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On the existence of extremal metrics for L^2-norm of scalar curvature on closed 3-manifolds

By

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Abstract

In this paper, based on Bochner formula, mass decay estimates and elliptic Moser iteration, we show the global existence and asymptotic convergence of a subsequence of solutions of Calabi flow on some closed 3-manifolds, and then the existence of extermal metrics of L^2-norm of scalar curvature functional on a fixed conformal class is claimed. In particular, we may re-solve part of the Yamabe conjecture on closed 3-manifolds.

1. Introduction

Let (M, g_0) be a closed smooth n-manifold with a given conformal class $[g_0]$ on M. Then the Euler-Lagrange equation of

$$Ss(g) = \frac{\int_M R^2 d\mu}{(\int_M d\mu)^{1-\frac{4}{n}}}, \quad g \in [g_0]$$

is given by

$$\Delta R - \beta R^2 + \beta r = 0, \tag{1.1}$$

where $d\mu = d\mu_g$, $\Delta = \Delta_g$, R is the scalar curvature with respect to the metric g, $r = \dfrac{\int_M R^2 d\mu}{\int_M d\mu}$ and $\beta = \dfrac{n-4}{4(n-1)}$. Now consider the negative gradient flow of $Ss(g)$ on a given conformal class $[g_0]$, that is, we consider the following initial value problem of fourth order parabolic equation:

$$\left\{\begin{array}{l} \dfrac{\partial \lambda}{\partial t} = \Delta R - \beta R^2 + \beta r, \\ g = e^{2\lambda} g_0\,;\ \lambda(p,0) = \lambda_0(p), \\ \int_{M^n} e^{n\lambda_0} d\mu_0 = \int_{M^n} d\mu_0\,, \end{array}\right. \tag{1.2}$$

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where $\lambda:M\times[0,\infty)\to\boldsymbol{R}$ is a smooth function and $d\mu_0$ is the volume element of g_0.

When $n=2$[1], if the background metric g_0 has the constant Gaussian curvature, P. Chruściel ([Chru]) proved the long time existence and asymptotic convergence of solutions of (1.3), and the first author generalized his results to any arbitrary background metric g_0 and then re-solve the uniformization theorem for surfaces ([Ch3]). Furthermore, we also proved some partial results for the long time existence of solutions of (1.3) when $n=4$ ([Ch2]).

When $n=3$, then $\beta=-\frac{1}{8}$ and we will consider the following flow:

$$(1.3)\qquad \begin{cases} \dfrac{\partial\lambda}{\partial t}=\Delta R+\dfrac{1}{8}R^2-\dfrac{1}{8}r, \\ g=e^{2\lambda}g_0;\ \lambda(p,0)=\lambda_0(p), \\ \int_{M^3}e^{3\lambda_0}d\mu_0=\int_{M^3}d\mu_0\,. \end{cases}$$

Although (1.3) is at heart a parabolic equation, due to equivariance under the group of diffeomorphisms, which makes it highly degenerate. On the other hand, Richard Hamilton's original proof of short time existence of the Ricci flow was involved and used the Nash-Moser inverse function theorem. Soon after, D. DeTurck simplified short time existence proof by "breaking the symmetry" (which causes difficulty in the directly applying standard theory) to prove short time existence ([De]). Then, by using the Deturck's trick, which was done by the first author's previous work ([Ch5, Lemma 4]) in general case, short time existence of (1.3) follows easily. We may also compare this to [LT].

In this paper, we will show the long-time existence and asymptotic convergence of solutions of (1.3) on $M^3\times[0,\infty)$.

Theorem 1.1. *Let $(M,\ g_0)$ be a closed 3-manifold and λ satisfy* (1.3) *on $[0,T)$ with*

$$\lambda\geq -H$$

for the positive constant H which is independent of t. Then the solution of (1.3) *exists on $M\times[0,\infty)$.*

Theorem 1.2. *The same assumptions as in Theorem* 1.1. *Then there exists a subsequence of solutions $\{e^{2\lambda(t)}g_0\}$ of* (1.3) *on $M\times[0,\infty)$ which converges smoothly to an extremal metric g_∞, i.e. its scalar curvature $R_\infty=R(g_\infty)$ satisfying $\Delta_\infty R_\infty+\frac{1}{8}R_\infty^2-\frac{1}{8}r_\infty=0$.*

Now consider the Yamabe constant which is conformal invariant

$$Q(M,g_0)=\inf_{\varphi\neq 0}\frac{E_{g_0}(\varphi)}{(\int|\varphi|^6d\mu_0)^{\frac{1}{3}}},$$

[1]For $n=2$, we consider the so-called Calabi flow $\dfrac{\partial\lambda}{\partial t}=\Delta R$ only.

where $E_{g_0}(\varphi)=\int|\overset{0}{\nabla}\varphi|^2d\mu_0+\frac{1}{8}\int R_0\varphi^2d\mu_0$.

