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Height and arithmetic intersection for a family of semi-stable curves

By

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Abstract

In this paper, we consider an arithmetic Hodge index theorem for a family of semi-stable curves, generalizing Faltings-Hriljac's arithmetic Hodge index theorem for an arithmetic surface.

1. Introduction

In papers [4] and [7], Faltings and Hriljac independently proved the arithmetic Hodge index theorem on an arithmetic surface. Moriwaki [12] subsequently proved a higher dimensional case of Faltings-Hriljac's arithmetic Hodge index therem. In this paper, we consider an arithmetic Hodge index theorem for a family of semi-stable curves. Namely, we prove the following theorem.

Theorem A (cf. Theorem 5.2). *Let K be a finitely generated field over $\mathbf{Q}$, X_K a geometrically irreducible regular projective curve over K, and L_K a line bundle on X_K with* $\deg L_K = 0$. *Let $\bar{B}=(B,\bar{H})$ be a polarization of K, i.e., B a normal projective arithmetic variety with the function field K, and $\bar{H}$ a nef C^∞-hermitian $\mathbf{Q}$-line bundle on B. Let $(X \xrightarrow{f} B, \bar{L})$ be a model of (X_K, L_K) (see §4 for terminology). We make the following assumptions on the model:*

(a) *f is semi-stable;*

(b) *$X_{\mathbf{C}}$ and $B_{\mathbf{C}}$ are non-singular and $f_{\mathbf{C}}: X_{\mathbf{C}} \to B_{\mathbf{C}}$ is smooth.*

Let J_K be the Jacobian of X_K and $\Theta_{\bar{K}}$ a divisor on $J_{\bar{K}}$ which is a translation of the theta divisor on $\mathrm{Pic}^{g-1}(X_{\bar{K}})$ *by a theta characteristic. Then we have*

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d) \le -2\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}([L_K]),$$

where $[L_K]$ denotes the point of J_K corresponding to L_K (For the definition of a height function $\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}$, see §4).

Furthermore, we assume that H is ample and $c_1(\bar{H})$ is positive. Then the equality

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holds if and only if $\bar{L}$ satisfies the following properties:

(a) *There is a Zariski open set B'' of B with* $\operatorname{codim}_B(B\backslash B'')\geq 2$ *such that* $\deg(L|_C)=0$ *for any fibral curves C lying over B''.*

(b) *The restriction of the metric of $\bar{L}$ to each fiber is flat.*

We note that when B is the spectrum of the ring of integers, the above theorem is nothing but the arithmetic Hodge index theorem for a semi-stable arithmetic surface.

Our proof uses arithmetic Riemann-Roch theorem, similar to that of Faltings on an arithmetic surface, although we must consider the Quillen metric. Now we outline the organization of this paper. In §1, we recall some properties of relative Picard functors. In §2, we recall some facts on determinant line bundles, especially for semi-stable curves. In §3, we deal with an arithmetic setting and give hermitian metrics to the results of §2. In §4, we quickly review (a part of) the theory of height functions over a finitely generated field over $\boldsymbol{Q}$, due to Moriwaki [13]. Finally in §5, we prove the main theorem.

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1. The Picard functor

The purpose of this section is to review some properties of the relative Picard functor, which we will use later. We refer to [2, §§8–9] for details. In this section, we only deal with schemes which are locally noetherian.

Let S be a locally noetherian base scheme, $f: X\to S$ a flat, projective morphism. The *relative Picard functor* $\operatorname{Pic}_{X/S}$ of X over S is the fppf-sheaf associated with the functor

$$P_{X/S}: (\text{locally noetherian } S\text{-schemes}) \to (\text{Sets}),\ T\mapsto \operatorname{Pic}(X\times_S T).$$

If we assume $f_*(\mathcal{O}_X)=\mathcal{O}_S$ holds universally, then for all locally noetherian S-schemes $g: T\to S$,

$$\operatorname{Pic}_{X/S}(T)=\operatorname{Pic}(X\times_S T)/\operatorname{Pic}(T).$$

Furthemore, if X/S admits a section $\epsilon: S\to X$, then one checks immediately,

$$\operatorname{Pic}_{X/S}(T)=\left\{\begin{array}{c}\text{group of isomorphism classes of}\\ \text{invertible sheaves } L \text{ on } X\times_S T,\\ \text{plus isomorphism } (\epsilon\circ g,\ 1_T)^*(L)\simeq\mathcal{O}_T\end{array}\right\} \tag{1.1}$$

Such invertible sheaves are said to be *rigidified* along the induced section $\epsilon_T=\epsilon\circ g$.

If S consists of a field, then $\operatorname{Pic}_{X/S}$ is a group scheme. Let $\operatorname{Pic}^0_{X/S}$ be its identity component. For a general locally noetherian scheme S, we introduce $\operatorname{Pic}^0_{X/S}$ as the subfunctor of $\operatorname{Pic}_{X/S}$ which consists of all elements whose restrictions to all fibers X_s, s being a point of S, belong to $\operatorname{Pic}^0_{X_s/k(s)}$.

If X is a proper curve over a field k, then $\operatorname{Pic}^0_{X/k}$ consists of all elements of $\operatorname{Pic}_{X/k}$ whose partial degree on each irreducible components of $X\otimes_k\bar{k}$ is zero, where

$\bar{k}$ is an algebraic closure of k.

We note that if $\mathrm{Pic}_{X/S}$ (resp. $\mathrm{Pic}^0_{X/S}$) is representable by a locally noetherian scheme, then for all locally noetherian S-schemes T,

$$\mathrm{Pic}_{X/S} \times_S T = \mathrm{Pic}_{X \times_S T/T} \quad (\text{resp. } \mathrm{Pic}^0_{X/S} \times_S T = \mathrm{Pic}^0_{X \times_S T/T}).$$

Now we introduce the notion of universal line bundles when $\mathrm{Pic}_{X/S}$ (resp. $\mathrm{Pic}^0_{X/S}$) is representable by a locally noetherian scheme. We assume that the structural morphism $f: X \to S$ admits a section ϵ and that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally, so that $\mathrm{Pic}_{X/S}$ is given by (1.1) for a locally noetherian S-scheme. If $\mathrm{Pic}_{X/S}$ (resp. $\mathrm{Pic}^0_{X/S}$) is representable by a locally noetherian scheme, then the identity on $\mathrm{Pic}_{X/S}$ (resp. $\mathrm{Pic}^0_{X/S}$) gives rise to a line bundle U (resp. U^0) on $X \times_S \mathrm{Pic}_{X/S}$ (resp. $X \times_S \mathrm{Pic}^0_{X/S}$) which is rigidified along the induced section. U (resp. U^0) is called the *universal* line bundle. The justification of the notion of "universal" is the following proposition (cf. [2, 8.2. Proposition 4]).

Proposition 1.1. *Let $f: X \to S$ be a flat morphism of locally noetherian schemes and let ϵ be a section of f. Assume that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally. If* $\mathrm{Pic}_{X/S}$ *(resp.* $\mathrm{Pic}^0_{X/S}$*) is representable by a locally noetherian scheme, then the universal line bundle U has the following property: For every locally noetherian scheme $g: T \to S$, and for every line bundle L' on $X' \times_S T$ which is regidified along the induced section $\epsilon' = \epsilon \circ g$, there exists a unique morphism $g: T \to$* $\mathrm{Pic}_{X/S}$ *such that L' is isomorphic to $(1 \times g)^*(U)$.*

If $\mathrm{Pic}^0_{X/S}$ *is representable by a locally noetherian scheme, the universal line bundle U^0 has a similar property for a line bundle L' on $X' = X \times_S T$ which is rigidified along the induced section and $L'_t \in$* $\mathrm{Pic}^0_{X_t/k(t)}$ *for all $t \in T$.*

Now we restrict ourselves to the case of semi-stable curves. We recall that a *semi-stable curve of genus g* is a proper flat morphism $f: X \to S$ whose fiber $X_{\bar{s}}$ over geometric point $\bar{s}$ of S is a reduced connected curve with at most ordinary double points such that $\dim_{k(s)} H^1(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}})$ equals to g.

Proposition 1.2. *Let $f: X \to S$ be a semi-stable curve of locally noetherian schemes. Then $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally.*

Proof. We have only to prove that $f_*(\mathcal{O}_X) = \mathcal{O}_S$. Let $\pi \circ \tilde{f}$ be the Stein factorization of f, where $\tilde{f}: X \to \tilde{S}$ is a proper morphism with connected fibers and $\pi: \tilde{S} \to S$ is a finite morphism. Since every fiber is geometrically reduced and geometrically connected, there is a section $\eta: S \to \tilde{S}$ such that $\tilde{f} = \eta \circ f$ by rigidity lemma ([14, Proposition 6.1]). Since $\mathcal{O}_{\tilde{S}} \simeq \tilde{f}_*(\mathcal{O}_X)$ factors through

$$\mathcal{O}_{\tilde{S}} \to \eta_*(\mathcal{O}_S) \to f_*\eta_*(\mathcal{O}_S) = \tilde{f}_*(\mathcal{O}_X),$$

$\mathcal{O}_{\tilde{S}} \to \eta_*(\mathcal{O}_S)$ is injective. On the other hand, since η is a closed immersion, $\mathcal{O}_{\tilde{S}} \to \eta_*(\mathcal{O}_S)$ is surjective, hence $\mathcal{O}_{\tilde{S}} = \eta_*(\mathcal{O}_S)$. Then, $f_*(\mathcal{O}_X) = \pi_*\tilde{f}_*(\mathcal{O}_X) = \pi_*(\mathcal{O}_{\tilde{S}})$

$=\pi_*(\eta_*(\mathcal{O}_S))=\mathcal{O}_S$.

We finish this section by quoting a result obtained by Deligne concerning the representability of the relative Picard functor (cf. [2, 9.4. Theorem 1] or [3, Proposition 4.3]).

Theorem 1.3. *Let $f:X\to S$ be a semi-stable curve of locally noetherian schemes. Then* $\mathrm{Pic}_{X/S}$ *is a smooth algebraic space over S. The identity component* $\mathrm{Pic}^0_{X/S}$ *is a semi-abelian scheme.*

2. Determinant line bundles

The purpose of this section is to review some properties of determinant line bundles. Since we are concerned about a family of curves in this paper, we only consider determinant line bundles in a restricted context. For a general treatment of determinant line bundles, we refer to [11]. For the next theorem, we refer to [11] or [10, VI §6].

Theorem 2.1. *Let us consider a morphism $f:X\to S$ of noetherian schemes with the following conditions:*

(i) *f is proper, $f_*(\mathcal{O}_X)=\mathcal{O}_S$, and* $\dim f=1$.

