J. Math. Kyoto Univ. (JMKYAZ)
39-3 (1999), 557-575

Symplectic volume of the moduli space of spatial polygons

By

Yasuhiko KAMIYAMA and Michishige TEZUKA

1. Introduction

Let $M_n (n \geq 3)$ be the moduli space of spatial polygons $P=(a_1, a_2, \cdots, a_n)$ whose edges are vectors $a_i \in \boldsymbol{R}^3$ of length $|a_i|=1$ $(1 \leq i \leq n)$. Two polygons are identified if they differ only by motions in $\boldsymbol{R}^3$. The sum of the vectors is assumed to be zero. Thus:

$$M_n = \{P=(a_1, \cdots, a_n) \in (S^2)^n : a_1 + \cdots + a_n = 0\}/SO(3). \tag{1.1}$$

It is known that M_n admits a symplectic structure such that the complex dimension of M_n is $n-3$[8], [11] (cf. Theorem 2.8). For odd n or $n=4$, M_n has no singular points. For even n with $n \geq 6$, $P=(a_1, a_2, \cdots, a_n)$ is a singular point if and only if all the a_i $(1 \leq i \leq n)$ lie on a line in $\boldsymbol{R}^3$ through O. Such singular points are cone-like singularities and have neighborhoods $C(S^{n-3} \times_{S^1} S^{n-3})$, where C denotes the cone and S^1 acts on both copies of S^{n-3} by the complex multiplication(see for example [8]).

For odd n, $H_*(M_n; \boldsymbol{R})$ was determined by Kirwan and Klyachko [9], [11]. Later the cohomology ring $H^*(M_n; \boldsymbol{R})$ was determined by Brion and Kirwan [1], [10] (cf. Theorem 2.2). In particular $H^*(M_n; \boldsymbol{R})$ is generated by certain two dimensional cohomology classes $z_1, \cdots, z_n \in H^2(M_n; \boldsymbol{R})$. But the intersection numbers $\int_{M_n} \alpha\beta$ are not yet known, where $\alpha \in H^p(M_n; \boldsymbol{R})$ and $\beta \in H^q(M_n; \boldsymbol{R})$ with $p+q=2n-6$.

In contrast with this, for even n, $H_*(M_n; \boldsymbol{R})$ is complicated and is not generated by two demensional cohomology classes nor does not obey Poincaré duality [7]. The cohomology ring $H^*(M_n; \boldsymbol{R})$ is not yet known.

The purposes of this paper are as follows. First we determine the intersection numbers $\int_{M_n} \alpha\beta$ for odd n, where $\alpha \in H^p(M_n; \boldsymbol{R})$ and $\beta \in H^q(M_n; \boldsymbol{R})$ with $p+q=2n-6$. Let ω_n be the symplectic form on M_n. Then secondly we determine the symplectic volume $\int_{M_n} \omega_n^{n-3}$ for all n.

In order to state our results, we prepare some notations. For a sequence $(d_1, \cdots, d_n)$ of nonnegative integers with $\Sigma_{i=1}^n d_i = n-3$, we define $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ by

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{M_n} z_1^{d_1} \cdots z_n^{d_n}, \tag{1.2}$$

Communicated by Prof. A. Kono, March 15, 1999

where $z_i \in H^2(M_n; \boldsymbol{R})$ $(1 \leq i \leq n)$ are the generators of $H^*(M_n; \boldsymbol{R})$, which will be specified in Theorem 2.2. In order to determine the intersection numbers for odd n, we need to determine $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ for all $(d_1, \cdots, d_n)$. To do this, we consider the following types of $(d_1, \cdots, d_n)$. We set $n = 2m+1$.

(i) $d_1 = \cdots = d_{n-3} = 1$ and $d_{n-2} = d_{n-1} = d_n = 0$.

(ii) $d_1 = 2k$, $d_2 = \cdots = d_{n-2k-2} = 1$ and $d_{n-2k-1} = \cdots = d_n = 0$, where $1 \leq k \leq m-1$ and $n = 2m+1$.

If $(d_1, \cdots, d_n)$ is of the type (i), then we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ by $\langle \rho_{n,0} \rangle$. On the other hand, if $(d_1, \cdots, d_n)$ is of the type (ii), then we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ by $\langle \rho_{n,2k} \rangle$. Thus:

$$\begin{cases} \langle \rho_{n,0} \rangle = \int_{M_n} z_1 \cdots z_{n-3} \\ \langle \rho_{n,2k} \rangle = \int_{M_n} z_1^{2k} z_2 \cdots z_{n-2k-2} \, (1 \leq k \leq m-1). \end{cases} \tag{1.3}$$

Then we first prove the following theorem. For a sequence $(d_1, \cdots, d_n)$ of nonnegative integers with $\Sigma_{i=1}^n d_i = n-3$, we set $d_i = 2\alpha_i + \epsilon_i$ $(1 \leq i \leq n)$, where $\epsilon_i = 0$ or 1.

Theorem A. *We have the following relations in $H^*(M_n; \boldsymbol{R})$.*

(i) *If $\alpha_i = 0$ for $1 \leq i \leq n$, then we have*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \rho_{n,0} \rangle.$$

(ii) *If $\alpha_i \neq 0$ for some i, then we have*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \rho_{n,2(\alpha_1 + \cdots + \alpha_n)} \rangle.$$

Thus it suffices to determine $\langle \rho_{n,2k} \rangle$ $(0 \leq k \leq m-1)$ in order to determine the intersection numbers. About this, we have the following theorem. Let $\binom{a}{b}$ be the binomial coefficient.

Theorem B. *When $n = 2m+1$, the number $\langle \rho_{n,2k} \rangle$ $(0 \leq k \leq m-1)$ is given as follows.*

$$\langle \rho_{n,2k} \rangle = (-1)^k \frac{\binom{m-1}{k}\binom{2m-1}{m}}{\binom{2m-1}{2k+1}}.$$

Example 1.4. We have the following examples:

(i) $M_5 : \langle \rho_{5,0} \rangle = 1$ and $\langle \rho_{5,2} \rangle = -3$.

(ii) $M_7 : \langle \rho_{7,0} \rangle = 2$, $\langle \rho_{7,2} \rangle = -2$ and $\langle \rho_{7,4} \rangle = 10$.

(iii) $M_9 : \langle \rho_{9,0} \rangle = 5$, $\langle \rho_{9,2} \rangle = -3$, $\langle \rho_{9,4} \rangle = 5$ and $\langle \rho_{9,6} \rangle = -35$.