As consequences of Theorem 1.1, we have

Theorem 1.3. *If* (M,g_0) *is a closed* 3*-manifold with* $Q<0$, *and let* λ *satisfy* (1.3) *on* $[0,T)$. *Then the solution of* (1.3) *exists on* $M\times[0,\infty)$.

Theorem 1.4. *If* (M,g_0) *is conformal equivalent to the standard sphere* $(S^3,\overline{g_0})$, *and let* λ *satisfy* (1.3) *on* $[0,T)$. *Then, up to conformal transformations* (*Lemma* 3.1), *the solution of* (1.3) *exists on* $M\times[0,\infty)$.

As consequences of Theorem 1.2, we have

Theorem 1.5. *If* (M,g_0) *is a closed* 3*-manifold with* $Q<0$. *Then there exists a subsequence of solutions* $\{g(t)\}$ *of* (1.3) *on* $M\times[0,\infty)$ *which converges smoothly to an extremal metric* g_∞, *i.e. its scalar curvature* $R_\infty=R(g_\infty)$ *satisfying* $\Delta_\infty R_\infty+\frac{1}{8}R_\infty^2-\frac{1}{8}r_\infty=0$.

Theorem 1.6. *If* (M,g_0) *is conformal equivalent to the standard sphere* $(S^3,\overline{g_0})$. *Then, up to conformal transformations, there exists a subsequence of solutions* $\{g(t)\}$ *of* (1.3) *on* $M\times[0,\infty)$ *which converges smoothly to an extremal metric* g_∞.

Remark 1.1. 1. The same assumptions as in Theorems 1.5. and 1.6. If R_∞ is constant, then we re-solve the Yamabe problem on closed 3-Manifolds. Otherwise, we get a nontrivial extremal metric of L^2-norm of scalar curvature on $(M^3,[g_0])$.

2. If $R_\infty\leq\left(\frac{\int R_\infty^2d\mu_\infty}{\int d\mu_\infty}\right)^{\frac{1}{2}}$, then R_∞ is constant. In particular, if $R_\infty\leq 0$, *then* R_∞ is constant.

One may think the problem here to be more difficult compare the second order parabolic equations, due to a lack of the maximum principle for fourth order parabolic equations. Then in order to estimate the C^0-bound, we will apply the elliptic Moser iteration method ([Ch1], [G]). However, we should point out that Theorem 2.4 is the starting point for applying the Moser iterations as in section 3.

We briefly describe the methods used in our proofs. In section 2, we will derive the key estimate of equation (1.3) from the Bochner formula (Theorem 2.4). In section 3, based on [Ch2], [Chru] and [G], we obtain the Harnack estimate for the equation (1.3). Then we have the C^0-bound and the higher order $W_{k,2}$-estimates of the solution for (1.3), which imply the long-time existence of solutions of (1.3).

In section 4, in fact we have the uniformly lower bound ([G]) for the solution λ of (1.3) plus the mass decay formula (Theorem 2.4), which implies the uniformly bounds on all higher-order derivatives. Then we are able to prove the convergence of the solution of (1.3).

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2. The mass decay estimates

In this section, based on the Bochner formula and bound of energy $Ss(g)$, we will derive the so-called mass decay estimate of equation (1.3) as in Theorem 2.4.

For $g=e^{2\lambda}g_0$, $R_0=R_{g_0}$, we have the following formulae for (1.2):

$$R=R_g=e^{-2\lambda}(R_0-2(n-1)\Delta_0\lambda-(n-1)(n-2)|\overset{0}{\nabla}\lambda|^2). \tag{2.1}$$

$$\Delta R=e^{-2\lambda}(\Delta_0 R+(n-2)\langle\overset{0}{\nabla}R,\overset{0}{\nabla}\lambda\rangle), \text{ where } \Delta_0=\Delta_{g_0},\ \Delta=\Delta_g. \tag{2.2}$$

$$d\mu=e^{n\lambda}d\mu_0, \text{ where } d\mu_0=d\mu_{g_0},\ d\mu=d\mu_g. \tag{2.3}$$

$$\frac{\partial}{\partial t}d\mu=n(\Delta R-\beta R^2+\beta r)d\mu. \tag{2.4}$$

$$\int_M d\mu=\int_M e^{n\lambda}d\mu_0=\int_M e^{n\lambda_0}d\mu_0=\int_M d\mu_0. \tag{2.5}$$

From now on, C denotes a generic constant which may vary from line to line. Then we have

Lemma 2.1. *Under the flow* (1.3), *for* $t\geq t_0$, *we have*

$$\int_M R^2 d\mu|_t\leq\int_M R^2 d\mu|_{t_0}.$$

Proof.

$$\begin{aligned}&\frac{d}{dt}\int_M R^2 d\mu\\ &=-8\int_M(\Delta R+\frac{1}{8}R^2-\frac{1}{8}r)^2 d\mu.\end{aligned}$$

Thus

$$\frac{d}{dt}\int_M R^2 d\mu\leq 0.$$

Corollary 2.2. *Under the flow* (1.3), *we have*

$$\int_M R^2 d\mu\leq C(R_0,\lambda_0),$$

for $0 \le t \le \infty$.