(ii) *There is an effective Cartier divisor D on X such that D is f-ample and flat over S.*

For every $f:X\to S$ satisfying the above conditions, for every line bundle L on X and isomorphism of sheaves $\phi: L \xrightarrow{\sim} L'$, one can uniquely construct a line bundle $\det Rf_*(L')$ *on S and an isomorphism* $\det Rf_*(L)\xrightarrow{\sim}\det Rf_*(L')$ *in such a way that* $\det Rf_*(L)$ *becomes a functor with the following properties:*

(a) *If $f_*(L)$ and $R^1f_*(L)$ are both locally free, then*

$$\det Rf_*(L)=\det f_*(L)\otimes(\det R^1f_*(L))^{-1};$$

(b) $\det Rf_*(L)$ *is compatible with a base change, i.e., if $g:T\to S$ is a morphism of noetherian schemes, then*

$$g^*(\det Rf_*(L))\cong\det R(f_T)_*(L_T);$$

(c) *If S is connected and M is a line bundle on S, then*

$$\det Rf_*(L\otimes f^*(M))\cong\det Rf_*(L)\otimes M^{\chi},$$

where $\chi=\chi(C_s,L_s)$ for some $s\in S$;

(d) *If D is an effective Cartier divisor on X which is flat over S, then*

$$\det Rf_*(L)\cong\det Rf_*(L(-D))\otimes\det f_*(L|_D).$$

Suppose now that $f:X\to S$ is a semi-stable curve of noetherian schemes and assume that f admits a section ϵ. Moreover, let A be a rigidified line bundle on X of degree $g-1$. By Theorem 1.3, $\mathrm{Pic}^0_{X/S}$ is a semi-abelian scheme and there exists a universal line bundle U^0 on $X \times_S \mathrm{Pic}^0_{X/S}$. Let P^a be the scheme which is the translation of $\mathrm{Pic}^0_{X/S}$ by A, i.e.,

$$P^a(T) = \left\{ \begin{array}{c} \text{rigidified line bundle } L \text{ on } X_T \\ \text{such that } L \otimes A^{-1} \text{ belongs to } \mathrm{Pic}^0_{X/S} \end{array} \right\}.$$

Moreover, let U^a be the line bundle on P^a which is the translation of U^0 by A. If $q^a: X \times_S P^a \to P^a$ is the second projection, then q^a satisfies the condition of Theorem 2.1, because $f: X \to S$ satisfies the condition of Theorem 2.1. Thus the determinant line bundle $\det Rq^a_*(U^a)$ on P^a is defined. To simplify the notation, let us denote $\det Rq^a_*(U^a)$ by $\mathscr{T}^{-1}$.

In the following, we will see that $\mathscr{T}^{-1}$ is related to the theta divisor. Here we further assume that $f: X \to S$ is smooth of genus $g \geq 1$. First, we define the theta divisor.

Let $(X/S)^{(g-1)}$ be the symmetric $(g-1)$-fold product, i.e.,

$$(X/S)^{(g-1)} = \overbrace{X \times_S \cdots \times_S X}^{(g-1)\text{ times}} / \mathfrak{S}_{g-1},$$

where the $(g-1)$-th symmetric group $\mathfrak{S}_{g-1}$ acts on $\overbrace{X \times_S \cdots \times_S X}^{(g-1)\text{ times}}$ naturally. Let

$$(X/S)^{(g-1)} \to \mathrm{Pic}^{g-1}_{X/S}, \qquad D_T \to [D_T]$$

be a morphism, where for any locally noetherian S-schemes T and for any T-valued point D_T of $(X/S)^{(g-1)}$ (i.e., for any effective Cartier divisors on $X \times_S T$ of degree $(g-1)$), we denote by $[D_T]$ the element of $\mathrm{Pic}^{g-1}_{X/S}$ corresponding to D_T. The schematic image of this morphism, which turns out an effective relative Cartier divisor on $\mathrm{Pic}^{g-1}_{X/S}$, is called the theta divisor for X/S and denoted by $\Theta_{X/S}$.

Proposition 2.2. *Let $f: X \to S$ be a projective smooth morphism of noetherian schemes whose geometric fibers are smooth projective curves of genus $g \geq 1$. We assume the existence of a section. Let $\mathrm{Pic}^{g-1}_{X/S}$ be a Picard scheme of degree $(g-1)$ and U a universal line bundle on $X \times_S \mathrm{Pic}^{g-1}_{X/S}$. Then*

$$\det Rq_*(U) \cong \mathcal{O}_{\mathrm{Pic}^{g-1}_{X/S}}(-\Theta_{X/S}),$$

where $\Theta_{X/S}$ is the theta divisor for X/S and $q: X \times_S \mathrm{Pic}^{g-1}_{X/S} \to \mathrm{Pic}^{g-1}_{X/S}$ is the second projection.

Proof. When the base scheme is a point, or an arithmetic surface, this is well-known (cf. [4, §5] or [10, VI Lemma 2.4]). The proof for a general base scheme is similar to that for a point, as we will see in the following.

Let $p: X \times_S \mathrm{Pic}^{g-1}_{X/S} \to \mathrm{Pic}^{g-1}_{X/S}$ be the first projection. Let D' be an effective relative Cartier divisor of sufficiently large degree on X (actually $\deg D' \geq g$ is enough) and put $D = p^*(D')$. Since

$$H^0(X_s, U(-D)_t) = 0$$

for all points t of $\mathrm{Pic}^{g-1}_{X/S}$ and the point s of S lying below t, $q_*(U(-D)) = 0$ by [6, Corollorary II. 12.9], and $R^1 q_*(U(-D))$ is locally free. Thus, by (a) and (d) of Theorem 2.1,

$$\det Rq_*(U) = \det q_*(U|_D) \otimes (R^1 q_*(U(-D)))^{-1}.$$

Since $q_*(U)$ is torsion-free and $H^0(X_s, U_t) = 0$ for a general point t of P, it follows that $q_*(U) = 0$. Also, since $D \to \mathrm{Pic}^{g-1}_{X/S}$ is finite, $R^1 q_*(U|_D) = 0$. Thus we get the exact sequence:

$$0 \to q_*(U|_D) \to R^1 q_*(U(-D)) \to R^1 q_*(U) \to 0.$$

We denote the homomorphism $q_*(U|_D) \to R^1 q_*(U(-D))$ by α. Since $R^2 q_*(U) = 0$, we get by [6, Theorem II. 12.11]

$$R^1 q_*(U) \otimes k(t) \cong H^1(X_s, U_t)$$

for all points of $\mathrm{Pic}^{g-1}_{X/S}$ and the point s of S lying below t. If $R^1 q_*(U) \otimes k(t) = 0$, then $R^1 q_*(U)$ is also zero for some neighborhood of t, and especially $R^1 q_*(U)$ is flat for some neighborhood of t. Thus

$$\begin{aligned} \alpha(t) \text{ is an isomophism} &\Leftrightarrow R^1 q_*(U) \otimes k(t) = 0 \\ &\Leftrightarrow H^1(X_s, U_t) = 0 \\ &\Leftrightarrow t \notin \Theta_{X/S}. \end{aligned}$$

Therefore if we put $E = \{t \in \mathrm{Pic}^{g-1}_{X/S} \mid (\det \alpha)(t) = 0\}$, then $E = a\Theta_{X/S}$ for some positive integer a. By considering the case that the base scheme is a point, we get $a = 1$.

Now we put everything together and get:

Theorem 2.3. *Let $f: X \to S$ be a semi-stable curve of genus $g \geq 1$ of noetherian schemes and assume that f admits a section ϵ. Let A be a rigidified line bundle of degree $(g-1)$ and (P^a, U^a) the translation of $(\mathrm{Pic}^0_{X/S}, U^0)$ by A. We put $\mathscr{T}^{-1} = \det Rq^a_*(U^a)$, where $q^a: X \times_S P^a \to P^a$ is the second projection. Then,*

(i) *If $T \to S$ be a morphism of noetherian schemes such that $f_T: X \times_S T \to T$ is smooth, then*

$$\mathscr{T}_T^{-1} = \mathcal{O}_{P^a_T}(-\Theta_{X_T/T})$$

where $\Theta_{X_T/T}$ is the theta divisor for X_T/T.

(ii) *If L is a rigidified line bundle on X which belongs to $P^a(S)$, then there is a canonical morphism $g^a: S \to P^a$ such that the induced morphism*

$$u_L: \det Rf_*(L) \to (g^a)^*(\mathscr{T}^{-1})$$

is canonically isomorphic.

Proof. Noting that determinant line bundles are compatible with a base change, we have already seen (i). Regarding as (ii), by the universal property of U^a, there exists a canonical morphism $g^a: S \to P^a$ such that

$$L \cong (1 \times g^a)^*(U^a).$$

On the other hand, since deteminant line bundles are compatible with a base change, we have canonically

$$(g^a)^*(\det Rq^a_*(U^a)) \cong \det Rf_*((1 \times g^a)^*(U^a)).$$

Combining above two isomorphisms, we get the desired isomorphism.

3. Arithmetic setting

In this section, we consider an arithmetic setting. An *arithmetic variety* is an integral scheme which is flat and quasi-projective over Spec($\boldsymbol{Z}$).

Let $f: X \to B$ be a semi-stable curve of genus $g \geq 1$ of arithmetic varieties and assume that f admits a section ϵ. We also assume that $f_{\boldsymbol{C}}: X_{\boldsymbol{C}} \to B_{\boldsymbol{C}}$ is a smooth morphism. Let A be a rigidified line bundle of degree $(g-1)$ and (P^a, U^a) the translation of $(\mathrm{Pic}^0_{X/S}, U^0)$ by A. We put $\mathscr{T}^{-1} = \det Rq^a_*(U^a)$ on P^a, where $q^a: X \times P^a \to P^a$ is the second projection. Then by theorem 2.3(ii), for a rigidified line bundle L which belongs to $\mathrm{Pic}^0_{X/S}$, we have a natural isomorphism

$$u_L: \det Rf_*(L \otimes A) \to (g^a)^*(\mathscr{T}^{-1}),$$

where $g^a: S \to P^a$ is an induced morphism by $L \otimes A$.

In this section we give metrics on the above line bundles, and consider the norm of u_L. Let $\Theta_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}}$ be the theta divisor for $X_{\boldsymbol{C}}/B_{\boldsymbol{C}}$, which is a relative Cartier divisor on $P^a_{\boldsymbol{C}} = \mathrm{Pic}^{g-1}_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}}$. Then by Theorem 2.3(i), $\mathscr{T}^{-1}_{\boldsymbol{C}} = \mathscr{O}_{\mathrm{Pic}^{g-1}_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}}}(-\Theta_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}})$.

