J. Math. Kyoto Univ. (JMKYAZ)
35-1 (1995) 105-114

Local smooth solutions of the relativistic Euler equation

By

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1. Introduction

The motion of a relativistic perfect fluid in the Minkowski space-time is governed by

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\Big(\dfrac{\rho c^2+p}{c^2-v^2}-\dfrac{p}{c^2}\Big)+\displaystyle\sum_{k=1}^{3}\dfrac{\partial}{\partial x_k}\Big(\dfrac{\rho c^2+p}{c^2-v^2}v_k\Big)=0\ , \\ \dfrac{\partial}{\partial t}\Big(\dfrac{\rho c^2+p}{c^2-v^2}v_i\Big)+\displaystyle\sum_{k=1}^{3}\dfrac{\partial}{\partial x_k}\Big(\dfrac{\rho c^2+p}{c^2-v^2}v_iv_k+p\delta_{ik}\Big)=0, \quad i=1,2,3\ . \end{cases}$$

Here c denotes the speed of light, p the pressure, (v_1,v_2,v_3) the velocity of the fiuid particle, ρ the mass-energy density of the fiuid (as measured in units of mass in a reference flame moving with the fluid particle) and $v^2=v_1{}^2+v_2{}^2+v_3{}^2$.

We assume the equation of state of the form

$$(1.2)\qquad p=a^2\rho\ ,$$

where a, the sound speed, is taken to be constant so that $0<a<c$. In particular, $a=c/\sqrt{3}$ arises in several important physical contexts. For detailed discussions of this setting, see J. Smoller and B. Temple [6].

Under the assumption (1.2), we can write the equation (1.1) as

$$(1.3)\qquad \begin{cases} \dfrac{\partial w_0}{\partial t}+\displaystyle\sum_{k=1}^{3}\dfrac{\partial w_k}{\partial x_k}=0\ , \\ \dfrac{\partial w_i}{\partial t}+\displaystyle\sum_{k=1}^{3}\dfrac{\partial f_i^k}{\partial x_k}=0, \quad i=1,2,3\ , \end{cases}$$

where

Communicated by Prof. Takaaki Nishida, May 25, 1994

$$w_0=\frac{c^4+a^2v^2}{c^2(c^2-v^2)}\rho\ ,\quad w_i=\frac{c^2+a^2}{c^2-v_2}\rho v_i\ ,\tag{1.4}$$
$$f_i^k=w_iv_k+a^2\rho\delta_{ik}\ ,\qquad i,k=1,2,3\ .$$

We shall solve the equation (1.3) for $t\geq 0$ and $x=(x_1,\ x_2,\ x_3)\in\mathbf{R}^3$ together with the initial conditions

$$\begin{cases}\rho|_{t=0}=\rho_0(x)\ ,\\ v_i|_{t=0}=v_{0i}(x),\quad i=1,2,3\ .\end{cases}\tag{1.5}$$

For the one-dimensional motions, Smoller and Temple [6] constructed global weak solutions, using Glimm's method [1]. However, no results have been known so far about the full-dimensional existence. Thus the aim of the present paper is to establish the existence of local smooth solutions of (1.3) and (1.5).

We note that in the limit $c\to\infty$, the system (1.3) reduces formally to the non-relativistic Euler equation

$$\begin{cases}\dfrac{\partial\rho}{\partial t}+\displaystyle\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(\rho v_k)=0\\ \dfrac{\partial}{\partial t}(\rho v_i)+\displaystyle\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(\rho v_iv_k+a^2\rho\delta_{ik})=0,\quad i=1,2,3\ .\end{cases}\tag{1.6}$$

It is well known that this system can be transformed to a symmetric hyperbolic system to which the Friedrichs-Lax-Kato existence theory of local smooth solutions is applicable, see, for example, Majda [4, §1.3]. Actually, several symmetrizers are known to (1.6), ([1], [2], [4], [5]), which lead to the local existence theorems in different function spaces.

