J. Math. Kyoto Univ. (JMKYAZ)
35-1 (1995) 35-41

The cohomology rings of $BO(n)$ and $BSO(n)$ with $\mathbf{Z}_{2m}$ coefficients

By

Martin ČADEK and Jiří VANŽURA

1. Introduction

The cohomology rings of the classifying spaces for the groups $O(n)$ and $SO(n)$ with $\mathbf{Z}_2$ and $\mathbf{Z}[1/2]$ coefficients have been known for a long time, see [MS]. In 1960, E. Thomas found the group structure of $H^*(BO(n))$ with integer and $\mathbf{Z}_{2m}$ coefficients [T]. The integer cohomology ring is much more complicated so that it lasted till the year 1982 than its structure was written down in terms of generators and relations independently by E. H. Brown [B] and M. Feshbach [F]. The aim of this note is to describe the cohomology rings of $BO(n)$ and $BSO(n)$ with $\mathbf{Z}_{2m}$ coefficients in a similar way.

2. Notation and main results

Let n be a positive interger or ∞. The letters w_i and p_i will stand for the i-th Stiefel-Whitney class and the i-th Pontrjagin class of the universal vector bundle over $BSO(n)$ or $BO(n)$. The Bockstein homomorphism associated with the exact sequence $0\to\mathbf{Z}\to\mathbf{Z}\to\mathbf{Z}_2\to 0$ will be denoted δ. The mappings θ: $H^*(X,\mathbf{Z}_2)\to H^*(X,\mathbf{Z}_{2m})$ and ρ_k:$H^*(X,\mathbf{Z})\to H^*(X,\mathbf{Z}_k)$ are induced from the inclusion $\mathbf{Z}_2\to\mathbf{Z}_{2m}$ and reduction mod k, respec-tively. For a fixed $m\geq 2$, we will write only ρ instead of ρ_{2m}. For the symmetric difference of two sets I and J we will use the symbol

$$\Delta(I,J)=(I\cup J)-(I\cap J) \ .$$

Definition. *Let $\mathcal{S}_n$ be the set consisting of the elements*

z_i, x_I, y_I and u_n if n is even ,

where $i\in\mathbf{Z}$, $1\leq i<n/2$ and I ranges over all finite nonempty subsets of the positive integers less than $n/2$.

Let $\mathcal{O}_n$ be the set consisting of the elements

z_i , x_I , y_I

1991 *Mathematics Subject Classification*. 55R40.
Key words and phrases. Simplicial cohomology, classifying spaces, characteristic classes.
Communicated by Prof K. Ueno, September 20, 1993

where $i\in\mathbf{Z}$, $1\leq i\leq n/2$ *and* I *ranges over all finite nonempty subsets of* $\{1/2\}\cup\{l\in\mathbf{Z};\ 1\leq l\leq n/2\}$ *with the exception that* I *does not contain both* $1/2$ *and* $n/2$ *for* $n>1$.

Next in the polynomial rings $\mathbf{Z}_{2^m}[\mathcal{S}_n]$ and $\mathbf{Z}_{2^m}[\mathcal{O}_n]$ we denote

$$x_\emptyset=y_\emptyset=0$$
$$z_{1/2}=x_{\{1/2\}}$$
$$z_\emptyset=1$$
$$z_I=\prod_{s=1}^{r} z_{i_s}$$

for $I=\{i_1, i_2,\dots, i_r\}$, and

$$x_M=x_{\{n/2\}}\cdot x_{M-\{1/2,n/2\}}$$
$$y_M=x_{\{n/2\}}\cdot y_{M-\{1/2,n/2\}}$$

for $\{1/2, n/2\}\subseteq M\subseteq \{1/2\}\cup\{l\in\mathbf{Z};1\leq l\leq n/2\}$.

Remark. In the following theorems the elements z_i, u_n, x_I, y_I, $I=\{i_1, i_2,\dots, i_r\}$ can be taken to be the reduced Pontrjagin class ρp_i, the reduced Euler class ρe_n, $\rho\delta(w_{2i_1}w_{2i_2}\cdots w_{2i_r})$ and $\theta(w_{2i_1}w_{2i_2}\dots w_{2i_r})$, respectively.