Next we give the symplectic volume of M_n for all n. As before, we denote the symplectic form of M_n by ω_n. Then, we set

$$v_n = \int_{M_n} \omega_n^{n-3}. \tag{1.5}$$

Then we have the following:

Theorem C. *The symplectic volume v_n is given as follows.*

$$v_n = \sum_{j=0}^{[\frac{n}{2}]-1} (-1)^j \binom{n-1}{j} (n-2-2j)^{n-3}.$$

Example 1.6. We have the following examples: $v_3 = 1$, $v_4 = 2$, $v_5 = 5$, $v_6 = 2^3 \cdot 3$, $v_7 = 2 \cdot 7 \cdot 11$, $v_8 = 2^8 \cdot 5$ and $v_9 = 3^2 \cdot 5 \cdot 17^2$.

This paper is organized as follows. In Section 2 we first recall the structure of $H^*(M_n; \boldsymbol{R})$ for odd n. Then we recall the results on the symplectic structure of M_n. In Section 3 we prove Theorems A and B.

In Section 4 we prove Theorem C. The method of the proof is as follows. By considering the moment map of the T^{n-3}-action on M_n', the subspace of M_n consisting of 'prodigal' polygons, it suffices to determine the volume of a convex polytope Δ_{n-3} in $\boldsymbol{R}^{n-3}$ in order to determine v_n (cf. Theorem 2.11). In Section 4 we detemine this volume by calculus.

For odd n, we can give a direct proof of Theorem C using the intersection numbers. The essential facts for the proof are the description of ω_n in terms of z_i [5] (cf. Theorem 2.12) and Theorems A and B. In Section 5 we give this proof.

2. Preliminaries

First we recall the structure of $H^*(M_n; \boldsymbol{R})$ for odd n, which was determined by Brion and Kirwan [1], [10]. For $i \in \{1, \cdots, n\}$, we define $A_{n,i} \subset (\boldsymbol{R}^3)^n$ by

$$A_{n,i} = \left\{ P = (a_1, \cdots, a_n) \in (S^2)^n : a_1 + \cdots + a_n = 0 \text{ and } a_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Let $SO(2)$ act on $\boldsymbol{R}^3$ by rotation about the z-axis. Then for odd n, the diagonal $SO(2)$-action on $(\boldsymbol{R}^3)^n$ is free on $A_{n,i}$ and we have $M_n = A_{n,i}/SO(2)$ (cf. (1.1)). Therefore, $A_{n,i} \to M_n$ is a principal $SO(2)$-bundle. Let $\xi_i \to M_n$ be a complex line bundle associated with $A_{n,i} \to M_n$:

$$\xi_i = (A_{n,i} \times \boldsymbol{C})/S^1,$$

where we identify $SO(2)$ with S^1 and let S^1 act on $A_{n,i} \times \boldsymbol{C}$ by

$$(P, \alpha) \cdot g = (Pg, \alpha g), \quad (P, \alpha) \in A_{n,i} \times \boldsymbol{C},\ g \in S^1.$$

Then we define $z_i \in H^2(M_n; \boldsymbol{R})$ to be the Chern class of ξ_i:

(2.1) $$z_i = c_1(\xi_i), \quad 1 \leq i \leq n.$$

Now we have the following theorem.

Theorem 2.2 ([1], [10]). *When $n=2m+1$, the algebra $H^*(M_n; \boldsymbol{R})$ is generated by $z_1, \cdots, z_n$ with the relations:*

(i) $z_1^2 = \cdots = z_n^2$.

(ii) $\Pi_{j \in J}(z_i + z_j) = 0$, *for all* $1 \leq i \leq n$ *and* $J \subset \{1, \cdots, n\}$ *such that* $i \notin J$ *and* $card(J) = m$, *where card denotes the cardinal.*

We take integers s and t with $1 \leq s, t \leq 2m+1$ and $s \neq t$. For such s and t, we define a divisor $D_{s,t}$ of M_n as follows.

(2.3) $$D_{s,t} = \{P = (a_1, \cdots, a_n) \in M_n : a_s = a_t\}.$$

Let $\gamma: H_{2n-8}(M_n; \boldsymbol{R}) \xrightarrow{\cong} H^2(M_n; \boldsymbol{R})$ be the Poincaré duality homomorphism. Then we have the following lemma, which will be used in Section 3 (cf. the proof of Theorem 3.7).

Lemma 2.4. *For $s \neq t$, we have*

$$\gamma(D_{s,t}) = \frac{z_s + z_t}{2}.$$

Proof. We describe $\gamma^{-1}(z_s) \in H_{2n-8}(M_n; \boldsymbol{R})$ in terms of submanifolds of real codimension two. We define a section σ of the line bundle $\xi_s \to M_n = A_{n,s}/SO(2)$ as follows. For $t \in \{1, \cdots, n\}$ with $t \neq s$, we set

$$\sigma(P) = (P, x_t^1 + \sqrt{-1}x_t^2) \in \xi_s,$$

where $P = (a_1, \cdots, a_n) \in M_n = A_{n,s}/SO(2)$ and $a_t = \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix}$.

Since $\gamma^{-1}(z_s) = \sigma^{-1}(0) \in H_{2n-8}(M_n; \boldsymbol{R})$, we have

(2.5)
$$\gamma^{-1}(z_s) = \left\{ P \in A_{n,s} : a_t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} / SO(2) + \left\{ P \in A_{n,s} : a_t = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\} / SO(2)$$
$$= \{P = (a_1, \cdots, a_n) \in M_n : a_s = a_t\} + \{P = (a_1, \cdots, a_n) \in M_n : a_s + a_t = 0\}.$$

We set

$$N_{s,t}=\left\{P\in A_{n,s}:a_t=\begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}/SO(2).$$

Then we choose an orientation of $N_{s,t}$ in (2.5) as follows. We define a map φ_t: $(S^2)^{n-1}\to(S^2)^n$ by

$$\varphi_t(a_1,\cdots,a_s,\cdots,\check{a}_t,\cdots,a_n)=(a_1,\cdots,a_s,\cdots,-a_s,\cdots,a_n),$$

where $\check{}$ means omitting the t-th coordinate. Then we define a subspace X_s of $(S^2)^{n-1}$ by

$$X_s=\left\{(a_1,\cdots,a_s,\cdots,\check{a}_t,\cdots,a_n)\in(S^2)^{n-1}:a_s=\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Note that $\varphi_t(X_s)$ has a natural orientation, and this orientation defines that of $N_{s,t}$. Thus as an orientation of $N_{s,t}$, we take the one induced from φ_t.