In the following, we introduce a metric on $\mathscr{O}_{\mathrm{Pic}^{g-1}_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}}}(-\Theta_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}})$. Put $J = \mathrm{Pic}^0_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}}$ and let

$$\lambda: \mathrm{Pic}^{g-1}_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}} \to J \quad [D_T] \mapsto (g-1)[\epsilon_T]$$

be an isomorphism, where for any $B_{\boldsymbol{C}}$-scheme T, $[\epsilon_T]$ is the class of the induced section by ϵ. Let $\Theta^0_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}}$ be the image of $\Theta_{X_{\boldsymbol{C}}/B_{\boldsymbol{C}}}$ by λ.

We need some definitions to proceed. The *Siegel upper-half space of deg2ree*

g, denoted by $\mathscr{H}_g$, is defined by

$$\mathscr{H}_g=\{\Omega=X+\sqrt{-1}Y\in \mathrm{GL}_g(\boldsymbol{C})\,|\,{}^t\Omega=\Omega,\ Y>0\}.$$

Moreover, the *symplectic group of degree* $2g$, denoted by $Sp_g(\boldsymbol{Z})$, is defined by

$$Sp_g(\boldsymbol{Z})=\{S\in GL_{2g}(\boldsymbol{Z})\,|\,{}^tSJS=J\},$$

where $J=\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$. An element $S=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $Sp_g(\boldsymbol{Z})$ acts on $\mathscr{H}_g$ by

$$S\cdot\Omega=(A\Omega+B)(C\Omega+D)^{-1}$$

and $Sp_g(\boldsymbol{Z})\backslash\mathscr{H}_g$ becomes a coarse moduli of principally polarized abelian varieties.

For $z=x+\sqrt{-1}y\in \boldsymbol{C}^g$ and $\Omega=X+\sqrt{-1}Y\in\mathscr{H}_g$, we define

$$\theta(z,\Omega)=\sum_{m\in\boldsymbol{Z}^g}\exp(\pi\sqrt{-1}\,{}^tm\Omega m+2\pi\sqrt{-1}\,{}^tm\cdot z),$$

$$\|\theta\|(z,\Omega)=\sqrt[4]{\det Y}\exp(-\pi\,{}^ty\,Yy)|\theta(z,\Omega)|.$$

Then θ becomes a holomorphic function on $\boldsymbol{C}^g\times\mathscr{H}_g$. Moreover $\|\theta\|$ becomes a C^∞-function which is periodic with respect to $\boldsymbol{Z}^g+\Omega\boldsymbol{Z}^g$, so that $\|\theta\|$ is seen as a C^∞-function on $\boldsymbol{C}^g/\boldsymbol{Z}^g+\Omega\boldsymbol{Z}^g$.

Going back to our situations, for any $b\in B(\boldsymbol{C})$, let us write analytically

$$J_b\cong\boldsymbol{C}^g/\boldsymbol{Z}^g+\Omega_b\boldsymbol{Z}^g$$

where $\Omega_b\in\mathscr{H}_g$. Then there is a unique element $t_b\in\boldsymbol{C}^g/\boldsymbol{Z}^g+\Omega\boldsymbol{Z}^g$ such that $\Theta^0_{X_b}=\mathrm{div}(\theta(z+t_b,\Omega_b))$, where $\theta(z+t_b,\Omega_b)$ is seen as a function of z.

Proposition 3.1. *With the notation being as above, let* $\mathbf{1}$ denote the section of $\mathcal{O}_J(\Theta_{X_C/B_C})$ *which corresponds to* $\Theta^0_{X_C/B_C}$. *For any* $p\in J$, *let* $b\in B(\boldsymbol{C})$ *be the point lying below* p *and write* $J_b\cong\boldsymbol{C}^g/\boldsymbol{Z}^g+\Omega_b\boldsymbol{Z}^g$ *and* $\Theta^0_{X_b}=\mathrm{div}(\theta(z+t_b,\Omega_b))$ *with* $t_b\in\boldsymbol{C}^g/\boldsymbol{Z}^g+\Omega\boldsymbol{Z}^g$. *Moreover, let* $z\in\boldsymbol{C}^g/\boldsymbol{Z}^g+\Omega\boldsymbol{Z}^g$ *correspond to* p. *Then, if we define*

$$\|\mathbf{1}\|_{\Theta^0_{X_C/B_C}}(p)=\|\theta\|(z+t_b,\Omega_b),$$

then $\|\cdot\|_{\Theta^0_{X_C/B_C}}$ *gives a* C^∞ *metric on* $\mathcal{O}_J(\Theta^0_{X_C/B_C})$

Proof. If the base space $B(\boldsymbol{C})$ is a point, the assertion is well-known (cf. [4, §3]). Thus all we need to prove is that $\|\mathbf{1}\|_{\Theta^0_{X_C/B_C}}$ varies smoothly as $b\in B(\boldsymbol{C})$ varies. However, since the morphism

$$\Phi: B(\boldsymbol{C})\to Sp_g(\boldsymbol{Z})\backslash\mathscr{H}_g,\quad b\mapsto\text{the class of } J_b$$

is holomorphic and t_b is given the difference of the section ϵ_C and a theta characteristic, $\|1\|_{\Theta^0_{X_C/B_C}}$ varies smoothly as $b\in B(C)$ varies.

Finally, $\mathcal{O}_{\mathrm{Pic}^{g-1}_{X_C/B_C}}(-\Theta_{X_C/B_C})$ is metrized by $(\mathcal{O}_J(\Theta^0_{X_C/B_C}),\|\cdot\|_{\Theta^0_{X_C/B_C}})$ through λ. We write this metric by $\|\cdot\|_{\Theta_{X_C/B_C}}$.

Next we give a C^∞ metric on L_C over X_C. Actually, there is a certain class of C^∞ metrics on L_C which is suitable for our purpose. We introduce this class in the following.

First we recall admissible metrics of line bundles on a compact Riemann surface. Let M be a compact Riemann surface of genus $g\geq 1$ and $\{\omega_1,\omega_2,\cdots,\omega_g\}$ a basis of $H^0(M,\Omega^1_M)$ with

$$\frac{\sqrt{-1}}{2}\int_M \omega_i\wedge\overline{\omega_j}=\delta_{ij}.$$

Let us put

$$\mu=\frac{\sqrt{-1}}{2g}\sum_{i=1}^{g}\omega_i\wedge\overline{\omega_i}.$$

Then μ is a positive (1,1)-form on M, and is called the canonical volume form on M. A C^∞-metric h_L of a line bundle L on M is said to be *admissible* if

$$c_1((L,h_L))=(\deg L)\mu.$$

For every line bundle on M, we can endow an admissible metric unique up to a constant multiplication.

Now let us go back to our situation, i.e., the case that $f:X_C\to B_C$ is a smooth family of curves of genus $g\geq 1$. A C^∞-metric h_L on L_C over X_C is said to be *admissible* if for any $b\in B(C)$, its restriction $(L_b,h_{L,b})$ on X_b is admissible. The following proposition guarantees the existence of an admissible metric.

Proposition 3.2. *Let X and B be smooth varieties over $\mathbf{C}$ and $f:X\to B$ a smooth projective morphism with a section whose fibers are curves of genus $g\geq 1$. Let L be a line bundle on X. Then there exists a (global) admissible metric on L over X.*

Proof. First we construct a suitable (1,1)-form on X. Let

$$j:X\to J=\mathrm{Pic}^0_{X/B}$$

is the embedding induced by the section. On J, we have a C^∞-hermitian line bundle $(\mathcal{O}_J(\Theta^0_{X/B}),\|\cdot\|_{\Theta^0_{X/B}})$ by Proposition 3.1. We consider

$$\omega=\frac{1}{g}j^*(c_1(\mathcal{O}_J(\Theta^0_{X/B}),\|\cdot\|_{\Theta^0_{X/B}})).$$

Then, for any $b \in B$, $\omega_b = \omega|_{X_b}$ is the canonical volume form on X_b (cf. [4, Thoerem 1]).

Let $\{U_i\}_{i=0}^{\infty}$ be an open covering of B. Let us set $X_{U_i} = f^{-1}(U_i)$. By taking suitable small open balls U_i, we may assume that $f|_{U_i}: X|_{U_i} \to U_i$ is differentiably trivial, i.e., there is a diffeomorphism $g_i: X_{U_i} \overset{\approx}{\to} X_{b_i} \times U_i$ over U_i with $b_i \in U_i$ ([9, Theorem 2.4]). Moreover, we take a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$.

Let h_0 be any C^∞-hermitian metric on L over X. We set $\eta = c_1(L, h_0)$, so that η is a d-closed real (1,1)-form on X. First, we claim that, for each i, $(\deg(L)\omega - \eta)|_{X_{U_i}}$ is d-exact over X_{U_i}. Indeed, $(\deg(L)\omega - \eta)|_{X_{b_i}} = 0$ in $H^2(X_{b_i}, \boldsymbol{C})$. On the other hand, $H^2(X_U, \boldsymbol{C}) = H^2(X_{b_i}, \boldsymbol{C})$ by Poincaré's lemma. Thus there is a real 1-form λ_i on X_{U_i} such that

$$(\deg(L)\omega - \eta)|_{X_{U_i}} = d(\lambda_i).$$

Now we set

$$\lambda = \sum_{i=0}^{\infty} f^*(\rho_i)\lambda_i$$

$$\tau = \sum_{i=0}^{\infty} f^*(d\rho_i) \wedge \lambda_i,$$

so that λ and τ are real forms on X. By definition, the equality

$$d(\lambda) = \deg(L)\omega - \eta + \tau$$

holds. If we denote by $\lambda^{(1,0)}$ (resp. $\lambda^{(0,1)}$) the $(1,0)$-part (resp. $(0,1)$-part) of λ and by $\tau^{(1,1)}$ the (1,1)-part of τ, then we have

$$\partial(\lambda^{(0,1)}) + \bar{\partial}(\lambda^{(1,0)}) = \deg(L)\omega - \eta + \tau^{(1,1)}.$$

Here, since X is projective, we can apply dd^c-lemma to $\partial(\lambda^{(0,1)})$ and $\bar{\partial}(\lambda^{(1,0)})$. Then there are C^∞-forms a, b on X with $\partial(\lambda^{(0,1)}) = dd^c(a)$ and $\bar{\partial}(\lambda^{(1,0)}) = dd^c(b)$. Since $\deg(L)\omega - \eta + \tau^{(1,1)}$ is a real form, if we set a C^∞-form on X by $\psi = \dfrac{a + b + \bar{a} + \bar{b}}{2}$, then we have

$$dd^c(\psi) = \deg(L)\omega - \eta + \tau^{(1,1)}.$$

Now if we set $h = \exp(-\psi)h_0$, then we have $c_1(L, h) = \deg(L)\omega + \tau^{(1,1)}$. On the other hand, since $\tau|_{X_b} = 0$ for any $b \in B$, we get $\tau^{(1,1)}|_{X_b} = 0$ for any $b \in B$. Therefore we obtain

$$c_1(L, h)|_{X_b} = \deg(L)\omega|_{X_b}$$

for any $b \in B$, which shows that h is an admissible metric on L over X.