In this paper, we will show that a symmetrizer exists also for the relativivtic case (1.3) which results in the

Theorem 1.1. *Suppose that the initial data ρ_0 and $(v_{01},\ v_{02},\ v_{03})$ belong to the uniformly local Sobolev space $H^s_{ul}=H^s_{ul}(\mathbf{R}^3)$, $s\geq 3$, and that there exist a positive constant $\delta(<1)$ such that*

$$\delta\leq\rho,\ v_0^2=v_{01}^2+v_{02}^2+v_{03}^2\leq(1-\delta)c^2\ .\tag{1.7}$$

Then, the system (1.3) has a unique solution

$$(\rho,v_1,v_2,v_3)\in C([0,T];H^s_{loc})\cap C^1([0,T];H^{s-1}_{loc})\ ,\tag{1.8}$$

with $\rho>0$ and $v^2<c^2$. Here $T>0$ depends only on δ and the H^s_{ul}-norm of the initial data.

To construct symmetrizers, instection is enough for the non-relativistic case (1.6), but it does not seen to work well for the present case (1.3). Instead, we shall follow the idea due to Godunov [1] which relies on the existence of a convex entropy function. Such an entropy function will be constructed in §3 and the symmetrization for (1.3) using this entropy function will be shown in §2. In §4, the convergence is established of solutions of the relativistic (1.3) to those of the non-relativistic (1.6) as the light speed c tends to infinity.

2. Symmetrization

Theorem 1.1 can be concluded if there is a change of variables

$$z=(\rho,v_1,v_2,v_3)\rightarrow(u_0,u_1,u_2,u_3) \quad (2.1)$$

which reduces the system (1.3) to a system of the form

$$A^0(u)\frac{\partial u}{\partial t}+\sum_{k=1}^{3}A^k(u)\frac{\partial u}{\partial x_k}=0 , \quad (2.2)$$

whose coefficent matrices $A^0(u)$ and $A^k(u)$, $k=1,2,3$ satisfy the condition

(2.3)
(*i*) *they are all read symmetric and smooth* u ,
(*ii*) $A^0(u)$ *is positive definite* .

The system (2.2) satisfying (2.3) is called a symmetric hyperbolic system.

We claim that one of such changes of variables is given by

$$\begin{cases} u_0=-\dfrac{c^3}{(c^2-v^2)^{1/2}}\rho^{-\theta}+c^2+a^2 , \\ u_j=\dfrac{c}{(c^2-v^2)^{1/2}}\rho^{-\theta}v_j, \quad j=1,2,3 , \end{cases} \quad (2.4)$$

where

$$\theta=\frac{a^2}{c^2+a^2} . \quad (2.5)$$

We shall check the condition (2.3). First, note that the map (2.1) with (2.4) is a diffeomorphism from $\Omega_z=\{\rho>0,\ v^2<c^2\}$ onto $\Omega_u=\{u_0<c^2+a^2,\ u^2=(u_1)^2+(u_2)^2+(u_3)^2<(u_0-c^2-a^2)^2/c^2\}$. By a straight but tedious computation, we can find the coefficients $A^0(u)=(A^0{}_{\alpha\beta})_{\alpha,\beta=0,1,2,3}$, $A^k(u)=(A^k{}_{\alpha\beta})_{\alpha\beta=0,1,2,3}$, $k=1,2,3$, as follows:

$$\begin{cases} A^0_{00}=A_1\rho^{\theta+1}, \quad A^0_{0i}=A^0_{i0}=A_2\rho^{\theta+1}v_i , \\ A^0_{ij}=A_3\rho^{\theta+1}v_iv_j+A_4\rho^{\theta+1}\delta_{ij}, \quad i,j=1,2,3 , \end{cases} \quad (2.6)$$