Similarly we have

$$\gamma^{-1}(z_t)=\left\{P\in A_{n,t}:a_s=\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}/SO(2)+\left\{P\in A_{n,t}:a_s=\begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}/SO(2).$$

Then it is seen that the orientation of the second term of the right side is induced from the map $\varphi_t\cdot(I^{s-1}\times(-I)\times I^{n-1-s})$, where I denotes the 3×3 unit matrix. Hence we have

(2.6)
$$\gamma^{-1}(z_t)=\{P=(a_1,\cdots,a_n)\in M_n:a_s=a_t\}+(-1)^3\{P=(a_1,\cdots,a_n)\in M_n:a_s+a_t=0\}.$$

Now from (2.5) and (2.6), we have

$$\begin{aligned}\frac{\gamma^{-1}(z_s)+\gamma^{-1}(z_t)}{2}&=\{P=(a_1,\cdots,a_n)\in M_n:a_s=a_t\}\\&=D_{s,t}.\end{aligned}$$

Thus Lemma 2.4 holds.

Next we recall the results on the symplectic structure of M_n for all n. Recall that the tangent space T_PM_n at $P=(a_1,\cdots,a_n)\in M_n$ consists of vectors $u=(u_1,\cdots,u_n)$ with $u_i\in\boldsymbol{R}^3$ $(1\le i\le n)$ under the following conditions:

(i) $(u_i,a_i)=0$ $(1\le i\le n)$, where (u_i,a_i) denotes the inner product.

(ii) $u_1+\cdots+u_n=0$.

(iii) Two systems of vectors $u=(u_1,\cdots,u_n)$ and $v=(v_1,\cdots,v_n)$ define the same tangent vector in T_PM_n if and only if there exists $w\in\boldsymbol{R}^3$ such that $u_i=v_i+[w,a_i]$ for $1\le i\le n$, where $[w,a_i]$ denotes the vector product.

We define a differential 2-form ω_n on M_n by the following formula:

$$\omega_n(u,v)=\sum_{i=1}^{n}\det(u_i,v_i,a_i), \tag{2.7}$$

where $P=(a_1,\cdots,a_n)\in M_n$, $u=(u_1,\cdots,u_n)$ and $v=(v_1,\cdots,v_n)$ are elements of T_PM_n. Then we have the following:

Theorem 2.8 ([8], [11]). *The differential 2-form ω_n defined by (2.7) gives a symplectic structure on the moduli space M_n.*

We define a map

$$\mu_n: M_n\to\boldsymbol{R}^{n-3}$$

as follows. Let $P=(a_1,\cdots,a_n)\in M_n$. Then we set

$$\mu_n(P)=(|a_1+a_2|,|a_1+a_2+a_3|,\cdots,|\sum_{i=1}^{n-2}a_i|). \tag{2.9}$$

Thus $\mu_n(P)$ is the lengths of the diagonals connecting the vertices to the origin. (Since $|a_1|=|\Sigma_{i=1}^{n-1}a_i|=1$, only these $n-3$ lengths are new.) As in [4], we call a polygon P 'prodigal' if none of these $n-3$ lengths vanish. Let M_n' be the open dense subspace of M_n consisting of prodigal polygons. Then as in [8] and [11], M_n' admits a T^{n-3}-action which is compatible with the symplectic structure on M_n, where T^{n-3} denotes the $(n-3)$-dimensional torus. We recall that the action is given as follows: The ith circle acts by rotating the part of the polygon, formed by the first $i+1$ edges, around the ith diagonal. (When that diagonal is length zero, there is no well-defined axis around which to be rotated, and indeed the action cannot be extended continuously over this subset. Thus to consider only prodigal polygons is essential.) This action preserves the level sets of the functions in (2.9).

Theorem 2.10 ([8], [11]). *The restriction $\mu_n|M_n': M_n'\to\boldsymbol{R}^{n-3}$ is a moment map for the T^{n-3}-action on M_n'.*

Thus we can understand $\mu_n: M_n\to\boldsymbol{R}^{n-3}$ in (2.9) as the extension of the moment map. We write the image of μ_n by Δ_{n-3}:

$$\Delta_{n-3}=\mu_n(M_n).$$

Note that Δ_{n-3} is a convex polytope in $\boldsymbol{R}^{n-3}$. We write its volume by

$\mathrm{Vol}(\Delta_{n-3})$. Since M_n' is an open dense subspace of M_n, $\mu_n(M_n')$ is also an open dense subspace of Δ_{n-3}. Hence we have $\mathrm{Vol}(\Delta_{n-3})=\mathrm{Vol}(\mu_n(M_n'))$. Note also that $\dim_C M_n=n-3$. Hence by Duistermaat-Heckman theorem [2], [3, §2], we have the following theorem from Theorem 2.10:

Theorem 2.11. *We have*

$$v_n=(n-3)!\mathrm{Vol}(\Delta_{n-3}),$$

where v_n is defined in (1.5).

Finally for odd n, we have the following description of ω_n:

Theorem 2.12 ([5]). *The class $[\omega_n]\in H^2(M_n;\boldsymbol{R})$ is given by*

$$[\omega_n]=\sum_{i=1}^{n} z_i,$$

where z_i is defined in (2.1).

3. Proofs of Theorems A and B

In this section we set $n=2m+1$. Recall (1.2), where we set $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle=\int_{M_n}z_1^{d_1}\cdots z_n^{d_n}$ for a sequence $(d_1,\cdots,d_n)$ of nonnegative integers with $\Sigma_{i=1}^n d_i=n-3$. Recall also (1.3), where we defined $\langle\rho_{n,2k}\rangle$ $(0\le k\le m-1)$ by setting $\langle\rho_{n,2k}\rangle=\langle\tau_{d_1}\cdots\tau_{d_n}\rangle$ for particular $(d_1,\cdots,d_n)$. As in Section 1 we set $d_i=2\alpha_i+\epsilon_i$ $(1\le i\le n)$, where $\epsilon_i=0$ or 1.

First we prove Theorem A. Note that the symmetric group S_n naturally acts on M_n such that $g_*[M_n]=[M_n]$ for all $g\in S_n$, where $[M_n]\in H_{2n-6}(M_n;\boldsymbol{R})$ denotes the fundamental class. Hence if $\alpha_i=0$ for $1\le i\le n$, then we can use the action to prove $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle=\langle\rho_{n,0}\rangle$.