Now we prove the main proposition of this section, which will be a key point to prove Proposition 5.1.

Proposition 3.3. *Let $f: X \to B$ be a semi-stable curve of genus $g \geq 1$ of arithmetic varieties and assume that f admits a section ϵ. We also assume that $f_{\mathbf{C}}: X_{\mathbf{C}} \to B_{\mathbf{C}}$ is a smooth morphism. Let*

$$u_L: \det Rf_*(L \otimes A) \to (g^a)^*(\mathcal{T}^{-1}),$$

be the isomorphism given at the beginning of this section. We endow C^∞ metrics on A and $\omega_{X/B}$, and an admissible metric on L, so that we have the Quillen metric on $\det Rf_*(L \otimes A)$ determined by these metrics. Moreover, we endow a metric $\|\cdot\|_{\Theta_{X_{\mathbf{C}}/B_{\mathbf{C}}}}^{-1}$ on $\mathcal{T}^{-1}$. Then the norm of u_L is independent of L.

Proof. Let $b \in B(\mathbf{C})$. Since determinant line bundles are compatible with a base change and since the Quillen metric is given fiberwise, we get

$$u_L: \det Rf_{b*}(L_b \otimes A_b) \to \mathcal{O}_{\mathrm{Pic}^{g-1}_{X_b}}(-\Theta_{X_b})|_{[L_b \otimes A_b]},$$

where $[L_b \otimes A_b]$ is the point corresponding to $L_b \otimes A_b$ on $\mathrm{Pic}^{g-1}_{X_b}$. Then by the following lemma, we obtain Proposition 3.3.

Lemma 3.4. *Let M be a compact Riemann surface of genus $g \geq 1$, L a line bundle of degree 0 on M. We endow a C^∞-metric h_A on A, a C^∞-metric $h_{\Omega^1_M}$ on Ω^1_M, and an admissible metric h_L on L. Then we have a canonical isomorphism*

$$u_L: \det \Gamma(L \otimes A) \to \mathcal{O}_{\mathrm{Pic}^{g-1}_M}(-\Theta_M)|_{[L \otimes A]},$$

where $\det \Gamma(L \otimes A)$ is the determinant line bundle of $L \otimes A$. We endow the Quillen metric on $\det \Gamma(L \otimes A)$ and $\|\cdot\|_{\Theta_M}^{-1}$ on $\mathcal{O}_{\mathrm{Pic}^{g-1}_M}(-\Theta_M)$. Then the norm of u_L is independent of L.

Proof. Let h'_A and $h'_{\Omega^1_M}$ be admissible metrics on A and Ω^1_M respectively. We write the Quillen metric defined by $(L \otimes A,\ h_L \otimes h_A)$ and $(\Omega^1_M, h_{\Omega^1_M})$ as $h_Q^{\bar{L} \otimes \bar{A}}$. We also write the Quillen metric defined by $(L \otimes A,\ h_L \otimes h'_A)$ and $(\Omega^1_M, h'_{\Omega^1_M})$ as $h_Q^{\bar{L} \otimes \bar{A}'}$. We decompose u_L into

$$(\det \Gamma(L \otimes A), h_Q^{\bar{L} \otimes \bar{A}}) \xrightarrow{\alpha} (\det \Gamma(L \otimes A), h_Q^{\bar{L} \otimes \bar{A}'})$$
$$\xrightarrow{\beta} (\det \Gamma(L \otimes A), h_F^{L \otimes A}) \xrightarrow{\gamma} \mathcal{O}_{\mathrm{Pic}^{g-1}_M}(-\Theta_M)|_{[L \otimes A]},$$

where $h_F^{L \otimes A}$ is the Faltings' metric on $L \otimes A$. By the definition of the Quillen metrics, the norm of α is independent of L, because we only change the metric of

A. The norm of β is the difference of the Quillen metric and the Faltings' metric for admissible line bundles, which is a constant depending only on M (cf. [15, 4.5]). Moreover, the norm of γ is also independent of L, which is actually given by $\exp(\delta(M)/8)$ with the Faltings' delta function $\delta(M)$ (Or rather, this is the definition of $\delta(M)$). Therefore the norm of u_L is independent of L.

4. Arithmetic height function over function fields

A. Moriwaki [13] has recently constructed a theory of arithmetic height function over function fields, with which he recovered the original Raynaud theorem (i.e., over a finitely generated field over $\boldsymbol{Q}$). In this section, we see a part of his theory.

Let K be a finitely generated field over $\boldsymbol{Q}$ with $\mathrm{tr.deg}_K(\boldsymbol{Q})=d$. Let B be a normal projective arithmetic variety with the function field K. Let $\bar{H}$ be a *nef* C^∞-hermitian $\boldsymbol{Q}$-line bundle on B, i.e., $\widehat{\deg}(\bar{H}|_C)\geq 0$ for any curve C and $c_1(\bar{H})$ is semi-positive on $B(\boldsymbol{C})$. A pair $\bar{B}=(B,\bar{H})$ with the above properties is called a *polarization* of K. Moreover, we say that a polarization $\bar{B}$ is *big* if $\mathrm{rk}\, H^0(B,H^{\otimes m})$ grows the order of m^d and that there is a non-zero section s of $H^0(B,H^{\otimes n})$ with $\|s\|_{\sup}<1$ for some positive integer n.

Let X_K be a projective variety over K and L_K a line bundle on X_K. By a *model* of (X_K,L_K) over B, we mean a pair $(X\xrightarrow{f}B,\ \bar{L})$ where $f: X\rightarrow B$ is a projective morphism of arithmetic varieties and $\bar{L}=(L,h_L)$ is a C^∞-hermitian $\boldsymbol{Q}$-line bundle on X such that, on the generic fiber, X and L coincide with X_K and L_K respectively.

By abbreviation, a model $(X\xrightarrow{f}B,\ \bar{L})$ is sometimes written as $(X,\bar{L})$. We note that although we use the notation X_K and L_K, a model of (X_K,L_K) is not a priori determined.

For $P\in X(\bar{K})$, we denote by Δ_P the Zariski closure of the Image $(\mathrm{Spec}(\bar{K})\xrightarrow{P}X_K)$ in X. Then we define the height of P with respect to $(X\xrightarrow{f}B,\ \bar{L})$ to be

$$h^{\bar{B}}_{(X,\bar{L})}(P)=\frac{1}{[K(P):K]}\widehat{\deg}(\hat{c}_1(\bar{L}|_{\Delta_P})\cdot\hat{c}_1(f^*\bar{H}|_{\Delta_P})^d).$$

If we change models of (X_K,L_K), then height functions differ by only bounded functions on $X_K(\bar{K})$. Namely, if $(X,\bar{L})$ and $(X',\bar{L}')$ are two models of (X_K,L_K), then there is a constant $C>0$ with

$$|h^{\bar{B}}_{(X,\bar{L})}(P)-h^{\bar{B}}_{(X',\bar{L}')}(P)|\leq C \tag{4.1}$$

for all $P\in X_K(\bar{K})$ ([13, Corollary 3.3.5]). Thus the height associated with L_K and $\bar{B}$ is well-defined up to bounded functions on $X_K(\bar{K})$. We denote $h^{\bar{B}}_{L_K}$ the class of $h^{\bar{B}}_{(X,\bar{L})}$ modulo bounded functions.

Now let $L_{\bar{K}}$ be a line bundle on $X_{\bar{K}}=X\otimes_K\bar{K}$. We would like to define $h^{\bar{B}}_{L_{\bar{K}}}: X_{\bar{K}}\rightarrow \boldsymbol{R}$. For this, we need the following proposition (cf. [13, Proposition 3.3.1]).

Proposition 4.1. *Let K' be a finite extension field of K, and let $g: B' \to B$ be a morphism of projective normal arithmetic varieties such that the function field of B' is K'. Let X' be the main component of $X \times_B B'$ and*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ B' & \xrightarrow{g} & B \end{array}$$

the induced morphism. Then $h^{(B', g^*(\bar{H}))}_{(X', g^*(\bar{L}))} = [K':K]\; h^{(B,\bar{H})}_{(X,\bar{L})}$.

Let $L_{\bar{K}}$ be a line bundle on $X_{\bar{K}}$. We take a finite extension field K' of K such that $L_{\bar{K}}$ is defined over $X_{K'}$. Take a projective normal arithmetic variety B' such that there is a morphism $g: B' \to B$ and that the function field of B' is K'. Let X' be the main component of $X \times'_B B'$. We take a blow-up $\tilde{X}' \to X'$ if necessary so that $L_{\bar{K}}$ extends to a line bundle $\tilde{L}'$ on $\tilde{X}'$.

Then we define

$$h^{\bar{B}}_{L_{\bar{K}}} = \frac{1}{[K':K]} h^{(B', g^*(\bar{H}))}_{(\tilde{X}', \tilde{L}')}$$

By (4.1) and Proposition 4.1, it is easy to see that $h^{\bar{B}}_{L_{\bar{K}}}$ is well-defined up to bounded functions on $X_{\bar{K}}(\bar{K})$. Moreover, if $L_{\bar{K}}$ is defined over X_K, then $h^{\bar{B}}_{L_{\bar{K}}}$ is equal to $h^{\bar{B}}_{L_K}$.

The next theorem shows some fundamental properties of $h^{\bar{B}}_{L_{\bar{K}}}$ (cf. [13, Proposition 3.3.6 and Theorem 4.3]).

Theorem 4.2. (i) (*positiveness*) *If we denote* Supp(Coker $(H^0(X_{\bar{K}}, L_{\bar{K}}) \otimes \mathcal{O}_{X_{\bar{K}}}) \to L_{\bar{K}}$) *by* $\mathrm{Bs}(L_{\bar{K}})$, *then $h^{\bar{B}}_{L_{\bar{K}}}$ is bounded below on $(X_{\bar{K}} \backslash \mathrm{Bs}(L_{\bar{K}}))$.*

(ii) (*Northcott*) *Assume $\bar{H}$ is big and that $L_{\bar{K}}$ is ample. Then for any $e \geq 1$ and $M \geq 0$,*

$$\{P \in X_{\bar{K}}(\bar{K}) \mid h^{\bar{B}}_{L_{\bar{K}}}(P) \leq M, \quad [K(P):K] \leq e\}$$

is a finite set.