Theorem 1. *For* $1\leq n\leq\infty$ *and* $m\geq 2$, *the cohomology ring* $H^*(BSO(n);\mathbf{Z}_{2^m})$ *is isomorphic to the polynomial ring over* $\mathbf{Z}_{2^m}$ *generated by the elements of* $\mathcal{S}_n$ *modulo the ideal generated by the following five general types of relations. In all the relations* I *and* J *are finite subsets of the positive integers less than* $n/2$, $I\neq\phi$ *but* J *can be empty.*

(1) $$2x_I=0$$
(2) $$2y_I=0$$
(3) $$y_I y_J=0$$
(4) $$x_I x_J=\sum_{i\in I} x_{\{i\}}x_{\Delta(I-\{i\},j)}z_{(I-\{i\})\cap j}$$
(5) $$x_I y_I=\sum_{i\in I} x_{\{i\}}y_{\Delta(I-\{i\},j)}z_{(I-\{i\})\cap j}$$

Theorem 2. *For* $1\leq n\leq\infty$ *and* $m\geq 2$, *the cohomology ring* $H^*(BO(n);\mathbf{Z}_{2^m})$ *is isomorphic to the polynomial ring over* $\mathbf{Z}_{2^m}$ *generated by the elements of* $\mathcal{O}_n$ *modulo the ideal generated for* n *odd or* ∞ *by relations* (1) - (5) *and for* n *even by relations* (1) - (5) *together with*

(6) $$x^2_{\{n/2\}}=x_{\{1/2\}}z_{n/2}$$
(7) $$x_{\{n/2\}}y_I=y_{(I-\{n/2\})\cup\{1/2\}}z_{n/2} \qquad \textit{if } n/2\in I,\ 1/2\notin I$$
(8) $$x_{\{n/2\}}y_I=x_{\{1/2\}}y_{(I-\{1/2\})\cup\{n/2\}} \qquad \textit{if } 1/2\in I,\ n/2\notin I$$

In the relations I *and* J *are finite subsets of* $\{1/2\}\cup\{l\in\mathbf{Z};1\leq l\leq n/2\}$ *which do* *not contain both* $1/2$ *and* $n/2$, $I \neq \phi$ *but* J *can be empty.*

Remark. The sets appearing on the right hand sides of relations (4) and (5) can be empty or can contain $1/2$ together with $n/2$. To get relations between generators in such a case it is necessary to use the notation introduced above.

3. Proofs

The proofs of both theorems follow the same lines. The proof of Theorem 2 is a little bit more difficult since $w_1 \neq 0$ in this case. We carry it out in details and at the end we outline the differences in the proof of Theorem 1.

Define φ: $\mathbf{Z}_{2^m}[\mathscr{O}_n] \to H^*(BO(n);\mathbf{Z}_{2^m})$ on generators in the following way

$$\begin{aligned}
&\varphi(z_i) = \rho p_i\\
&\varphi(x_I) = \rho\delta w_I = \rho\delta(w_{2i_1}w_{2i_2}\dots w_{2i_r})\\
&\varphi(y_I) = \theta w_I = \theta(w_{2i_1}w_{2i_2}\dots w_{2i_r}) \quad ,
\end{aligned}$$

$I = \{i_1, i_2, \dots, i_r\}$ and extend it into a ring homomorphism. Denote $\mathscr{I}_n$ the ideal in $\mathbf{Z}_{2^m}[\mathscr{O}_n]$ generated by relations (1) - (8). We will show that $\varphi(\mathscr{I}_n) = 0$. (Here we use the convention $w_\emptyset = 1$, $p_\emptyset = 1$, $p_{1/2} = \rho\delta w_1$.) Relatios (1), (4) and (6) in $H^*(BO(n);\mathbf{Z}_{2^m})$ arise from the relations which hold in $H^*(BO(n);\mathbf{Z})$ after application of the mapping ρ. See [B] and [F]. Relations (2) and (3) are consequences of the definition of θ. To prove (5), (7) and (8) in $H^*(BO(n);\mathbf{Z}_{2^m})$, we use the formula