Assume that $\alpha_i\ne 0$ for some i. Since $d_1+\cdots+d_n=n-3$, we must have $d_j=0$ for some j. Then by using Theorem 2.2(i), we have

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle=\langle\tau_{\varepsilon_1}\cdots\tau_{\varepsilon_{j-1}}\tau_{2(\alpha_1+\cdots+\alpha_n)}\tau_{\varepsilon_{j+1}}\cdots\tau_{\varepsilon_n}\rangle. \tag{3.1}$$

Then by the S_n-action, we see that the right side of (3.1) is equal to

$$\langle\tau_{2(\alpha_1+\cdots+\alpha_n)}\overbrace{\tau_1\cdots\tau_1}^{(n-3-2(\alpha_1+\cdots+\alpha_n))-\mathrm{fold}}\overbrace{\tau_0\cdots\tau_0}^{(2(\alpha_1+\cdots+\alpha_n)+2)-\mathrm{fold}}\rangle=\langle\rho_{n,2(\alpha_1+\cdots+\alpha_n)}\rangle.$$

Next we prove Theorem B. First we describe $\langle\rho_{n,2k}\rangle$ $(1\le k\le m-1)$ in terms of $\langle\rho_{n,0}\rangle$.

Proposition 3.2. *When $n=2m+1$, the number $\langle\rho_{n,2k}\rangle$ $(1\le k\le m-1)$ is given as follows.*

(i) *For* $1 \leq k \leq m-2$, *we have*

$$\langle \rho_{n,2k} \rangle = (-1)^k (2m-2) \frac{\binom{m-2}{k}}{\binom{2m-2}{2k+1}} \langle \rho_{n,0} \rangle.$$

(ii) $\langle \rho_{n,2m-2} \rangle = (-1)^{m+1} (2m-1) \langle \rho_{n,0} \rangle$.

Proof. We shall assume the truth of the following lemma.

Lemma 3.3. *When* $n = 2m+1$, *we have the following equations.*

(i) *When m is even, we set* $m = 2a$. *Then*

$$\begin{cases} \sum_{j=0}^{a} \binom{m+1}{2j+1} \langle \rho_{n,2j+2p} \rangle = 0 & \text{for } 0 \leq p \leq a-1 \\ \sum_{j=0}^{a} \binom{m+1}{2j} \langle \rho_{n,2j+2p} \rangle = 0 & \text{for } 0 \leq p \leq a-2. \end{cases} \tag{3.4}$$

(ii) *When m is odd, we set* $m = 2a+1$. *Then*

$$\begin{cases} \sum_{j=0}^{a} \binom{m+1}{2j+1} \langle \rho_{n,2j+2p} \rangle = 0 & \text{for } 0 \leq p \leq a-1 \\ \sum_{j=0}^{a+1} \binom{m+1}{2j} \langle \rho_{n,2j+2p} \rangle = 0 & \text{for } 0 \leq p \leq a-1. \end{cases} \tag{3.5}$$

Then it is easy to see that $\langle \rho_{n,2k} \rangle$ in Proposition 3.2 is the general solution of (3.4) or (3.5). Hence Proposition 3.2 follows.

Proof of Lemma 3.3. We prove (3.4). By Theorem 2.2 (ii), we have

$$\prod_{i=2}^{m+1} (z_1 + z_i) = 0.$$

We expand this and write as

$$\sum_{i=0}^{m} f_{m-i}(z_2, \cdots, z_{m+1}) z_1^i = 0, \tag{3.6}$$

where $f_j(z_2, \cdots, z_{m+1})$ denotes a polynomial of degree j (here we give the degree 1 to z_i) with variables $z_2, \cdots, z_{m+1}$. In particular, we have $f_0(z_2, \cdots, z_{m+1}) = 1$.

Now let $m = 2a$. For every $0 \leq p \leq a-1$, we multiply $z_1^{2p} z_{m+2} \cdots z_{2m-1-2p}$ to (3.6). Then by Theorem 2.2 (ii) and the S_n-action, we have

$$\left(z_1^{2a} + \sum_{j=0}^{a-1} (z_1 f_{m-2j-1}(z_2, \cdots, z_{m+1}) + f_{m-2j}(z_2, \cdots, z_{m+1})) z_1^{2j} \right)$$
$$\times (z_1^{2p} z_{m+2} \cdots z_{2m-1-2p}) = 0.$$

Note that the numbers of monomials of $f_{m-2j-1}(z_2, \cdots, z_{m+1})$ and $f_{m-2j}(z_2, \cdots, z_{m+1})$ are $\binom{m}{m-2j-1}$ and $\binom{m}{m-2j}$. Hence we have

$$(z_1 f_{m-2j-1}(z_2, \cdots, z_{m+1}) + f_{m-2j}(z_2, \cdots, z_{m+1}))(z_1^{2j+2p} z_{m+2} \cdots z_{2m-1-2p})$$

$$=\left(\binom{m}{m-2j-1}+\binom{m}{m-2j}\right)\langle\rho_{n,2j+2p}\rangle$$
$$=\binom{m+1}{2j+1}\langle\rho_{n,2j+2p}\rangle$$

Thus the first equation of (3.4) follows. The second equation of (3.4) is proved similarly by multiplying $z_1^{2p+1}z_{m+2}\cdots z_{2m-2p-2}$ $(0\leq p\leq a-2)$ to (3.6).

By Proposition 3.2, we need to determine $\langle\rho_{n,0}\rangle$ in order to complete the proof of Theorem B.

Theorem 3.7. *When $n=2m+1$, we have the following*:

$$\langle\rho_{n,0}\rangle=\frac{\binom{2m-1}{m}}{2m-1}.$$

Proof. Recall that for integers s and t with $1\leq s,t\leq 2m+1$ and $s\neq t$, we defined a divisor $D_{s,t}$ of M_n as follows (cf. (2.3)).

(3.8) $$D_{s,t}=\{P=(a_1,\cdots,a_n)\in M_n: a_s=a_t\}$$

We set $N_1=D_{1,2}\cap D_{1,3}\cap\cdots\cap D_{1,m}$ and $N_2=D_{m+1,m+2}\cap D_{m+1,m+3}\cap\cdots\cap D_{m+1,2m}$. Since $M_3=\{\text{point}\}$, we have

(3.9) $$\begin{aligned}N_1\cap N_2&=\{P=(a_1,\cdots,a_n)\in M_n: a_1=\cdots=a_m \text{ and } a_{m+1}=\cdots=a_{2m}\}\\&=\{\text{point}\}.\end{aligned}$$

Let $\gamma: H_{2n-8}(M_n;\boldsymbol{R})\xrightarrow{\cong}H^2(M_n;\boldsymbol{R})$ be the Poincaré duality homomorphism. Then (3.9) tells us that

(3.10) $$\int_{M_n}\prod_{p=2}^{m}\gamma(D_{1,p})\prod_{q=m+2}^{2m}\gamma(D_{m+1,q})=1.$$

By Lemma 2.4, we have

(3.11) $$\gamma(D_{s,t})=\frac{z_s+z_t}{2}.$$

Using (3.11), we can write (3.10) as

(3.12) $$\frac{1}{2^{2m-2}}\int_{M_n}\prod_{p=2}^{m}(z_1+z_p)\prod_{q=m+2}^{2m}(z_{m+1}+z_q)=1.$$

By the same argument as in the proof of Lemma 3.3, we can describe the left side of (3.12) in terms of $\langle\rho_{n,2k}\rangle$ $(0\leq k\leq m-1)$. Thus (3.12) is equivalent to the following.