$$(2.7)\qquad \begin{cases} A^k_{00}=A_2\rho^{\theta+1} , \\ A^k_{0i}=A^k_{i0}=A_3\rho^{\theta+1}v_iv_k+A_4\rho^{\theta+1}\delta_{ik} , \\ A^k_{ij}=A_3\rho^{\theta+1}v_iv_jv_k+A_4\rho^{\theta+1}(v_i\delta_{jk}+v_i\delta_{ik}+v_k\delta_{ij}), \quad i,j=1,2,3 , \end{cases}$$

where

$$(2.8)\qquad \begin{cases} A_1=\dfrac{c^4+3a^2v^2}{c^3\theta(c^2-v^2)^{3/2}} , \quad A_2=\dfrac{c^4+2a^2c^2+a^2v^2}{c^3\theta(c^2-v^2)^{3/2}} , \\ A_3=\dfrac{c^2+3a^2}{c\theta(c^2-v^2)^{3/2}} , \quad A_4=\dfrac{c^2-a^2}{c(c^2-v^2)^{1/2}} . \end{cases}$$

These coefficents can be calculated by the chain rule and the formula

$$(2.9)\qquad \begin{cases} \dfrac{\partial\rho}{\partial u_0}=\dfrac{A_4}{a^2}\rho^{\theta+1} , \quad \dfrac{\partial\rho}{\partial u_j}=\dfrac{A_4}{a^2}\rho^{\theta+1}v_j , \\ \dfrac{\partial v_i}{\partial u_0}=A_5\rho^\theta v_i , \quad \dfrac{\partial v_i}{\partial u_j}=c^2A_5\rho^\theta\delta_{Ij}, \quad i, \ j=1,2,3 , \end{cases}$$

with

$$(2.10)\qquad A_5=c^{-3}(c^2-v^2)^{1/2} .$$

Clearly, (2.6) and (2.7) show that the matrices $A^0(u)$ and $A^0(u)$ are real symmetric and smooth in Ω_u. Let us show that $A^0(u)$ is positive definite Let $\Xi=(\xi_0,\xi)\in\mathbf{R}^4$ be a 4-vector with $\xi\in\mathbf{R}^3$ and $\|\Xi\|=\sqrt{\xi_0^2+\xi^2}$. We should calculate the inner product

$$(2.11)\qquad (A^0(u)\Xi|\Xi)=\rho^{\theta+1}J$$

where

$$(2.12)\qquad J=A_1\xi_0^2+2A_2\xi_0(v|\xi)+A_3(v|\xi)^2+A_4\xi^2 ,$$

A_j being those in (2.8). It is sufficient to show that

$$(2.13)\qquad J\geq\kappa\|\Xi\|^2 ,$$

with some positive constant κ. First, we write

$$J=A_1\Big(\xi_0+\frac{A_2}{A_1}(v|\xi)\Big)^2-\frac{1}{A_1}(A_2^2-A_1A_3)(v|\xi)^2+A_4\xi^2 .$$

Since

$$A_6\equiv\frac{1}{A_1}(A_2^2-A_1A_3)=\frac{(c^2+a^2)(c^4+4a^2c^2-a^2v^2)}{c(c^2-v^2)^{1/2}(c^4+3a^2v^2)}>0$$

and by Schwarz' inequality $(v|\xi)\leq v^2\xi^2$, we get

(2.14) $\quad J \geq \kappa_1 \xi^2$,

with

$$\kappa_1 = A^4 - A_6 v^2 = \frac{(c^2-v^2)^{1/2}(c^4-a^2v^2)}{c^3\theta(c^4+3a^2v^2)} > 0 .$$