If $X_{\bar{K}}$ is an abelian variety, we can choose the good representative of a class $h^{\bar{B}}_{L_{\bar{K}}}$. For a line bundle $L_{\bar{K}}$ on $X_{\bar{K}}$ and a point $P \in X_{\bar{K}}(\bar{K})$, define $q^{\bar{B}}_{L_{\bar{K}}}(P, P)$ and $l^{\bar{B}}_{L_{\bar{K}}}(P)$ to be

$$q^{\bar{B}}_{L_{\bar{K}}}(P, P) = \lim_{n \to \infty} \frac{1}{4^n} h^{\bar{B}}_{L_{\bar{K}}}(2^n P)$$

$$l^{\bar{B}}_{L_{\bar{K}}}(P)=\lim_{n\to\infty}\frac{1}{2^n}\left(\frac{1}{4^n}h^{\bar{B}}_{L_{\bar{K}}}(2^nP)-q^{\bar{B}}_{L_{\bar{K}}}(P,P)\right).$$

Then $q^{\bar{B}}_{L_{\bar{K}}}$ is a bilinear form, while $l^{\bar{B}}_{L_{\bar{K}}}$ is a linear form. We define $\hat{h}^{\bar{B}}_{L_{\bar{K}}}$ by

$$\hat{h}^{\bar{B}}_{L_{\bar{K}}}(P)=q^{\bar{B}}_{L_{\bar{K}}}(P,P)+l^{\bar{B}}_{L_{\bar{K}}}(P),$$

and call it the *canonical height* of $L_{\bar{K}}$ with respect to a polarization $\bar{B}$.

Proposition 4.3. *Let $X_{\bar{K}}$ be an abelian variety.*
(i) *If $L_{\bar{K}}$ is ample and symmetric, then $\hat{h}^{\bar{B}}_{L_{\bar{K}}}\geq 0$.*
(ii) *If $L_{\bar{K}}$ and $M_{\bar{K}}$ are two line bundles on $X_{\bar{K}}$, then*

$$\hat{h}^{\bar{B}}_{L_{\bar{K}}\otimes M_{\bar{K}}}(P)=\hat{h}^{\bar{B}}_{L_{\bar{K}}}(P)+\hat{h}^{\bar{B}}_{M_{\bar{K}}}(P)$$

(iii) *If P is a torsion point, then $\hat{h}^{\bar{B}}_{L_{\bar{K}}}(P)=0$. If we assume $\bar{H}$ is big, then $\hat{h}^{\bar{B}}_{L_{\bar{K}}}(P)=0$ if and only if P is a torsion point.*

Proof. The first assertion follows from Theorem 4.2(i). The second assertion can be readily checked. The third assertion is an easy consequence of Theorem 4.2(ii). We note that in (i) we need the symmetricity of a line bundle.

We need the next lemma to prove Proposition 5.1.

Lemma 4.4. *Let $L_{\bar{K}}$ is an ample symmetric line bundle on an abelian variety $X_{\bar{K}}$, P an element of $X_{\bar{K}}(\bar{K})$. Let t be an element of $X_{\bar{K}}(\bar{K})$ and $T_t: X_{\bar{K}}\to X_{\bar{K}}$ the translation by t. Then there is a constant C such that*

$$|\hat{h}^{\bar{B}}_{T_t^*(L_{\bar{K}})}(nP)-n^2\hat{h}^{\bar{B}}_{L_{\bar{K}}}(P)|=Cn$$

for any positive integers n.

Proof. Let $T_{-t}: X_{\bar{K}}\to X_{\bar{K}}$ be the translation by $-t$. We write $T_t^*(L_{\bar{K}})^{\otimes 2}$ as

$$T_t^*(L_{\bar{K}})^{\otimes 2}=(T_t^*(L_{\bar{K}})\otimes T_{-t}^*(L_{\bar{K}}))\otimes(T_t^*(L_{\bar{K}})\otimes(T_{-t}^*(L_{\bar{K}}))^{-1})$$

Since $T_t^*(L_{\bar{K}})\otimes T_{-t}^*(L_{\bar{K}})=L_{\bar{K}}^{\otimes 2}$ by the theorem of square, we obtain

$$T_t^*(L_{\bar{K}})^{\otimes 2}=(L_{\bar{K}}^{\otimes 2})\otimes(T_t^*(L_{\bar{K}})\otimes(T_{-t}^*(L_{\bar{K}}))^{-1}).$$

Thus we get $4\hat{h}^{\bar{B}}_{T_t^*(L_{\bar{K}})}=4\hat{h}^{\bar{B}}_{L_{\bar{K}}}+\hat{h}^{\bar{B}}_{T_t^*(L_{\bar{K}})\otimes(T_{-t}^*(L_{\bar{K}}))^{-1}}$. Since $L_{\bar{K}}$ is symmetric and $T_t^*(L_{\bar{K}})\otimes(T_{-t}^*(L_{\bar{K}}))^{-1}$ is anti-symmetric, $\hat{h}^{\bar{B}}_{L_{\bar{K}}}$ is quadric, while $\hat{h}^{\bar{B}}_{T_t^*(L_{\bar{K}})\otimes(T_{-t}^*(L_{\bar{K}}))^{-1}}$ is linear. Thus if we set $C=|\hat{h}^{\bar{B}}_{T_t^*(L_{\bar{K}})\otimes(T_{-t}^*(L_{\bar{K}}))^{-1}}(P)$, then we obtain the lemma.

5. Height and intersection

By a *big* Zariski open set of a noethrian scheme B, we mean a Zariski open set B' of B with $\operatorname{codim}_B(B\setminus B')\geq 2$.

We first prove the following proposition, which is a special case of the main theorem (Theorem 5.2).

Proposition 5.1. *Let K be a finitely generated field over Q, X_K a geometrically irreducible regular projective curve over K, and L_K a line bundle on X_K with $\deg L_K=0$. Let $\bar{B}=(B,\bar{H})$ be a polarization of K, and $(X\xrightarrow{f}B,\ \bar{L})$ a model of (X_K, L_K). We make the following assumptions on the model:*

(a) *B is regular;*

(b) *f is semi-stable with a section ϵ;*

(c) *X_C and B_C are non-singular and $f_C: X_C\to B_C$ is smooth.*

Let J_K be the Jacobian of X_K and $\Theta_{\bar{K}}$ a divisor on $J_{\bar{K}}$ which is a translation of the theta divisor on $\operatorname{Pic}^{g-1}(X_{\bar{K}})$ *by a theta characteristic. If there is a big Zariski open set $B'\subset B$ such that* $\deg(L|_C)=0$ *for any fibral curve C lying over B' and if the metric of $\bar{L}$ is flat along fibers, then*

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2\cdot\hat{c}_1(f^*(\bar{H}))^d)=-2\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}([L_K]), \tag{5.1}$$

where $[L_K]$ denotes the point of J_K corresponding to L_K.

Proof. We note that since $\deg L=0$, the admissibility of $\bar{L}$ means that the metric of $\bar{L}$ is flat along fibers. Since $\deg(L_K)=0$, if we change $\bar{L}$ to $\bar{L}\otimes f^*(\bar{M})$ with $\bar{M}$ being a hermitian line bundle on B, then each side of (5.1) does not change. Thus we may assume that L is rigidified along the section ϵ. Let us set $A=\mathcal{O}_X((g-1)[\epsilon])$. Then A is a rigidified line bundle of degree $(g-1)$ on X. Let (P^a, U^a) be the translation of $(\operatorname{Pic}^0_{X/B}, U^0)$ by A, where U^0 is the universal line bundle on $X\times_B\operatorname{Pic}^0_{X/B}$. We put $\mathcal{T}^{-1}=\det Rq^a_*(U^a)$, where $q^a: X\times_B P^a\to P^a$ is the second projection.

We give an admissible metric h_A on A and an admissible metric $h_{\omega_{X/B}}$ on $\omega_{X/B}$ and then give $\det Rf_*(L^{\otimes n}\otimes A)$ the Quillen metric $h_Q^{\bar{L}^{\otimes n}\otimes\bar{A}}$ with respect to $\bar{L}^{\otimes n}\otimes\bar{A}=(L^{\otimes n}\otimes A,\ h_L^n\cdot h_A)$ and $\overline{\omega_{X/B}}=(\omega_{X/B}, h_{\omega_{X/B}})$. Moreover we endow $\|\cdot\|^{-1}_{\Theta_{X_C/B_C}}$ on $\mathcal{T}^{-1}$ (cf. Proposition 3.1).

Let us put $X'=f^{-1}(B')$, $f'=f|_{X'}$ and $A'=A|_{X'}$. Moreover Let $(P^{a'}, U^{a'})$, $(\operatorname{Pic}^0_{X'/B'}, U^{0'})$, $q^{a'}$ and $\mathcal{T}^{-1'}=\det Rq^{a'}_*(U^{a'})$ be the restriction of (P^a, U^a), $(\operatorname{Pic}^0_{X/B}, U^0)$, q^a and $\mathcal{T}^{-1}=\det Rq^a_*(U^a)$ over B', respectively.

Now we consider $L'^{\otimes n}\otimes A'$ for a positive integer n. Since $\deg(L'|_C)=0$ for any fibral curve lying over B', L' belongs to $\operatorname{Pic}^0_{X'/B'}$. Thus by Theorem 2.3(ii), there is a canonical morphism $g'_n: B'\to P^{a'}$ such that

$$u'_n: \det Rf'_*(L'^{\otimes n}\otimes A')\tilde{\to} g'^*_n(\mathcal{T}^{-1'})$$

is canonically isomorphic over B'. Since both sides are metrized, we can consider the norm α_n of u'_n. Then

$$u'_n:(\det Rf'_*(L'^{\otimes n}\otimes A'), h_Q^{\bar{L}^{\otimes n}\otimes\bar{A}}) \tilde{\to} g'^*_n(\mathscr{T}^{-1\prime}, \|\cdot\|^{-1}_{\Theta_{X_C/B_C}})\otimes\mathcal{O}_{B'}(\alpha_n^{-1})$$

is an isometry. Moreover, by Proposition 3.3, the function $\alpha_n : B_C(C)\to \boldsymbol{R}_{>0}$ is independent of n.