(i) When m is even, we set $m=2a$. Then

$$\frac{1}{2^{2m-2}}\sum_{i,j=1}^{a}\binom{2a}{2i-1}\binom{2a}{2j-1}\langle\rho_{n,4a-2i-2j}\rangle=1. \tag{3.13}$$

(ii) When m is odd, we set $m=2a+1$. Then

$$\frac{1}{2^{2m-2}}\sum_{i,j=0}^{a}\binom{2a+1}{2i}\binom{2a+1}{2j}\langle\rho_{n,4a-2i-2j}\rangle=1. \tag{3.14}$$

Proposition 3.15. (3.13) *or* (3.14) *is equivalent to*

$$\frac{1}{2^{2m-2}}\frac{2^{2m-2}(2m-1)}{\binom{2m-1}{m}}\langle\rho_{n,0}\rangle=1.$$

Hence $\langle\rho_{n,0}\rangle=\dfrac{\binom{2m-1}{m}}{2m-1}$ and Theorem 3.7 follows.

Proof of Proposition 3.15. We shall prove the case (3.13). It is easy to see that (3.13) is equivalent to

$$\frac{1}{2^{2m-2}}\sum_{k=2}^{2a}\langle\rho_{n,4a-2k}\rangle\sum_{i=1}^{k-1}\binom{2a}{2i-1}\binom{2a}{2k-2i-1}=1. \tag{3.16}$$

From Proposition 3.2, we have

$$\langle\rho_{n,4a-2k}\rangle=(-1)^k\frac{(4a-2)\binom{2a-2}{k-2}}{\binom{4a-2}{2k-3}}\langle\rho_{n,0}\rangle$$

for $2\le k\le 2a$. It is easy to see that

$$\sum_{i=1}^{k-1}\binom{2a}{2i-1}\binom{2a}{2k-2i-1}=\frac{\binom{4a}{2k-2}+(-1)^k\binom{2a}{k-1}}{2}.$$

Hence (3.16) is equivalent to

$$\frac{1}{2^{2m-2}}\left(\sum_{k=2}^{2a}(A_k+B_k)\right)\langle\rho_{n,0}\rangle=1, \tag{3.17}$$

where

$$A_k=(-1)^k\frac{(2a-1)\binom{2a-2}{k-2}\binom{4a}{2k-2}}{\binom{4a-2}{2k-3}} \tag{3.18}$$

and

$$B_k=\frac{(2a-1)\binom{2a-2}{k-2}\binom{2a}{k-1}}{\binom{4a-2}{2k-3}} \tag{3.19}$$

Lemma 3.20. *We have the following equations*:

(a) $\Sigma_{k=2}^{2a} A_k = 4a-1$.

(b) $\Sigma_{k=2}^{2a} B_k = \dfrac{2^{4a-2}(2a-1)}{\binom{4a-2}{2a}} - (4a-1)$.

Proof of Lemma 3.20. First we prove (a). From (3.18), we have

$$A_k = (-1)^k \frac{4a-1}{2}\binom{2a}{k-1}.$$

Then (a) follows easily.

Next we prove (b). From (3.19), we have

$$B_k = \frac{(2a-1)}{\binom{4a-2}{2a}}\binom{2k-3}{k-1}\binom{4a-2k+1}{2a-k+1}. \tag{3.21}$$

Note that

$$\sum_{k=2}^{2a}\binom{2k-3}{k-1}\binom{4a-2k+1}{2a-k+1} = \frac{1}{4}\sum_{k=2}^{2a}\binom{2k-2}{k-1}\binom{4a-2k+2}{2a-k+1}$$
$$= \frac{1}{4}\left(2^{4a} - 2\binom{4a}{2a}\right).$$

Hence (b) follows from (3.21).

Now from (3.17) and Lemma 3.20, we have

$$\frac{1}{2^{2m-2}}\frac{2^{4a-2}(2a-1)}{\binom{4a-2}{2a}}\langle \rho_{n,0}\rangle = 1.$$

Since $m = 2a$, this is equivalent to

$$\frac{1}{2^{2m-2}}\frac{2^{2m-2}(2m-1)}{\binom{2m-1}{m}}\langle \rho_{n,0}\rangle = 1.$$

Thus Proposition 3.15 holds for the case (3.13). The case (3.14) is proved similarly.

4. Proof of Theorem C

We prove Theorem C using Theorem 2.11. Recall that in Section 2, we set $\Delta_{n-3} = \mu_n(M_n)$, where $\mu_n : M_n \to \boldsymbol{R}^{n-3}$ is given in (2.9). First we describe Δ_{n-3} (cf. (4.2)). We set $\boldsymbol{R}_+ = \{x \in \boldsymbol{R} : x \geq 0\}$. We use the following notation.

Definition 4.1. For $x, y \in \boldsymbol{R}_+$, we use the symbol $\Delta(x, y, 1)$ to denote that x and y satisfy the conditions

$$x \leq y+1, \quad y \leq x+1 \quad \text{and} \quad 1 \leq x+y.$$

Thus $\Delta(x, y, 1)$ denotes the conditions that there exists a triangle whose edges have lengths x, y and 1.

Let $P=(a_1, \cdots, a_n) \in M_n$. For $1 \leq j \leq n-3$, we set $x_j = |\Sigma_{i=1}^{j+1} a_i|$. Then from (2.9), it is easy to see that Δ_{n-3} is given as follows.

$$\text{(4.2)} \qquad \Delta_{n-3} = \{(x_1, \cdots, x_{n-3}) \in (\boldsymbol{R}_+)^{n-3} : 0 \leq x_1 \leq 2, 0 \leq x_{n-3} \leq 2 \text{ and}$$
$$\Delta(x_1, x_2, 1),\ \Delta(x_2, x_3, 1), \cdots, \Delta(x_{n-4}, x_{n-3}, 1)\}.$$

Note that for $(x_1, \cdots, x_{n-3}) \in \Delta_{n-3}$, (4.2) tells us that $0 \leq x_j \leq j+1$.