On the other hand, decomposition $\xi^2 = |\xi - (\tilde{v}|\xi)\tilde{v}|^2 + (\tilde{v}|\xi)^2$ where $\tilde{v} = v/|v| \in S^2$ gives

$$\begin{aligned} J &= A_1\xi_0^2 + 2A_2\xi_0(\tilde{v}|\xi) + \left(A_3 + \frac{A^4}{v^2}\right)(\tilde{v}|\xi)^2 + A_4|\xi - (\tilde{v}|\xi)\tilde{v})|^2 \\ &\geq (A_3v^2 + A_4)\left((\tilde{v}|\xi) + \frac{A_2|v|\xi_0}{A_3v_2 + A^4}\right)^2 + \left(A_1 - \frac{A^2{}_2v^2}{A_3v^2 + A_4}\right)\xi_0^2 \\ &\geq \kappa_2\xi^2{}_0 , \end{aligned}$$

where

$$\kappa_2 = A_1 - \frac{A_2^2v^2}{A_3v^2 + A^4} = \frac{(c^2+a^2)(c^2-v^2)^{1/2}(c^4-a^2v^2)}{c^5(c^2a^2 + (c^2+2a^2)v^2)} > 0 .$$

This and (2.14) now given

$$J \geq \frac{1}{2}(\kappa_1\xi^2 + \kappa_2\xi_0^2) .$$

This shows that (2.3) (ii) is also astisfied, and hence, the Friedlichs-Kato-Lax theory works for the system (2.2) . Since the map (2.1) with (2.4) defines a diffeomorhism, we then conclude Theorem 1.1. We can say more, however. Given $\delta \in (0,1)$ and $c > 0$, put

(2.15) $\quad \Omega(\delta,c) = \{\delta \leq \rho \leq \delta^{-1}, v^2 \leq (1-\delta)c^2\}$.

It is seen that, for any $\delta \in (0,1)$ and $c_0 > 0$, κ_1 and κ_2 are bounded and bounded away from 0 uniformly for $c \geq c_0$ as well as for $z = (\rho, v_1, v_2, v_3) \in \Omega(\delta, c_0)$. Also, $A^\alpha(u)$ and any of their derivatives are uniformly bounded both for $c \geq c_0$ and $z \in \Omega(\delta, c_0)$. Hence, we have a strengthen version of Theorem 1.1.

Theorem 2.1. *For any numbers a_0, $c_0 > 0$ and $\delta_0 \in (0,1)$, there exist posi-tive constants C and T such that for each initial data $z_0 = (\rho_0, v_{01}, v_{02}, v_{03}) \in H^s$ satisfying*

$$\|z_0\|_{H^s_{ul}} \leq a_0, \quad z_0 \in \Omega(\delta_0, c_0) \textit{ for all } x \in \mathbf{R}^3 ,$$

and for each $c \geq c_0$, the Cauchy problem (1.3) with (1.5) posesses a unique solu-tion $z = (\rho, v_1, v_2, v_3)$ belonging to the class (1.8) and satisfying

(2.16) $\quad \|z(t)\|_{H^s_{ul}} \leq C,\ z(t) \in \Omega(\delta_0/2, c_0)$ *for all* $x \in \mathbf{R}^3$,

for all $t \in [0,T]$.

3. Strictly convex entropy function

Let us consider the system of conservation laws

$$(3.1)\qquad w_t+\sum_{k}^{N}(f^k(w))_{x_k}=0,\quad w=(w_1,w_2,\dots,w_m)\ .$$

A scalar function $\eta=\eta(w)$ is called an entropy function to (3.1) if there exist scalar functions, $q^k(w)$, $k=1,2,\dots, N$, satisfying

$$(3.2)\qquad D_w\eta(w)D_wf^k(w)=D_wq^k\ .$$

Here and in the sequel, D_wh is taken as a row vector in case h is a scalar function and is the Jacobi matrix case h is a vector valued function.

According to Godunov [1], (see also Kawashima-Shizuta [3]), if a strictly convex entropy function exists, then the transformation

$$(3.3)\qquad w\to u=D_w\eta(w)\ ,$$

is well-defined and reduces the system (3.1) to a symmetric hyperbolic system of the form (2.2) whose coefficients

$$(3.4)\qquad \begin{cases} A^0(u)=D_uw=(D_w^2\eta)^{-1}\ , \\ A^k(u)=D_uf^k=D_wf^kD_uw\ , \end{cases}$$

satisfy the condition (2.3).