$$\theta(\rho_2 x \cdot y) = \rho x \cdot \theta y \quad \text{for } x \in H^*(X;\mathbf{Z}),\ y \in H^*(X;\mathbf{Z}_2) \tag{9}$$

which is an easy consequence of the definition of cup product and θ. Realizing that $\rho_2 p_i = w_{2i}^2$ and $\rho_2\delta = Sq^1$ we get

$$\begin{aligned}
\rho\delta w_I \theta w_J &= \theta(\rho_2\delta w_I \cdot w_J) = \theta(Sq^1 w_I \cdot w_J)\\
&= \theta\left(\sum_{i\in I} Sq^1 w_{2i} \cdot w_{I-\{i\}} \cdot w_J\right)\\
&= \theta\left(\sum_{i\in I} \rho_2\delta w_{2i} \cdot \rho_{2P(I\cap J)-\{i\}} \cdot w_{\Delta(I-\{i\},J)}\right)\\
&= \sum_{i\in I} \rho\delta w_{2i} \cdot \rho p_{(I\cap J)-\{i\}} \cdot \theta w_{\Delta(I-\{i\},J)} \quad .
\end{aligned}$$

If $n/2$ I and $1/2 \notin I$ we have

$$\begin{aligned}
\rho\theta w_n \theta w_I &= \theta(\rho_2\delta w_n \cdot w_I) = \theta(Sq^1 w_n \cdot w_I)\\
&= \theta(w_1 w_n w_I) = \theta(w_{(I-\{n/2\})\cup\{1/2\}} w_n^2)\\
&= \theta(w_{(I-\{n/2\})\cup\{1/2\}}\rho_2 p_{n/2}) = \theta(w_{(I-\{n/2\})\cup\{1/2\}})\,\rho p_{n/2} \quad .
\end{aligned}$$

Finally, for $1/2 \in I$ and $n/2 \notin I$ we obtain

$$\begin{aligned}\rho\delta w_n\theta w_I &= \theta(\rho_2\delta w_n\cdot w_I)=\theta(Sq^1w_n\cdot w_I)\\ &=\theta(w_1w_nw_I)=\theta(w_1^2w_{(I-\{1/2\})\cup\{1/2\}})\\ &=\theta(\rho_2\delta w_1w_{(I-\{1/2\})\cup\{n/2\}})=\rho\delta w_1\theta(w_{(I-\{1/2\})\cup\{n/2\}})\ .\end{aligned}$$

Put $\mathcal{R}_n=\mathbf{Z}_{2^m}[\mathcal{O}_n]/\mathcal{I}_n$. Since $\varphi(\mathcal{I}_n)=0$, φ induces the ring homomorphism Φ: $\mathcal{R}_n\to H^*(BO(n);\mathbf{Z}_{2^m})$. Now we will split $\mathcal{R}_n$ and $H^*(BO(n);\mathbf{Z}_{2^m})$ as groups into direct sums $\mathcal{T}\oplus\mathcal{U}$ and $\rho H^*(BO(n);\mathbf{Z})\oplus S_{2^m}$ such that

$$\begin{aligned}&\Phi/\mathcal{T}\colon \mathcal{T}\to\rho H^*(BO(n);\mathbf{Z})\\ &\Phi/\mathcal{U}\colon \mathcal{U}\to S_{2^m}\end{aligned}$$

will be isomorphisms of groups. To make these decompositions possible we will find the complement of ker Sq^1 in the $\mathbf{Z}_2$-vector space $H^*(BO(n);\mathbf{Z})$ explicitly.

Consider the following slight modification of the Stiefel-Whitney classes

$$v_1=w_1,\ v_{2i}=w_{2i},\ v_{2i+1}=w_{2i+1}+w_1w_{2i}\ .$$

We have $Sq^1v_1=v_1^2$, $Sq^1v_{2i+1}=0$ if $i\geq 1$, $Sq^1v_{2i}=v_{2i+1}$ if $i<n/2$ and $Sq^1v_n=v_1v_n$ if n is even. If k ranges over all the multiindices $(k_1, k_2,\dots, k_n)$, $k_i\geq 0$, the elements $v^k=\prod_{i=1}^n v_i^{k_i}$ form a basis of $H^*(BO(n);\mathbf{Z}_2)$. (For $n=\infty$, the multiindex is a sequence with only finite number of $k_i>0$. Moreover, we put $k_\infty=0$.)