Let $k \in \boldsymbol{N}$ and let $t \in \boldsymbol{R}_+$ satisfy $0 \leq t \leq k+2$. For such t, we define a convex polytope $\Omega_{k,t}$ in $\boldsymbol{R}^k$ as follows.

$$\text{(4.3)} \qquad \Omega_{k,t} = \{(x_1, \cdots, x_k) \in (\boldsymbol{R}_+)^k : 0 \leq x_1 \leq 2 \text{ and}$$
$$\Delta(x_1, x_2, 1),\ \Delta(x_2, x_3, 1), \cdots, \Delta(x_{k-1}, x_k, 1), \Delta(x_k, t, 1)\}.$$

We write the volume of $\Omega_{k,t}$ in $\boldsymbol{R}^k$ by $V_k(t)$. Thus:

$$V_k(t) = \mathrm{Vol}(\Omega_{k,t}).$$

Let $t=1$ and we consider $\Omega_{k,1}$. In this case, the condition $\Delta(x_k, 1, 1)$ implies that $0 \leq x_k \leq 2$. Thus from (4.2) and (4.3), we have

$$\Omega_{k,1} = \Delta_k.$$

Hence by Theorem 2.11, we have

$$\text{(4.4)} \qquad v_n = (n-3)!\, V_{n-3}(1).$$

In the following, we determine $V_k(t)$ for $k \in \boldsymbol{N}$ and $t \in \boldsymbol{R}_+$ with $0 \leq t \leq k+2$. The method of calculations is as follows. We prove a recursion formula which gives $V_{k+1}(t')(0 \leq t' \leq k+3)$ from $V_k(t)$ $(0 \leq t \leq k+2)$ (cf. Lemma 4.5 and Theorem 4.6). Then we solve this (cf. Theorem 4.8). For that purpose, it is convenient to decompose the interval $[0, k+2]$ as follows.

(i) When $k=2l+1$. We decompose

$$[0, k+2] = [0, 1] \cup [1, 3] \cup \cdots \cup [k-2i, k+2-2i] \cup \cdots \cup [k, k+2].$$

(ii) When $k=2l$. We decompose

$$[0, k+2] = [0, 2] \cup [2, 4] \cup \cdots \cup [k-2i, k+2-2i] \cup \cdots \cup [k, k+2].$$

When $k=2l+1$, we define $V_{k,l+1}(t)$ or $V_{k,i}(t)$ $(0\le i\le l)$ to be the restriction of $V_k(t)$ (with respect to the variable t) to $[0,1]$ or $[k-2i,k+2-2i]$ $(0\le i\le l)$. When $k=2l$, we define $V_{k,i}(t)$ $(0\le i\le l)$ to be the restriction of $V_k(t)$ to $[k-2i,k+2-2i]$ $(0\le i\le l)$. Thus:

(i) When $k=2l+1$.

$$V_k(t)=\begin{cases} V_{k,l+1}(t) & 0\le t\le 1 \\ V_{k,l}(t) & 1\le t\le 3 \\ \cdots & \\ V_{k,i}(t) & k-2i\le t\le k+2-2i \\ \cdots & \\ V_{k,1}(t) & k-2\le t\le k \\ V_{k,0}(t) & k\le t\le k+2. \end{cases}$$

(ii) When $k=2l$.

$$V_k(t)=\begin{cases} V_{k,l}(t) & 0\le t\le 2 \\ V_{k,l-1}(t) & 2\le t\le 4 \\ \cdots & \\ V_{k,i}(t) & k-2i\le t\le k+2-2i \\ \cdots & \\ V_{k,1}(t) & k-2\le t\le k \\ V_{k,0}(t) & k\le t\le k+2. \end{cases}$$

Now we give the recursion formula. For the initial condition, we have the following:

Lemma 4.5. *We have the following formula for* $V_1(t)$.

$$\begin{cases} V_{1,1}(t)=2t & 0\le t\le 1 \\ V_{1,0}(t)=3-t & 1\le t\le 3. \end{cases}$$

Proof. By the definition, we have

$$\Omega_{1,t}=\{x_1: 0\le x_1\le 2 \text{ and } \Delta(x_1,t,1)\}.$$

Consider the domain in (x_1,t)-plane surrounded by four lines $t=-x_1+1$, $t=x_1+1$, $t=x_1-1$ and $x_1=2$. For each t, we cut this domain by a line through $(0,t)$, which is parallel to the x_1-axis.

(i) If $0\le t\le 1$, then we must have $1-t\le x_1\le 1+t$. Hence $V_{1,1}(t)=(1+t)-(1-t)=2t$.

(ii) If $1\le t\le 3$, then we must have $t-1\le x_1\le 2$. Hence $V_{1,0}(t)=2-(t-1)=3-t$.

Hence the result follows.

Next we give the recursion formula which gives $V_{k+1,i'}(t')$ from $V_{k,i}(t)$.

Theorem 4.6.

(i) *When* $k=2l+1$. (*In this case, we have* $k+1=2(l+1)$.)

(a) *When* $i=l+1$.

$$V_{k+1,l+1}(t)=\begin{cases}\int_{1-t}^{1}V_{k,l+1}(x_{k+1})dx_{k+1}+\int_{1}^{1+t}V_{k,l}(x_{k+1})dx_{k+1} & 0\leq t\leq 1\\ \int_{t-1}^{1}V_{k,l+1}(x_{k+1})dx_{k+1}+\int_{1}^{1+t}V_{k,l}(x_{k+1})dx_{k+1} & 1\leq t\leq 2.\end{cases}$$

(b) *When* $1\leq i\leq l$.

$$V_{k+1,i}(t)=\int_{t-1}^{k+2-2i}V_{k,i}(x_{k+1})dx_{k+1}+\int_{k+2-2i}^{t+1}V_{k,i-1}(x_{k+1})dx_{k+1}$$
$$k+1-2i\leq t\leq k+3-2i.$$

(c) *When* $i=0$.

$$V_{k+1,0}(t)=\int_{t-1}^{k+2}V_{k,0}(x_{k+1})dx_{k+1}\qquad k+1\leq t\leq k+3.$$

(ii) *When* $k=2l$. (*In this case, we have* $k+1=2l+1$.)