Next we consider a compactification of P^a. Since there is a relatively ample line bundle on P^a, we first embed P^a into a large projective space $\boldsymbol{P}^N_B$ and then take its closure. If $\mathscr{T}^{-1}$ does not extend to a line bundle on this closure, then we make blow-ups along the boundary. Then we get a projective arithmetic variety $\underline{P}^a$ with $\pi:\underline{P}^a\to B$ and a line bundle $\underline{\mathscr{T}}^{-1}$ on $\underline{P}^a$ with $\underline{\mathscr{T}}^{-1}|_{P^a}=\mathscr{T}^{-1}$. We note that since f_C is smooth, $\underline{P}^a_C=P^a_C$

Let Δ_n be the Zariski closure of the Image$(g'_n: B'\to P^{a\prime})$ in $\underline{P}^a$. Now we claim the following equation;

$$\text{(5.2)}\quad \widehat{\deg}(\hat{c}_1(\det Rf_*(L^{\otimes n}\otimes A), h_Q^{\bar{L}^{\otimes n}\otimes\bar{A}})\cdot\hat{c}_1(\bar{H})^d)$$
$$=\widehat{\deg}(\hat{c}_1(\mathcal{O}_{P^a}(\mathscr{T}^{-1}), \|\cdot\|^{-1}_{\Theta_{X_C/B_C}})|_{\Delta_n}\cdot\hat{c}_1(\pi^*(\bar{H}))^d|_{\Delta_n})-\frac{1}{2}\int_{B_C(C)}(\log\alpha_n)\wedge c_1(\bar{H})^d.$$

Actually, since B is regular and B' is big, a line bundle on B' extends uniquely to a line bundle on B. The line bundle $\det Rf'_*(L'^{\otimes n}\otimes A')$ on B' extends to $\det Rf_*(L^{\otimes n}\otimes A)$ and the line bundle $g'^*_n(\mathscr{T}^{-1\prime})$ on B' extends to a line bundle on B, which we denote by M_n. Let us set $\overline{M_n}=(M_n, g'^*_n(\|\cdot\|^{-1}_{\Theta_{X_C/B_C}}))$. Since $\pi|_{\Delta_n}:\Delta_n\to B$ is an isomorphism over B' and $\operatorname{codim}_B(B\backslash B')\geq 2$, $\overline{M_n}$ is actually equal to $(\pi|_{\Delta_n})_*(\mathcal{O}_{\underline{P}^a}(\underline{\mathscr{T}}^{-1}), \|\cdot\|^{-1}_{\Theta_{X_C/B_C}})$. Then since the infinite part is not changed at all, we get the isometry

$$u_n:(\det Rf_*(L^{\otimes n}\otimes A), h_Q^{\bar{L}^{\otimes n}\otimes\bar{A}}) \tilde{\to} \overline{M_n}\otimes\mathcal{O}_B(\alpha_n^{-1}).$$

Then by intersecting $\hat{c}_1(\bar{H})^d$ and taking degrees on both sides, we get

$$\widehat{\deg}(\hat{c}_1(\det Rf_*(L^{\otimes n}\otimes A), h_Q^{\bar{L}^{\otimes n}\otimes\bar{A}})\cdot\hat{c}_1(\bar{H})^d)$$
$$=\widehat{\deg}(\hat{c}_1(\overline{M_n})\cdot\hat{c}_1(\bar{H})^d)-\frac{1}{2}\int_{B_C(C)}(\log\alpha_n)\wedge c_1(\bar{H})^d$$
$$=\widehat{\deg}(\hat{c}_1(\mathcal{O}_{\underline{P}^a}(\underline{\mathscr{T}}^{-1}), \|\cdot\|^{-1}_{\Theta_{X_C/B_C}})|_{\Delta_n}\cdot\hat{c}_1(\pi^*(\bar{H}))^d|_{\Delta_n})-\frac{1}{2}\int_{B_C(C)}(\log\alpha_n)\wedge c_1(\bar{H})^d,$$

where we use the projection formula in the second equality.

First we conmpute the left hand side of (5.2). By the arithmetic Riemann-Roch theorem established by Gillet and Soulé [5], we have

$$\hat{c}_1(\det Rf_*(L^{\otimes n}\otimes A), h_Q^{\bar{L}^{\otimes n}\otimes \bar{A}})$$

$$=\frac{1}{2}f_*(\hat{c}_1(\bar{L}^{\otimes n}\otimes \bar{A})^2-\hat{c}_1(\bar{L}^{\otimes n}\otimes \bar{A})\cdot \hat{c}_1(\overline{\omega_{X/B}}))+\hat{c}_1(\det Rf_*(\mathcal{O}_X), h_Q^{\bar{\mathcal{O}}_X})$$

$$=\frac{1}{2}f_*(\hat{c}_1(\bar{L})^2)n^2+O(n).$$

Thus, we obtain

$$\widehat{\deg}(\hat{c}_1(\det Rf_*(L^{\otimes n}\otimes A), h_Q^{\bar{L}^{\otimes n}\otimes \bar{A}})\cdot \hat{c}_1(\bar{H})^d)\tag{5.3}$$

$$=\frac{1}{2}\widehat{\deg}(f_*(\hat{c}_1(\bar{L})^2)\cdot \hat{c}_1(\bar{H})^d)n^2+O(n)$$

$$=\frac{1}{2}\widehat{\deg}(\hat{c}_1(\bar{L})^2\cdot \hat{c}_1(f^*(\bar{H}))^d)n^2+O(n)$$

Next we compute the right hand side of (5.2). Let $\lambda_a:\mathrm{Pic}^0_{X/B}\xrightarrow{\sim}P^a$ be the isomorphism which is given by the translation by A. By way of this identification, let $\underline{P}^0$ be the compactification of $\mathrm{Pic}^0_{X/B}$ which corresponds to $\underline{P}^a$. Similarly, we define $(\underline{\mathscr{T}}^0)^{-1}$, Δ_n^0 and π^0 which correspond to $\underline{\mathscr{T}}^{-1}$, Δ_n and π respectively. We note that a metric on $(\underline{\mathscr{T}}^0)^{-1}$ induced from λ_a is nothing but $\|\cdot\|^{-1}_{\bar{\Theta}^0_{X_C/B_C}}$ by Proposition 3.1. Then we have $\underline{\mathscr{T}}^0_K=\mathcal{O}_{J_K}(\Theta'_K)$, where

$$\Theta'_K=\Theta_K+[\text{a theta characteristic}]-(g-1)[\epsilon_K].$$

Since $(\pi^0:\underline{P}^0\to B, ((\underline{\mathscr{T}}^0)^{-1}, \|\cdot\|^{-1}_{\bar{\Theta}^0_{X_C/B_C}}))$ is a model of $(J_K, \mathcal{O}_{J_K}(-\Theta'_K))$, (4.1) shows that there is a constant C such that

$$|\widehat{\deg}(\hat{c}_1(\mathcal{O}_{\underline{P}^a}(\underline{\mathscr{T}}^{-1}), \|\cdot\|^{-1}_{\bar{\Theta}_{X_C/B_C}})|_{\Delta_n}\cdot \hat{c}_1(\pi^*(\bar{H}))^d|_{\Delta_n})-\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(-\Theta'_{\bar{K}})}([L_{\bar{K}}^{\otimes n}])|$$

$$=|\widehat{\deg}(\hat{c}_1(\mathcal{O}_{\underline{P}^0}((\underline{\mathscr{T}}^0)^{-1}), \|\cdot\|^{-1}_{\bar{\Theta}^0_{X_C/B_C}})|_{\Delta_n}\cdot \hat{c}_1(\pi^*(\bar{H}))^d|_{\Delta_n})-\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(-\Theta'_{\bar{K}})}([L_{\bar{K}}^{\otimes n}])|\le C$$

Then using Lemma 4.4, we get

$$|\widehat{\deg}(\hat{c}_1(\mathcal{O}_{\underline{P}^a}(\underline{\mathscr{T}}^{-1}), \|\cdot\|^{-1}_{\bar{\Theta}_{X_C/B_C}})|_{\Delta_n}\cdot \hat{c}_1(\pi^*(\bar{H}))^d|_{\Delta_n})-n^2\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}([L_K])|=O(n).\tag{5.4}$$

Taking into consideration (5.3) and (5.4) and the fact that α_n is independent of n, if we divede (5.2) by n^2 and let n goes to ∞, we get (5.1).

Now we prove the main theorem of this paper.

Theorem 5.2. *Let K be a finitely generated field over $\mathbf{Q}$, X_K a geometrically irreducible regular projective curve over K, and L_K a line bundle on X_K with $\deg L_K=0$. Let $\bar{B}=(B,\bar{H})$ be a polarization of K, and $(X\xrightarrow{f}B,\ \bar{L})$ a model of*

(X_K, L_K). *We make the following assumptions on the model:*

(a) *f is semi-stable;*

(b) *X_C and B_C are non-singular and $f_C: X_C \to B_C$ is smooth.*

Then we have

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d) \leq -2\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}([L_K]),$$

where $[L_K]$ denotes the point of J_K corresponding to L_K.

Furthermore, we assume that H is ample and $c_1(\bar{H})$ is positive. Then the equality holds if and only if $\bar{L}$ satisfies the following properties:

(a) *There is a big Zariski open set B'' of B such that* $\deg(L|_C) = 0$ *for any fibral curves C lying over B''.*

(b) *The metric of $\bar{L}$ is flat along fibers.*

The next corollary is an immediate consequence of the main theorem and Proposition 4.3(iii).

Corollary 5.3. *Let the notation and the assumption be as in Theorem* 5.2. *We assume that $\bar{H}$ is big, H is ample and $c_1(H)$ is positive. Then*

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d) = 0$$

if and only if the following properties hold:

(a) *There is a big Zariski open set B'' of B such that* $\deg(L|_C) = 0$ *for any fibral curves C lying over B'';*

(b) *The restriction of the metric of $\bar{L}$ to each fiber is flat;*

(c) *There is a positive integer m with $L_K^{\otimes m} = \mathcal{O}_{X_K}$.*

We need three lemmas to prove the theorem.

Lemma 5.4. *Let $\tilde{K}$ be a finite extension field of K, and let $g: \tilde{B} \to B$ be a morphism of projective normal arithmetic varieties such that the function field of $\tilde{B}$ is $\tilde{K}$. Let $\tilde{X} = X \times_B \tilde{B}$ and*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle f}\downarrow \\ \tilde{B} & \xrightarrow{g} & B \end{array}$$

the induced morphism. Then

$$\widehat{\deg}(\hat{c}_1(\tilde{g}^*\bar{L})^2 \cdot \hat{c}_1(\tilde{f}^*g^*(\bar{H}))^d) = [\tilde{K}:K]\widehat{\deg}(\hat{c}_1(\bar{L})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d).$$

Proof. It is an easy consequence of the projection formula.