Let U be the set of all multiindices k which satisfy one of the following conditions:

a) There exist an index i, $1\leq i<n/2$, with k_{2i} odd and if i_0 is the biggest such integer, then $k_{2j+1}=0$ for all $j>i_0$.

b) For all $1\leq i<n/2$, $k_{2i+1}=0$, k_{2i} are even and k_1+k_n is odd.

Denote S_2 the vector space over $\mathbf{Z}_2$ spanned by the monomials v^k, $k\in U$. We show that S_2 is a complement of ker Sq^1 in $H^*(BO(n);\mathbf{Z}_2)$.

If $k\in U$ then $Sq^1v^k=v^{\bar{k}}+$elements from S_2, where $\bar{k}_1=k_1+1$, $\bar{k}_i=k_i$, $i\geq 2$, if k satisfies b) and $\bar{k}_{2i_0}=k_{2i_0}-1$, $\bar{k}_{2i_0+1}=k_{2i_0+1}+1$, $\bar{k}_j=k_j$ otherwise, if k satisfies a). In both cases $\bar{k}\notin U$ and it is uniquely determined by $k\in U$. It means that Sq^1v^k, $k\in U$, are linearly independent in $H^*(BO(n);\mathbf{Z}_2)$ and ker $Sq^1\cap S_2=\{0\}$.

Let v^k be a monomial not lying in S_2, i.e. $k\notin U$. First suppose that all k_{2i}, $1\leq i<n/2$ are even. If k_1+k_n is even for n even or k_1 is even for n odd or ∞, then $v^k\in$ker Sq^1. If k_1+k_n is odd for n even or k_1 is odd for n odd or ∞, then there is j, $1\leq j<n/2$ such that $k_{2j+1}\neq 0$. Let j_0 be the biggest such j. There is a multiindex l, $Sq^1v^l=0$, such that for n even we get

$$\begin{aligned}v^k&=v_1^{k_1}v_n^{k_n}v^lv_{2j_0+1}=v_1^{k_1}v_n^{k_n}v^lSq^1v_{2j_0}\\ &=Sq^1(v_1^{k_1}v_n^{k_n}v^lv_{2j_0})+Sq^1(v_1^{k_1}v_n^{k_n})v^lv_{2j_0}\\ &=Sq^1(v_1^{k_1}v_n^{k_n}v^lv_{2j_0})+v_1^{k_1+1}v_n^{k_n}v^lv_{2j_0}\end{aligned}$$

where the first summand belongs to ker Sq^1 and the second one to S_2. For n odd or ∞ it is not necessary to single out the v_n. In this case it can be incorporated into v^l.

Now suppose that at least one k_{2i}, $1\leq i<n/2$, is odd. Since $k\notin U$ there are $j_0>i_0$ such that i_0 is the biggest integer $i<n/2$ with k_{2i} odd and j_0 is the biggest integer $j>i_0$ such that $k_{2j+1}\neq 0$. If n is even put $k_{2i}=2m_i+\varepsilon_i$, $\varepsilon_i=0$ or 1, for $1\leq i<n/2$, $l_1=0$, $l_n=0$, $l_{2i}=2m_i$, $l_{2j+1}=k_{2j+1}$ for $1\leq j<n/2$, $j\neq j_0$, $l_{2j_0+1}=k_{2j_0+1}-1$, $I=\{i;\varepsilon_i=1\}$. We get