(a) *When* $i=l+1$

$$V_{k+1,l+1}(t)=\int_{1-t}^{1+t}V_{k,l}(x_{k+1})dx_{k+1}\qquad 0\leq t\leq 1.$$

(b) *When* $1\leq i\leq l$.

$$V_{k+1,i}(t)=\int_{t-1}^{k+2-2i}V_{k,i}(x_{k+1})dx_{k+1}+\int_{k+2-2i}^{t+1}V_{k,i-1}(x_{k+1})dx_{k+1}$$
$$k+1-2i\leq t\leq k+3-2i.$$

(c) *When* $i=0$.

$$V_{k+1,0}(t)=\int_{t-1}^{k+2}V_{k,0}(x_{k+1})dx_{k+1}\qquad k+1\leq t\leq k+3.$$

Proof. This theorem is proved in the same way as in Lemma 4.5. As an example, we show (i)(b). Consider the domain in (x_{k+1}, t)-plane surrounded by four lines $t=-x_{k+1}+1$, $t=x_{k+1}+1$, $t=x_{k+1}-1$ and $x_{k+1}=k+2$. For each t with

$k+1-2i \leq t \leq k+3-2i$, we cut this domain by a line through $(0, t)$, which is parallel to the x_{k+1}-axis. Then we must have $t-1 \leq x_{k+1} \leq t+1$. Hence we have

$$V_{k+1,i}(t)=\int_{t-1}^{t+1} V_k(x_{k+1})dx_{k+1} \tag{4.7}$$

(We think of t as x_{k+1} in the definition of $\Omega_{k,t}$ in (4.3).) Note that we have

$$k-2i \leq t-1 \leq k+2-2i \leq t+1 \leq k+4-2i$$

Then by the definition of $V_{k,i}(t)$, we can write (4.7) in the form of (i)(b). Hence the result follows.

Now the solution of the recursion formula in Theorem 4.6 under the initial condition Lemma 4.5 is given as follows.

Theorem 4.8. *For $k \in N$, $V_{k,i}(t)$ is given as follows.*

$$V_{k,i}(t)=\frac{1}{k!}\sum_{p=0}^{i}(-1)^p\binom{k+2}{p}(k+2-2p-t)^k.$$

Proof. This theorem is proved easily by induction on k. As an example, we assume the truth of Theorem 4.8 for $k=2l+1$ and show the case $V_{k+1,l+1}(t)$. We must treat the cases $0 \leq t \leq 1$ and $1 \leq t \leq 2$. But as the calculations are similar, we treat the former case. By Theorerm 4.6 (i)(a) and the inductive hypothesis, we have

(4.9)
$$\begin{aligned}
V_{k+1,l+1}(t)&=\int_{1-t}^{1}\frac{1}{k!}\sum_{p=0}^{l+1}(-1)^p\binom{k+2}{p}(k+2-2p-x_{k+1})^k dx_{k+1}\\
&\quad+\int_{1}^{1+t}\frac{1}{k!}\sum_{p=0}^{l}(-1)^p\binom{k+2}{p}(k+2-2p-x_{k+1})^k dx_{k+1}\\
&=\frac{1}{(k+1)!}(A+B),
\end{aligned}$$

where

$$A=\sum_{p=0}^{l+1}(-1)^p\binom{k+2}{p}(k+1-2p+t)^{k+1} \tag{4.10}$$

and

$$B=-\sum_{p=0}^{l}(-1)^p\binom{k+2}{p}(k+1-2p-t)^{k+1}. \tag{4.11}$$

About (4.11), it is easy to see that

$$B=\sum_{q=1}^{l+1}(-1)^q\binom{k+2}{q-1}(k+3-2q-t)^{k+1}. \tag{4.12}$$

About (4.10), it is easy to see that

$$A=-\sum_{q=l+2}^{k+2}(-1)^q\binom{k+2}{q}(k+3-2q-t)^{k+1}. \tag{4.13}$$

The following lemma is proved easily.

Lemma 4.14. *Let x be a variable and $n \in \boldsymbol{N}$. Let $r \in \boldsymbol{Z}$ satisfy $0 \leq r \leq n-1$. Then we have the following equation.*

$$\sum_{q=0}^{n}(-1)^q\binom{n}{q}(x-2q)^r=0.$$

Now we use Lemma 4.14 for $x=k+3-t$, $n=k+2$ and $r=k+1$. Then from (4.13), we have

$$A=\sum_{q=0}^{l+1}(-1)^q\binom{k+2}{q}(k+3-2q-t)^{k+1}. \tag{4.15}$$

From (4.9), (4.12) and (4.15), we see that

$$V_{k+1,l+1}(t)=\frac{1}{(k+1)!}\sum_{q=0}^{l+1}(-1)^q\left\{\binom{k+2}{q}+\binom{k+2}{q-1}\right\}(k+3-2q-t)^{k+1}$$
$$=\frac{1}{(k+1)!}\sum_{q=0}^{l+1}(-1)^q\binom{k+3}{q}(k+3-2q-t)^{k+1}.$$

This completes the proof of Theorem 4.8.

Now we prove Theorem C. From (4.4), we have $v_n=(n-3)!V_{n-3}(1)$. If $n=2m+1$, then $n-3=2(m-1)$. By the definition of $V_k(t)$, we have $V_{n-3}(1)=V_{n-3,m-1}(1)$. Hence by Theorem 4.8, we have

$$v_n=\sum_{p=0}^{m-1}(-1)^p\binom{n-1}{p}(n-2-2p)^{n-3}.$$

Thus Theorem C holds for $n=2m+1$. The case for $n=2m$ is proved similarly.

5. Alternative proof of Theorem C for odd n

In this section we set $n=2m+1$. By Theoorem 2.12, we have $[\omega_n]=\Sigma_{i=1}^n z_i$. Hence in order to calculate $v_n=\int_{M_n}\omega_n^{n-3}$, it suffices to determine $\int_{M_n}(z_1+\cdots+z_n)^{n-3}$. The essential ideas for calculations are first to expand $(z_1+\cdots+z_n)^{n-3}$, then to apply Theorems A and B. These calculations are somewhat long, but each step is easy. So we just mention the steps for calculations.