In our case, (1.3) is of the form (3.1) with

$$(3.5)\qquad w=(w_0,w_1,w_2,w_3),\ f^k(w)=(w_k,f_1^k,f_2^k,f_3^k)\ ,$$

where w_0, w_k, f_i^k, $(i,k=1,2,3)$ are those in (1.4). Recall $z=(\rho, v_1, v_2, v_3)$. The map $z\to w$ is a diffeomorphism from $\Omega_z=\{\rho>0,\ v^2<c^2\}$ onto $\Omega_w=\{w_0>0,$ $(w_1, w_2, w_3)\in\mathbf{R}^3$. Specifically, using the formula,

$$(3.6)\qquad \begin{cases} \dfrac{\partial\rho}{\partial w_0}=M_1\ , & \dfrac{\partial\rho}{\partial w_j}=M_2v_j\ , \\ \dfrac{\partial v_i}{\partial w_0}=M_3\rho^{-1}v_i\ , & \dfrac{\partial v_i}{\partial w_j}=M_4\rho^{-1}v_iv_j+M_5\rho^{-1}\delta_{ij}\ , \end{cases}$$

where

$$(3.7)\qquad \begin{cases} M_1=\dfrac{c^2(c^2+v^2)}{c^4-a^2v^2}\ , \quad M_2=-\dfrac{2c^2}{c^4-a^2v^2}\ , \quad M_3=-\dfrac{c^2(c^2-v^2)}{c^4-a^2v^2}\ , \\ M_4=\dfrac{2\theta(c^2-v^2)}{c^4-s^2v^2}\ , \quad M_5=\dfrac{(c^2-v^2)}{c^2+a^2}\ , \end{cases}$$

we see that the Jacobi matrix D_wz is nonsingular with

(3.8) $$\det(D_w z) = \frac{1}{\rho^3} \frac{c^2 (c^2 - v^2)^4}{(c^2 + a^2)^3 (c^4 - a^2 v^2)} > 0 \ .$$

Rewrite (3.2) as

(3.9) $$D_z \eta B^k = D_z q^k, \ k = 1,2,3 \ ,$$

where

(3.10) $$B^k = D_w z D_z f^k = (b^k_{\alpha\beta})_{\alpha' \beta = 0,1,2,3} \ ,$$

are computed using (3.5) and (3.6) as

(3.11) $$\begin{cases} b^k_{00} = B_1 v_k, \ b^k_{0j} = B_2 \rho \delta_{kj} \ , \\ b^k_{i0} = B_3 \rho^{-1} v_k v_i + B_4 \rho^{-1} \delta_{ki} \ , \\ b^k_{ij} = B_5 v_i \delta_{kj} + v_k \delta_{ij} \ , \end{cases}$$

with

(3.12) $$\begin{cases} B_1 = \dfrac{c^2 (c^2 - a^2)}{c^4 - a^2 v^2} \ , \quad B_2 = \dfrac{c^2 (c^2 + a^2)}{c^4 - a^2 v^2} \ , \\ B_3 = -\dfrac{a^2 (c^2 - a^2)(c^2 - v^2)}{(c^2 + a^2)(c^4 - a^2 v^2)} \ , \\ B_j = \dfrac{a^2 (c^2 - v^2)}{c^2 + a^2} \ , \quad B_k = -\dfrac{a^2 (c^2 - v^2)}{c^4 - a^2 v^2} \ . \end{cases}$$

We shall solve (3.9) assuming that our entropy pair (η, q^k) is of the form

(3.13) $$\eta = H(\rho, v^2), \quad q^k = Q(\rho, v^2) v_k \ .$$

Then, setting $y = v^2$, the condition (3.9) reduces to the following set of equations for the functions H and Q.