$$v^k=v_1^{k_1}\ v_n^{k_n}\ v^l v_{2j_0+1}\prod_{i\in I} v_{2i}=v_1^{k_1}\ v_n^{k_n}\ v^l\prod_{i\in I} v_{2i}Sq^1 v_{2j_0}$$
$$=Sq^1\left(v_1^{k_1}\ v_n^{k_n}\ v^l v_{2j_0}\prod_{i\in I} v_{2i}\right)+(k_1+k_n)v_1^{k_1+1}\ v_n^{k_n}\ v^l v_{2j_0}\prod_{i\in I} v_{2i}$$
$$+\sum_{r\in I} v_1^{k_1}\ v_n^{k_n}\ v^l v_{2j_0}v_{2r+1}\prod_{i\in I-\{r\}} v_{2i}$$

where the first summand lies in ker Sq^1 and the rest is a sum of v^p, $p\in U$. For n odd or ∞, we put $l_n=k_n$ and omit k_n and v_n in the computations. So every v^k can be expressed as a sum of elements from ker Sq^1 and S_2.

Define $S_{2m}=\theta S_2$. Since $2x=0$ for every torsion element of $H^*(BO(n);\mathbf{Z})$, we can use Lemma 6.7 from [T]. It implies

$$H^*(BO(n);\mathbf{Z}_{2m})=\rho H^*(BO(n);\mathbf{Z})\oplus S_{2m}\ ,$$

which is one of the required splittings. Moreover, as an easy consequence we get that θ: $S_2\to S_{2m}$ is an isomorphism (see also [T], Lemma 6.11).

Let $\mathcal{T}$ be the subring in $\mathcal{R}_n$ generated by z_i, $1\leq i\leq n/2$ and x_I, where I ranges over all finite nonempty subsets of $\{1/2\}\cup\{l\in\mathbf{Z};1\leq l\leq n/2\}$ which do not contain both 1/2 and $n/2$ for n>1. From the description of $H^*(BO(n);\mathbf{Z})$ in [B] and [F] it follows that $\Phi/\mathcal{T}$: $\mathcal{T}\to\rho H^*(BO(n);\mathbf{Z})$ is a ring isomorphism.

Let $k_{2i}=2m_i+\varepsilon_i, \varepsilon_i=0$ or 1, $i\in\{1/2\}\cup\{l\in\mathbf{Z};\ 1\leq l\leq n/2\}$, for a multiindex k. Let $I_k=\{i;\ \varepsilon_i=1\}$. We define s^k in the following way. Put

$$s^k=x_{\{1/2\}}^{m_{1/2}}\prod_{i=1}^{[\frac{n}{2}]} z_i^{m_i}\prod_{j=1}^{[\frac{n-1}{2}]} x_{\{j\}}^{k_{2j+1}}\cdot y_{I_k}$$

Here again the notation preceding Theorem 1 must be taken into account. In any case we have

$$\Phi(s^k)=\theta v^k\ .$$

(It can be verfined using formula (9).)

Let $\mathcal{U}$ be a subgroup of $\mathcal{R}_n$ generated by s^k where $k\in U$. Since $\theta: S_2\to S_{2m}$ is a group isomorphism and v^k, $k\in U$, form a basis of S_2, we get that $\Phi/\mathcal{U}$: $\mathcal{U}\to S_{2m}$ is a group isomorphism as well.

$\rho H^*(BO(n);\mathbf{Z})\cap S_{2m}=\{0\}$ and $\Phi/\mathcal{T}$, $\Phi/\mathcal{U}$ are group isomorphisms, hence $\mathcal{T}\cap\mathcal{U}=\{0\}$. It suffices to show that every element of $\mathcal{R}_n$ is a sum of elements from subgroups $\mathcal{T}$ and $\mathcal{U}$.

First of all, every element of $\mathcal{R}_n$ is a sum of elements from $\mathcal{T}$ and elements of the form

$$\prod_{i=1}^{[\frac{n}{2}]} z_i^{m_i} \prod_{r=1}^{p} x_{J_r} \cdot y_I .$$

Using successively relations (5) and, if necessary (6) - (8), we get that this element is a sum of s^k. Let us deal with an element s^k with k not lying in U. First, suppose that all k_{2i}, $1 \leq i < n/2$, are even. If $k_1 + k_n$ is even for n even or k_1 is even for n odd or ∞, then $s^k = 0$ (see the definition of s^k and the notation preceding Theorem 1). If $k_1 + k_n$ is odd for n even on k_1 is odd for n odd or ∞, there is j, $1 \leq j < n/2$, such that $k_{2j+1} \neq 0$. Denote the biggest such j by j_0 and put $l_{2j+1} = k_{2j+1}$ for $1 \leq j < n/2$, $j \neq j_0$, and $l_{2j_0+1} = k_{2j_0+1} - 1$. We have $I_k = \{1/2\}$ or $\{n/2\}$ and using the relation (5) for $x_{I \cup \{j_0\}} y_{\emptyset}$, we get