Lemma 5.5. *Let $\bar{L}=(L,h_L)$ be a C^∞-hermitian line bundle on X and $\bar{L}'=(L,h'_L)$ be a hermitian line bundle whose metric is flat along fibers. Then*

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2\cdot\hat{c}_1(f^*(\bar{H}))^d)\le\widehat{\deg}(\hat{c}_1(\bar{L}')^2\cdot\hat{c}_1(f^*(\bar{H}))^d).$$

If $c_1(\bar{H})$ is positive over a dense open subset of $B(\mathbf{C})$, then the equality holds if and only if the metric of $\bar{L}$ is flat along fibers.

Proof. Let us write $h_{L'}=uh_L$. Then u is a positive smooth function on $X_{\mathbf{C}}(\mathbf{C})$. Since

$$\hat{c}_1(\bar{L})=\hat{c}_1(\bar{L}')+(0,\log u),$$

we have

$$\hat{c}_1(\bar{L})^2=\hat{c}_1(\bar{L}')^2+(0,2c_1(\bar{L}')\log u)+(0,\log u)dd^c(\log u)).$$

Thus

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2\cdot\hat{c}_1(f^*(\bar{H}))^d)=\widehat{\deg}(\hat{c}_1(\bar{L}')^2\cdot\hat{c}_1(f^*(\bar{H}))^d)$$
$$-\int_{X_{\mathbf{C}}(\mathbf{C})}(\log u)c_1(\bar{L}')\wedge c_1(f^*(\bar{H}))^d+\frac{1}{2}\int_{X_{\mathbf{C}}(\mathbf{C})}(\log u)dd^c(\log u)\wedge c_1(f^*(\bar{H}))^d.$$

Now the assertion follows the following two claims.

Claim 5.5.1. $\int_{X_{\mathbf{C}}(\mathbf{C})}(\log u)c_1(\bar{L}')\wedge c_1(f^*(\bar{H}))^d=0$

Proof. For $b\in B_{\mathbf{C}}(\mathbf{C})$, $c_1(\bar{L}')|_b=0$. Then

$$\int_{X_{\mathbf{C}}(\mathbf{C})}(\log u)c_1(\bar{L}')\wedge c_1(f^*(\bar{H}))^d=\int_{B_{\mathbf{C}}(\mathbf{C})}\left(\int_{f_{\mathbf{C}}:X_{\mathbf{C}}\to B_{\mathbf{C}}}(\log u)c_1(\bar{L}')\right)c_1(\bar{H})^d=0.$$

Claim 5.2.2. $\int_{X_{\mathbf{C}}(\mathbf{C})}(\log u)dd^c(\log u)\wedge c_1(f^*(\bar{H}))^d\le 0$. *Moreover, if $c_1(\bar{H})$ is positive over a dense open set of $B(\mathbf{C})$, then the equality holds if and only if $u=f^*(v)$ with some C^∞ function v on $B_{\mathbf{C}}(\mathbf{C})$.*

Proof. We have

$$(\log u)dd^c(\log u)=\frac{\sqrt{-1}}{2\pi}(\log u)\partial\bar{\partial}(\log u)$$
$$=\frac{\sqrt{-1}}{2\pi}\partial(\log u\cdot\bar{\partial}(\log u))-\frac{\sqrt{-1}}{2\pi}\partial(\log u)\wedge\bar{\partial}(\log u).$$

Since $c_1(f^*(\bar{H}))^d$ is a closed (d,d)-form, by Stokes' lemma, we get

$$\int_{X_C(C)} (\log u) dd^c(\log u) \wedge c_1(f^*(\bar{H}))^d$$

$$= -\frac{1}{2\pi} \int_{X_C(C)} (\sqrt{-1}\partial(\log u) \wedge \bar{\partial}(\log u)) \wedge c_1(f^*(\bar{H}))^d.$$

By the definition of the polarization of $\bar{B}=(B,\bar{H})$, $c_1(\bar{H})$ is semipositive. Moreover, $\partial(\log u) \wedge \bar{\partial}(\log u)$ is semipositive. Thus we get the first assertion.

Suppose now $c_1(\bar{H})$ is positive over a dense open set of $B(C)$. We have

$$\int_{X_C(C)} (\sqrt{-1}\partial(\log u) \wedge \bar{\partial}(\log u)) \wedge c_1(f^*(\bar{H}))^d$$

$$= \int_{B_C(C)} \left(\int_{f_C: X_C \to B_C} \sqrt{-1}\partial(\log u) \wedge \bar{\partial}(\log u) \right) c_1(\bar{H})^d.$$

If this value is zero, then, for any $b \in B_C$, $\sqrt{-1}\partial(\log u) \wedge \bar{\partial}(\log u)|_{X_b} = 0$. Then $u|_{X_b}$ is a constant function on $X_b(C)$. This shows the second assertion.

Lemma 5.6. *We assume that B is regular. Let Δ be the set of critical values of f, i.e., $\Delta = \{b \in B \mid f$ is not smooth over $b\}$. Let $\Delta = \cup_{i=1}^{I} \Delta_i$ be the irreducible decomposition of Δ such that $\Delta_1, \cdots, \Delta_{I_1}$ are divisors on B while $\mathrm{codim}_B(\Delta_i) \geq 2$ for $i \geq I_1 + 1$. Let us set $\Gamma_i = f^{-1}(\Delta_i)$ for $i = 1, \cdots, I_1$ and write $\Gamma_i = \cup_{i=1}^{J_i} \Gamma_{ij}$ as its irreducible decomposition. Note that Γ_{ij} are all divisors on X for $1 \leq i \leq I_1$, $1 \leq j \leq J_i$. Then there are a big Zariski open set B' of B, integers e_{ij} $(1 \leq i \leq I_1, 1 \leq j \leq J_i)$ and a positive integer such that $L^{\otimes m} \otimes \mathcal{O}_X(-\Sigma_{ij} e_{ij} \Gamma_{ij})|_{B'}$ belongs to $\mathrm{Pic}^0_{f^{-1}(B')/B'}$.*

Proof. If $I_1 = 0$, then we heve nothing to prove. Thus, we assume $I_1 \geq 1$. To ease the notation, we first assume the irreducibility of Δ. Since f_C is smooth, Δ is defined over the finite field $\boldsymbol{F}_p$ for some prime number p. Let $k(\Delta)$ be the rational function of Δ and write $\eta = \mathrm{Spec}(k(\Delta))$. Moreover, let $\overline{k(\Delta)}$ be an algebric closure of $k(\Delta)$ and write $\bar{\eta} = \mathrm{Spec}(\overline{k(\Delta)})$.

Let $X_{\bar{\eta}} = \cup_{1 \leq j \leq J} \cup_{1 \leq \alpha \leq \alpha(j)} C_j^{\alpha}$ be the irreducible decomposition of $X_{\bar{\eta}}$ such that C_j^{α} and C_j^{β} are $\mathrm{Gal}(\overline{k(\Delta)}/k(\Delta))$-conjugate to each other for $1 \leq \alpha, \beta \leq \alpha(j)$. We denote by Γ_j the Zariski closure of C_j^{α} in X for some (hence all) α.

We put $c_j^{\alpha} = \deg(L_{\eta}|_{C_j^{\alpha}})$. Since L is defined over X, $c_j^{\alpha} = c_j^{\beta}$ for $1 \leq \alpha, \beta \leq \alpha(j)$. Moreover, since the degree of L is zero, $\Sigma_{1 \leq j \leq J, 1 \leq \alpha \leq \alpha(j)} c_j^{\alpha} = 0$.

We put $q_{jk}^{\alpha\beta} = \dim_{k(\Delta)}(C_j^{\alpha} \cap C_k^{\beta})$ for $(j,\alpha) \neq (k,\beta)$, and $q_{jj}^{\alpha\alpha} = -\Sigma_{(k,\beta) \neq (j,\alpha)} q_{jk}^{\alpha\beta}$. Then by Zariski's lemma ([1, I, Lemma (2.10)]), there are rational numbers a_j^{α} $(1 \leq j \leq J$, $1 \leq \alpha \leq \alpha(j))$ such that $a_j^{\alpha} = a_j^{\beta}$ and that $\Sigma_{j,\alpha} a_j^{\alpha} q_{jk}^{\alpha\beta} = c_k^{\beta}$ for $1 \leq k \leq J$ and $1 \leq \beta \leq \alpha(k)$. Moreover, $\Sigma_{j,k,\alpha,\beta} a_j^{\alpha} q_{jk}^{\alpha\beta} = 0$ if and only if $a_j^{\alpha} = a_k^{\beta}$ for any (j,α) and (k,β).

Let Y be the subset of $|\Delta|$ consisting of $\overline{F}_p$-valued points b such that:

(a) The irreducible decomposition of X_b is of form $X_b = \cup_{1 \le j \le J} \cup_{1 \le \alpha \le \alpha(j)} C(b)_j^\alpha$ such that $\Gamma_j \cap X_b = \cup_{1 \le \alpha \le \alpha(j)} C(b)_j^\alpha$;

(b) $\deg(L|_{C(b)_j^\alpha}) = c_j^\alpha$;

(c) $\Gamma_j \cdot C(b)_k^\beta = \Sigma_{1 \le \alpha \le \alpha(j)} q_{jk}^{\alpha\beta}$.

Then there is a divisor Z on Δ such that $Y \subset |Z|$. We set $B' = B - |Z|$.

Now we set $e_j = m a_j^\alpha$ $(1 \le j \le J)$ for sufficiently divisible m and $L' = L^{\otimes m} \otimes \mathcal{O}_X(-\Sigma_{j=1}^J e_j \Gamma_j)$. We claim that $L'|_{B'}$ belongs to $\mathrm{Pic}^0_{f^{-1}(B')/B'}$. Indeed, if $b \notin \Delta$, then X_b is a smooth connected curve and $\deg(L'|_{X_b}) = 0$. Thus $L'|_{X_b}$ belongs to $\mathrm{Pic}^0_{X_b}$. Next, if $b \in \Delta \backslash |Z|$, then $X_b = \cup_{j,\alpha} C(b)_j^\alpha$ is the irreducible decomposition of X_b and

$$\deg(L'|_{C(b)_k^\beta}) = m\left(c_k^\beta - \sum_{1 \le j \le J, 1 \le \alpha \le \alpha(j)} q_{jk}^{\alpha\beta} a_k^\beta\right) = 0$$

for any j and β. Thus also in this case, $L'|_{X_b}$ belongs to $\mathrm{Pic}^0_{X_b}$. Therefore $L'|_{B'}$ belongs to $\mathrm{Pic}^0_{f^{-1}(B')/B'}$.