(3.14) $$H_y = Q_y \ ,$$

(3.15) $$B_1 H_\rho + 2(B_3 y + B_4) \frac{1}{\rho} H_y = Q_\rho \ ,$$

(3.16) $$B_2 \rho H_\rho + 2 B_5 y H_y = Q \ .$$

From (3.14), there should exist a function $G = G(\rho)$ of ρ only such that $H = Q(\rho, y) + G(\rho)$. On the other hand, eliminating ρH_ρ from (3.15) and (3.16), and using (3.14), we have

(3.17) $$(c^2 + a^2) \rho Q_\rho - (c^2 - a^2) Q = 2a^2 (c^2 - y) Q_y \ .$$

This and (3.15) then yield

(3.18) $$\rho G_\rho = \frac{c^2 - y}{c^2 + a^2} \left(Q - \frac{c^2 + a^2}{c^2} \rho Q_\rho \right) \ .$$

or putting $q=(c^2-y)Q$,

$$(1-\theta)q-\rho q_\rho=c^2\rho G_\rho \ . \tag{3.19}$$

Since the right hand side is a function of ρ only, q must be of the form

$$q=\rho^{1-\theta}[g(\rho)+h(y)] \ , \tag{3.20}$$

where g and h are arbitrary functions. Substituting (3.20) into (3.17) or

$$\rho q_\rho-q=2\theta(c^2-y)q_y \ , \tag{3.21}$$

we get, with a constant K_0,

$$\rho g'(p)-\theta g(\rho)=\theta\{2(c^2-y)h'(y)+h(y)\}=-\theta K_0 \ , \tag{3.22}$$

whose solutions are

$$g(\rho)=K_2\rho^\theta+K_0,\ h(y)=K_1(c^2-y)^{1/2}-K_0 \ , \tag{3.23}$$

K_j's being arbitray constants. Now, substitution of (3.23) into (3.20) and then into (3.19) yields

$$G=-\frac{K_2\theta}{c^2}\rho+K_3 \ . \tag{3.24}$$

Thus we get

$$\eta=H=\frac{K_1}{(c^2-v^2)^{1/2}}\rho^{1-\theta}+K^2\left(\frac{1}{c^2-v^2}-\frac{\theta}{c^2}\right)\rho+K_3 \ , \tag{3.25}$$

$$Q=\frac{K_1}{(c^2-v^2)^{1/2}}\rho^{1-\theta}+\frac{K^2}{c^2-v^2}\rho \ . \tag{3.26}$$

For the later purpuse, we wish to choose the constants K_j, $j=1,2,3$, so that (3.25) converges, as $c\to\infty$, to the entropy function for the non-relativistic case (1.6) given by

$$\bar{\eta}=\frac{1}{2}\rho v^2+a^2\rho\log\rho \ , \tag{3.27}$$

which can be obtained exactly in the same way. The right choice is then found to be

$$K_1=-c(c^2+a^2),\quad K_2=c^4-a^4,\ K_3=0 \ , \tag{3.28}$$

with which (3.25) becomes

$$\eta=-\frac{c(c^2+a^2)}{(c^2-v^2)^{1/2}}\rho^{1-\theta}+\frac{(c^2+a^2)(c^4+a^2v^2)}{c^2(c^2-v^2)}\rho \ . \tag{3.29}$$

The change of variables (2.4) was derived from this η via (3.3), using (3.6). This η is strictly convex due to (3.4) since $A^0(u)$ is positive definite as was seen in the previous section.

4. Non relativistic limit

Now for the non-relativistic case (1.6), the symmetrizing variables associated with the entropy function (3.27) are

$$\text{(4.1)} \qquad \begin{cases} \bar{u}_0 = -\frac{1}{2}v^2 + a^2 \log\rho + a^2 , \\ \bar{u}_j = v_j, \qquad\qquad j=1,2,3 , \end{cases}$$

and the resulting system is

$$\text{(4.2)} \qquad \bar{A}^0(\bar{u})\bar{u}_t + \sum_{k=1}^{3} \bar{A}^k(\bar{u})\bar{u}_{xk} = 0 ,$$

with

$$\text{(4.3)} \qquad \begin{cases} \bar{A}^0_{00} = a^{-2}\rho,\ \bar{A}^0_{0j} = \bar{A}^0_{j0} = a^{-2}\rho v_j , \\ \bar{A}^0_{ij} = a^{-2}\rho v_i v_j + \rho\delta_{ij} , \end{cases}$$

and so on. The condition (2.3) can be easily checked to hold, so that the Friedlichs-Kato-Lax theory applies also to the system (4.2) and, as a conse-quence, to the non-relativistic Euler equation (1.6).

