty}\mathrm{Re}\int\int_D\varphi_n\mu_f dxdy=\|\mu_f\|_\infty \ ,$$

which is impossible by the above Lemma.

Thus we conclude $\bar{k}_0(f)<k_0(f)$, which is equivalent that $\bar{d}(0,\pi([f]))<d(0,[f])$.

Proof of Corollary. Suppose that $[f]\in\sum_\theta$. Then Theorem 1 implies that

$$\bar{d}(0,\pi([f])<d(0,[f])=d_p(0,k) \ ,$$

namely, that $\lambda<1$.

Thus the assertion follows by Principle of Teichmüller contraction.

3. Examples

Suppose that $\varphi\in B_1$ has positive real part, namely, $\mathrm{Re}\varphi(z)>0$ for every $z\in D$. Let f be the quasiconformal mapping of D onto itself such that $\mu_f=t\overline{\varphi}/|\varphi|$ with positive $t<1$. Then by Theorem 1, we have

$$\bar{d}(0,\pi([f]))<d(0,[f])=d_p(0,t).$$

Typical examples of such φ are

$$\frac{e^{ia}+z}{e^{ia}-z}$$

with real a, and positive linear combinations of them. Another examples are

$$1+\psi/\|\psi\|_\infty \quad \text{and} \quad \exp(\pi\psi/2\|\psi\|_\infty) \ ,$$

where ψ is a bounded holomorphic function on D.

Next fix θ as in Theorem 1, and set

$$\varphi(z)=\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)^{2\theta/\pi}$$

or

$$\varphi(z)=\exp(\theta z^n)$$

with $n\geq 1$. As before, let f be the quasiconformal mapping of D onto itself such that $\mu_f=t\overline{\varphi}/|\varphi|$ with positive $t<1$. Then $[f]\in\Sigma_\theta$, but belongs to no $\Sigma_{\theta'}$ for every θ' with $(\pi/2<)\,\theta'<\theta$.

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