$$\begin{aligned} s^k &= x_{\{1/2\}}^{m_{1/2}} \prod_{i=1}^{[\frac{n}{2}]} z_i^{m_i} \prod_{j=1}^{j_0} x_{\{j\}}^{l_{2j+1}} x_{\{j_0\}} y_{I_k} \\ &= x_{\{1/2\}}^{m_{1/2}} \prod_{i=1}^{[\frac{n}{2}]} z_i^{m_i} \prod_{j=1}^{j_0} x_{\{j\}}^{l_{2j+1}} x_{I_k} y_{\{j_0\}} = s^p \end{aligned}$$

where $p \in U$ and consequently, $s^k = s^p \in \mathcal{U}$. (If $I_k = \{n/2\}$, then $x_{\{n/2\}} y_{\{j_0\}} = y_{\{1/2, n/2, j_0\}}$ and $I_p = \{1/2, n/2, j_0\}$.)

Now suppose that some k_{2i}, $1 \leq i < n/2$ is odd. Since $k \notin U$, there are $i_0 < j_0$ such that i_0 is the biggest i, $1 \leq i < n/2$, with k_{2i} odd and j_0 is the biggest $j > i_0$ such that $k_{2j+1} \neq 0$. Put $l_{2j+1} = k_{2j+1}$ for $1 \leq j < n/2$, $j \neq j_0$, $l_{2j_0+1} = k_{2j_0+1} - 1$, $I_k = \{i;\ \varepsilon_i = 1\}$. Then

$$\begin{aligned} s^k &= x_{\{1/2\}}^{m_{1/2}} \prod_{i=1}^{[\frac{n}{2}]} z_i^{m_i} \prod_{j=1}^{j_0} x_{\{j\}}^{l_{2j+1}} x_{\{j_0\}} y_{I_k} \\ &= \sum_{r \in I_k} x_{\{1/2\}}^{m_{1/2}} \prod_{i=1}^{[\frac{n}{2}]} z_i^{m_i} \prod_{j=1}^{j_0} x_{\{j\}}^{l_{2j+1}} \cdot x_{\{r\}} \cdot y_{(I_k \cup \{j_0\}) - \{r\}} , \end{aligned}$$

which is a sum of s^p, $p \in U$. Here we have again applied relation (5) for $x_{I \cup \{j_0\}} y_{\emptyset}$.

Thus Φ: $\mathcal{R}_n = \mathcal{T} \oplus \mathcal{U} \to H^*(BO(n);\ \mathbf{Z}_{2^m})$ is a group isomorphism and a ring homorphism, which completes the proof of Theorem 2.

Finally, we mention some changes which are necessary for the proof of Theorem 1 and which simplify it. Since $w_1 = 0$ in $H^*(BSO(n);\ \mathbf{Z}_2)$, we have $v_i = w_i$, $Sq^1 w_{2i} = w_{2i+1}$ if $i < n/2$ and $Sq^1 w_i = 0$ otherwise. All multiindeces k have $k_1 = 0$ and the set U is defined only by condition a), which is the most substantial change. So all the parts of the previous proof concerning condition b) are omitted. In some other parts the cases n even and n odd or ∞ must be treated separately. Nevertheless, the proof is principally the same as that of

Theorem 2.

ACADEMY OF SCIENCES OF THE CZECH REPUBLIC,
INSTITUTE OF MATHEMATICS,
ŽIŽKOVA 22, 616 62 BRNO,
CZECH REPUBLIC.
E-MAIL: CADEK @DRS. IPM. CZ
VANZURA @DRS. IPM. CZ

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