Let $z=(\rho,v_1,v_2,v_3)$ and $\bar{z}=(\bar{\rho},\bar{v}_1,\bar{v}_2,\bar{v}_3)$ be the solutions to (1.3) and (1.6), repectively, both for the same initial data $z_0=(\rho_0,v_{01},v_{02},v_{03})$. Let z_0 be as in Theorem 2.1. Then, we may conclude that $\bar{z}$ exists on the same time interval $[0,T]$ as z, $c>c_0$, belongs to the same class (1.8) and enjoys the same estimate (2.16). We shall show the

Theorem 4.1. *As $c\longrightarrow\infty$, z converges to $\bar{z}$ uniformly on $[0,T]$ in $H^{s-\varepsilon}_{loc}$ for any $\varepsilon>0$.*

Proof. It suffices to prove the theorem for the solutions u to (2.2) and $\bar{u}$ to (4.2). Put $\psi=u-\bar{u}$. Subtracting (4.2) from (2.2), we have

$$\text{(4.4)} \qquad A^0(u)\psi_t + \sum_{k=1}^{3} A^k(u)\psi_{xk} = \\ -\{A^0(u) - \bar{A}^0(\bar{u})\}\bar{u}_t - \sum_{k=1}^{3}\{A^k(u) - \bar{A}^k(\bar{u})\}\bar{u}_{xk} .$$

First, we know from the remark made above that the uniform estimates

$$\text{(4.5)} \qquad \begin{cases} \|u(t)\|_{H^s_{ul}},\ \|\bar{u}(t)\|_{H^s_{ul}},\ \|\bar{u}(t)\|_{H^{s-1}_{ul}} \le C_0 , \\ A^0(u(t)\Xi|\Xi),\ (\bar{A}^0(\bar{u}(t))\Xi|\Xi) \ge \kappa_0\|\Xi\| , \\ z(t),\ \bar{z}(t) \in \Omega(\delta_0/2,c_0) \text{ for all } x\in\mathbf{R}^3 , \end{cases}$$

hold for all $c\ge c_0$ and for all $t\in[0,T]$, with some constants C_0, $\kappa_0>0$. On the other hand, it is easily seen that the maps $u=u(z)$ defined by (2.4) and $\bar{u}=\bar{u}(z)$ by (4.1) satisfy

$$u(z)=\bar{u}(z)+O(c^{-2}) \ , \tag{4.6}$$

whereas

$$A^{\alpha}(u(z))=\bar{A}^{\alpha}(\bar{u}(z))+O(c^{-2}),\ \alpha=0,1,2,3 \ , \tag{4.7}$$

and similarly for their derivatives, as $c\to\infty$, where the remainders $O(c^{-2})$ are all uniform for $z\in\Omega(\delta_0/2, c_0)$. Owing to (2.16), (4.5), (4.6) and (4.7), the L^2 norm of the right hand side of (4.4) is majorized by $C(\|\psi\|_{L^2}+c^{-2})$ with some positive constant C, uniformly for $c\geq c_0$, and, hence, (4.4) gives, by integration by parts and using Gronwall's inequality,

$$\|\psi(t)\|_{L^2}=O(c^{-2}) \ ,$$

which then yields, after interpolation with (4.5),

$$\|\psi(t)\|_{H^{s-\varepsilon}}=O(c^{-2\varepsilon}) \ ,$$

with any $\varepsilon>0$. Thus we are done.

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