We have just shown the lemma when Δ is irreducible. Now we consider a general case, i.e., $\Delta = \cup_{i=1}^{I_1} \Delta_i$. For each Δ_i $(1 \le i \le I_1)$, take a divisor Z_i of Δ_i and $\Sigma_{1 \le i \le I_1, 1 \le j \le J_i} e_{ij} \Gamma_{ij}$ in the same way as above. If we set

$$B' = B - (|Z_1| \cup \cdots \cup |Z_{I_1}| \cup (\bigcup_{i,j} |\Delta_i| \cap |\Delta_j|)),$$

then B' is a big open set, and it is easy to see that $L^{\otimes m} \otimes \mathcal{O}_X(-\Sigma_{ij} e_{ij} \Gamma_{ij})|_{B'}$ belongs to $\mathrm{Pic}^0_{f^{-1}(B')/B'}$.

Proof of Theorem 5.2. First we prove the first assertion of the theorem. In virtue of Lemma 5.4, by taking a suitable generically finite cover of B, we may assume that $f: X \to B$ has a section. Moreover, by [8, Theorem 8.2], there is a surjective generically finite morphism $\tilde{B} \to B$ of arithmetic varieties such that $\tilde{B}$ is regular. Thus, by Lemma 5.4, we may also assume that B is regular.

We follow the notation of lemma 5.6, and let $L^{\otimes m} \otimes \mathcal{O}_X(-\Sigma_{ij} e_{ij} \Gamma_{ij})$ be a line bundle on B whose restriction to a big open set B' of B belongs to $\mathrm{Pic}^0_{f^{-1}(B')/B'}$. For simplicity, we set $E = -\Sigma_{ij} e_{ij} \Gamma_{ij}$. Then

$$\begin{aligned}
&\widehat{\deg}(\hat{c}_1(\overline{L^{\otimes m}})^2 \cdot \hat{c}_1(f^*(\bar{H})^d)) \\
&= \widehat{\deg}((\hat{c}_1(\overline{L^{\otimes m} \otimes \mathcal{O}_X(E)}) - \hat{c}_1(\overline{\mathcal{O}_X(E)}))^2 \cdot \hat{c}_1(f^*(\bar{H}))^d) \\
&= \widehat{\deg}(\hat{c}_1(\overline{L^{\otimes m} \otimes \mathcal{O}_X(E)})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d) \\
&\quad - 2\widehat{\deg}(\hat{c}_1(\overline{L^{\otimes m} \otimes \mathcal{O}_X(E)}) \cdot \hat{c}_1(\overline{\mathcal{O}_X(E)}) \cdot \hat{c}_1(f^*(\bar{H}))^d) + \widehat{\deg}(\hat{c}_1(\overline{\mathcal{O}_X(E)})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d).
\end{aligned}$$

Since $\deg(L^{\otimes m} \otimes \mathcal{O}_X(E)|_C) = 0$ for any vertical curve C lying over B', the second term in the last expression becomes zero. Moreover, for the third term in the last expression, we have

$$\widehat{\deg}(\hat{c}_1\overline{\mathcal{O}_X(E)})^2\cdot\hat{c}_1(f^*(\bar{H})^d))=\sum_{i=1}^{I_1}\deg_H(\Delta_i)\cdot\left(\sum_{1\le j,k\le J_i}e_{ij}e_{ik}q^i_{jk}\right),$$

where $q^i_{jk}=\dim_{k(\Delta_i)}(\Gamma_{j,k(\Delta_i)}\cap\Gamma_{k,k(\Delta_i)})$. From the proof of lemma 5.6, this value is non-positive. Moreover the equality holds if and only if $e_{i1}=\cdots=e_{iJ_i}$ for $1\le i\le I_1$. To sum up, we get

$$\widehat{\deg}(\hat{c}_1(\overline{L^{\otimes m}})^2\cdot\hat{c}_1(f^*(\bar{H})^d))\le\widehat{\deg}((\hat{c}_1(\overline{L^{\otimes m}\otimes\mathcal{O}_X(E)}))^2\cdot\hat{c}_1(f^*(\bar{H})^d)).$$

Next let h'_L be an admissible line bundle on L. Then by Lemma 5.5 and Proposition 5.1,

$$\begin{aligned}&\widehat{\deg}((\hat{c}_1(\overline{L^{\otimes m}\otimes\mathcal{O}_X(E)}))^2\cdot\hat{c}_1(f^*(\bar{H}))^d).\\ &\le\widehat{\deg}((\hat{c}_1(L^{\otimes m}\otimes\mathcal{O}_X(E),h'^m_L))^2\cdot\hat{c}_1(f^*(\bar{H}))^d).\\ &=-2m\hat{h}^{\bar{B}}_{\mathcal{O}_{J_K}(\Theta_K)}([L_K]).\end{aligned}$$

Thus we get the first assertion of Theorem 5.2.

Now assuming that H is ample and $c_1(\bar{H})$ is positive, we consider when the equality holds.

Let $g:\tilde{B}\to B$ be a surjective generically finite morphism of arithmetic varieties such that $\tilde{B}$ is regular and $\tilde{f}:\tilde{X}\to\tilde{B}$ has a section, where $\tilde{X}=X\times_B\tilde{B}$ and

$$\begin{tikzcd}\tilde{X} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ \tilde{B} \arrow[r, "g"] & B\end{tikzcd}$$

is the induced morphism. Let us set $\tilde{L}=\tilde{g}^*(L)$ and $\tilde{H}=g^*(H)$.

Now let us assume the condition (a) and (b) in the second assertion of the theorem. By Lemma 5.6 there are a big open set $\tilde{B}'$ of $\tilde{B}$, a positive integer and a vertical divisor Γ of $\tilde{X}$ such that $g\circ\tilde{f}(\Gamma)\subset B\backslash B''$ and that $\tilde{L}^{\otimes m}\otimes\mathcal{O}_{\tilde{X}}(\Gamma)|_{\tilde{B}'}$ belongs to $\mathrm{Pic}^{\mathbb{Q}}_{\tilde{X}/\tilde{B}}|_{\tilde{B}'}$.

Claim 5.6.1. *If $\tilde{K}$ denotes the function field of $\tilde{B}$, then*

$$\widehat{\deg}(\hat{c}_1(\bar{\tilde{L}})^2\cdot\hat{c}_1(\tilde{f}^*(\bar{\tilde{H}}))^d)=-2[\tilde{K}:K]\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\tilde{K}}}(\Theta_{\tilde{K}})}([L_K]).$$

Proof. By Proposition 5.1, we get

$$\begin{aligned}\widehat{\deg}(\hat{c}_1(\bar{\tilde{L}}^{\otimes m}\otimes\mathcal{O}_{\tilde{X}}(\Gamma))^2\cdot\hat{c}_1(\tilde{f}^*(\bar{\tilde{H}}))^d)&=-2m^2\hat{h}^{(\tilde{B},g_*(\bar{H}))}_{\mathcal{O}_{J_{\tilde{K}}}(\Theta_{\tilde{K}})}([L_K])\\ &=-2m^2[\tilde{K}:K]\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\tilde{K}}}(\Theta_{\tilde{K}})}([L_K]).\end{aligned}$$

On the other hand, since

$$\hat{c}_1(\bar{\tilde{L}}^{\otimes m} \otimes \mathcal{O}_{\tilde{X}}(\Gamma))^2 = m^2 \hat{c}_1(\tilde{L})^2 + 2m\hat{c}_1(\tilde{L}) \cdot \hat{c}_1(\mathcal{O}_{\tilde{X}}(\Gamma)) + \hat{c}_1(\mathcal{O}_{\tilde{X}}(\Gamma))^2$$

and $\tilde{f}^*(\tilde{H}) = f^*(g^*(H))$, we get

$$\widehat{\deg}(\hat{c}_1(\bar{\tilde{L}}^{\otimes m} \otimes \mathcal{O}_{\tilde{X}}(\Gamma))^2 \cdot \hat{c}_1(\tilde{f}^*(\bar{\tilde{H}}))^d) = m^2 \widehat{\deg}(\hat{c}_1(\bar{\tilde{L}})^2 \cdot \hat{c}_1(\tilde{f}^*(\bar{\tilde{H}}))^d).$$

by projection formula (Note that $g \circ \tilde{f}(\Gamma) \subset B \backslash B''$). Thus we obtain the claim.

From the claim, we get

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d) = -2\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}([L_K])$$

by projection formula.

Next we assume that

$$\widehat{\deg}(\hat{c}_1(\bar{L})^2 \cdot \hat{c}_1(f^*(\bar{H}))^d) = -2\hat{h}^{\bar{B}}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}([L_K]).$$

Then by projection formula, we have

$$\widehat{\deg}(\hat{c}_1(\bar{\tilde{L}})^2 \cdot \hat{c}_1(\tilde{f}^*(\bar{\tilde{H}}))^d) = -2\hat{h}^{(\tilde{B}, g_*(\bar{H}))}_{\mathcal{O}_{J_{\bar{K}}}(\Theta_{\bar{K}})}([L_K]).$$

Let $\tilde{\Delta}$ be the set of critical values of $\tilde{f}$ and $\tilde{\Delta} = \cup_{i=1}^{I} \tilde{\Delta}_i$ be the irreducible decomposition of $\tilde{\Delta}$, where $\tilde{\Delta}_1, \cdots, \tilde{\Delta}_{I_1}$ are divisors on $\tilde{B}$ such that $g(\tilde{\Delta}_i))$ are also divisors on B for $1 \le i \le I_1$, $\tilde{\Delta}_{I_1+1}, \cdots, \Delta_{I_2}$ are divisors on $\tilde{B}$ such that $\operatorname{codim}_B(g(\tilde{\Delta}_i)) \ge 2$ for $I_1 + 1 \le i \le I_2$, and $\tilde{\Delta}_i$ $(i \ge I_2)$ satisfy $\operatorname{codim}_{\tilde{B}}(\tilde{\Delta}_i) \ge 2$. Then we take $\Sigma_{1 \le i \le I_2, 1 \le j \le J_i} e_{ij} \Gamma_{ij}$ as in Lemma 5.6 (which is applied to $\tilde{f}: \tilde{X} \to \tilde{B}$). If we look back closely the proof of the first assertion of the theorem, we find that the equality holds if and only if (a) $e_{i1} = \cdots = e_{iJ_i}$ for $1 \le i \le I_1$ and (b) $\bar{L}$ is flat along fibers (Note that the reason we need to consider I_1 and I_2 is that $\deg_{g_*(H)}(\tilde{\Delta}_i) = 0$ for $I_1 + 1 \le i \le I_2$). Moreover the condition (a) is equivalent to the existence of a big open set B'' of B such that $\deg(L|_C) = 0$ for any fibral curves C lying over B''. This proves the second assertion.

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