STEP 1. Since $\dim_{\mathbf{C}} M_n = n-3$, the equation $v_n = \int_{M_n}(z_1 + \cdots + z_n)^{n-3}$ is equivalent to

$$(5.1) \qquad v_n = (n-3)! \int_{M_n} \exp(z_1 + \cdots + z_n).$$

We set $f(z) = \Sigma_{i=0}^{\infty} \frac{z^{2i}}{(2i)!}$ and $g(z) = \Sigma_{i=0}^{\infty} \frac{z^{2i}}{(2i+1)!}$, where z is a variable. Then we have

$$\exp z = f(z) + zg(z).$$

Since $z_i^2 = z_j^2$ by Theorem 2.2 (i), we have

$$(5.2) \qquad \exp(z_1 + \cdots + z_n) = \prod_{i=1}^{n} (f(z_1) + z_i g(z_1)).$$

Since n is odd, $\dim_{\mathbf{C}} M_n = n-3$ is even. Hence using the S_n-action (as in Section 3, S_n denotes the symmetric group), (5.1) and (5.2) imply the following:

$$(5.3) \qquad v_n = (n-3)! \sum_{i=0}^{m-1} \binom{n}{2i} \int_{M_n} f(z_1)^{n-2i} g(z_1)^{2i} z_2 z_3 \cdots z_{2i+1}$$

STEP 2. Let $a_{n-2i,2i}$ be the coefficient of z^{n-3-2i} in $f(z)^{n-2i} g(z)^{2i}$, which is regarded as a formal power series. Then we can describe the right side of (5.3) in terms of $\langle \rho_{n,2m-2-2i} \rangle$ and $a_{2m+1-2i,2i}$. As $\langle \rho_{n,2k} \rangle$ is given in Theorem B, we can calculate the right side of (5.3). To state the result, we define A_m and B_m as follows.

$$A_m = (-1)^{m+1} \frac{(2m-2)!(2m+1)!}{(m-1)!m!} \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \frac{a_{2m+1-2i,2i}}{2m-2i}$$

and

$$B_m = (-1)^m \frac{(2m-2)!(2m+1)!}{(m-1)!m!} \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \frac{a_{2m+1-2i,2i}}{2m+1-2i}.$$

Then we have from (5.3) that

$$(5.4) \qquad v_n = A_m + B_m.$$

STEP 3. Note that $f(z) = \cosh z$ and $g(z) = \frac{\sinh z}{z}$. Hence we can regard $a_{2m+1-2i,2i}$ as the coefficient of z^{2m-2} in $\cosh^{2m+1-2i} z \sinh^{2i} z$. Since

$$\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \frac{1}{2m-2i} \cosh^{2m+1-2i} z \sinh^{2i} z$$

$$=\frac{\cosh z}{2m}\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}\cosh^{2m-2i}z\sinh^{2i}z$$

$$=\frac{\cosh z}{2m}(1+(-1)^{m+1}\sinh^{2m}z),$$

we calculate A_m as follows.

$$A_m=\frac{(-1)^{m+1}}{2}\frac{(2m+1)!}{m!m!}. \tag{5.5}$$

STEP 4. We determine B_m. We set

$$\phi_m(z)=\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\frac{\cosh^{2m+1-2i}z\sinh^{2i}z}{2m+1-2i}. \tag{5.6}$$

Since

$$\frac{d\phi_m(z)}{dz}=\sinh z\cosh^2 z-(m-1)(\sinh 2z)\phi_{m-1}(z),$$

we can prove the following equation by induction on m.

$$\phi_m(z)=\sum_{j=0}^{m}\alpha_{m,2j+1}\cosh(2j+1)z \tag{5.7}$$

with

$$\alpha_{m,2j+1}=\begin{cases}\dfrac{(-1)^{j+1}}{2}\dfrac{(m-1)!m!}{(2m+1)!}\dbinom{2m+1}{m-j} & 1\leq j\leq m\\[2ex] \dfrac{1}{2(m+1)} & j=0.\end{cases}$$

Note that $a_{2m+1-2i,2i}$ is the coefficient of z^{2m-2} in $\cosh^{2m+1-2i}z\sinh^{2i}z$ (cf. STEP 3). Hence (5.6) tells us that the term $\Sigma_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\frac{a_{2m+1-2i,2i}}{2m+1-2i}$ in B_m is equal to the coefficient of z^{2m-2} in $\phi_m(z)$. Then by (5.7), we can write B_m as follows.

$$B_m=\frac{(-1)^m}{2}\frac{(2m+1)!}{(m-1)!(m+1)!}+\frac{(-1)^{m+1}}{2}\sum_{j=1}^{m}(-1)^j\binom{2m+1}{m-j}(2j+1)^{2m-2}. \tag{5.8}$$

STEP 5. By (5.5), we see that

$$A_m+\text{the first term of (5.8)}=\frac{(-1)^{m+1}}{2}\binom{2m+1}{m}.$$

Hence by (5.4), we have

$$(5.9) \qquad v_n=\frac{(-1)^{m+1}}{2}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{m-j}(2j+1)^{2m-2}.$$

It is easy to see that (5.9) is equivalent to

$$(5.10) \qquad v_n=-\frac{1}{2}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}(2m+1-2j)^{2m-2}.$$

Using Lemma 4.14, it is easy to see that (5.10) equivalent to Theorem C for $n=2m+1$.

Department of Mathematics
University of the Ryukyus
e-mail: kamiyama@sci.u-ryukyu.ac.jp

References

[1] M. Brion, Cohomologie èquivariante des points semi-stables, J.Reine Angew. Math., **421** (1991), 125–140.
[2] J. Duistermaat and G. Heckman, On the variation in the cohomology of the symplectic form of the reduced phase space, Invent. Math., **69** (1982), 259–268.
[3] V. Guillemin, Moment maps and combinatorial invariants of Hamiltonian T^n-spaces, Progress in Mathematics 122, Birkhäuser, Boston, 1994.
[4] J-C. Hausmann and A. Knutson, Polygon spaces and Grassmannians, Enseign. Math., **43** (1997), 173–198.
[5] J-C. Hausmann and A. Knutson, The cohomology ring of polygon spaces, Ann. Inst. Fourier (Grenoble), **48** (1998), 281–321.
[6] Y. Kamiyama, Topology of equilateral polygon linkages, Topology Appl., **68** (1996), 13–31.
[7] Y. Kamiyama, The homology of singular polygon spaces, Canad. J. Math., **50** (1998), 581–594.
[8] M. Kapovich and J. Millson, The symplectic geometry of polygons in Euclidean space, J. Differential Geom., **44** (1996), 479–513.
[9] F. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Mathematical Notes 31, Princeton University Press, Princeton, 1984.
[10] F. Kirwan, The cohomology rings of moduli spaces of bundles over Riemann surfaces, J. Amer. Math. Soc., **5** (1992), 853–906.
[11] A. Klyachko, Spatial polygons and stable configurations of points in the projective line, Algebraic geometry and its applications (Yaroslavl', 1992), Aspects Math. E 25, Vieweg, Braunschweig, 1994, pp. 67–84.