Firstly, from Proposition 2.1 of [G], we have

Lemma 2.3. *For* $g \in [g_0]$, *say* $g = e^{2\lambda} g_0$. *If* $\int d\mu \le V$ *and* $\int_M R^2 d\mu \le \beta^2$, *for some positive constants* V, β. *Then*[2], *for* $0 \le \alpha$

$$\int_M e^{\alpha\lambda} R^2 d\mu_0 \le C(\alpha, \beta, V) + C(\alpha, \beta, V) \int_M e^{\alpha\lambda} d\mu_0 .$$

We will postpone its proof until the end of the section.

Theorem 2.4. (*i*) *For any background metric* g_0, *under the flow* (1.3), *we have*

$$\frac{d}{dt} \int_M e^{4\lambda} d\mu_0 \le I_0(\beta, g_0) + I_1(\beta, g_0) \int_M e^{4\lambda} d\mu_0 .$$

(ii) *For any background metric* g_0, *under the flow* (1.3), *we have*

$$\frac{d}{dt} \int_M e^{\alpha\lambda} d\mu_0 \le I_2 + I_3 \int_M e^{\alpha\lambda} d\mu_0 - I_4 \int_M e^{(\alpha-4)\lambda} |\overset{0}{\nabla} \lambda|^4 d\mu_0 , \qquad 5 < \alpha < \frac{11}{2}$$

where the positive constants I_i *are independent of* t.

Remark 2.1. *We will show the long time existence of solution of* (1.3) *in the next section based on* (*i*) *of Theorem* 2.4, *and show the convergence based on the* (*ii*) *of Theorem* 2.4.

Proof. Compute

$$\frac{1}{\alpha} \frac{d}{dt} \int_M e^{\alpha\lambda} d\mu_0 = \int_M e^{\alpha\lambda} \frac{\partial \lambda}{\partial t} d\mu_0 \le \int_M e^{\alpha\lambda} (\Delta R + \frac{1}{8} R^2) d\mu_0 .$$

Firstly, we will estimate the term $\int_M e^{\alpha\lambda} \Delta R d\mu_0$. Since, for $n = 3$, we have

$$R = e^{-2\lambda}(R_0 - 4\Delta_0 \lambda - 2|\overset{0}{\nabla} \lambda|^2),$$

$$\Delta R = e^{-2\lambda}(\Delta_0 R + \langle \overset{0}{\nabla} R, \overset{0}{\nabla} \lambda \rangle).$$

Integrating by parts, it follows

---

[2] It may work only for a subsequence $\{t_j\}$.

$$
\begin{aligned}
&\int_M e^{\alpha\lambda}\Delta R\,d\mu_0 \\
&= \int_M e^{(\alpha-2)\lambda}(\Delta_0 R + \langle \overset{0}{\nabla} R, \overset{0}{\nabla}\lambda\rangle)d\mu_0 \\
&= \int_M (\alpha-3)e^{(\alpha-2)\lambda}R(\Delta_0\lambda+(\alpha-2)|\overset{0}{\nabla}\lambda|^2)d\mu_0 \\
&= \int_M (\alpha-3)e^{(\alpha-4)\lambda}(R_0-4\Delta_0\lambda-2|\overset{0}{\nabla}\lambda|^2)(\Delta_0\lambda+(\alpha-2)|\overset{0}{\nabla}\lambda|^2)d\mu_0 .
\end{aligned}
\tag{2.6}
$$

Now let $f=e^{\gamma\lambda}$, then

$$
\begin{aligned}
&|\overset{0}{\nabla}\lambda|^2=\gamma^{-2}f^{-2}|\overset{0}{\nabla}f|^2, \\
&\Delta_0\lambda=\gamma^{-1}(f^{-1}\Delta_0 f-f^{-2}|\overset{0}{\nabla}f|^2).
\end{aligned}
$$

This and (2.6) imply

$$
\begin{aligned}
&\frac{\gamma^2}{\alpha-3}\int_M e^{\alpha\lambda}\Delta R\,d\mu_0 \\
&=\gamma\int_M R_0 f^{\frac{\alpha-4}{\gamma}-1}\Delta_0 f\,d\mu_0+(\alpha-2-\gamma)\int_M R_0 f^{\frac{\alpha-4}{\gamma}-2}|\overset{0}{\nabla}f|^2d\mu_0 \\
&-4\int_M f^{\frac{\alpha-4}{\gamma}-2}(\Delta_0 f)^2d\mu_0+2\gamma^{-2}(2\gamma-1)(\alpha-2-\gamma)\int_M f^{\frac{\alpha-4}{\gamma}-4}|\overset{0}{\nabla}f|^4d\mu_0 \\
&+2\gamma^{-1}(4\gamma+3-2\alpha)\int_M f^{\frac{\alpha-4}{\gamma}-3}\Delta_0 f|\overset{0}{\nabla}f|^2d\mu_0 .
\end{aligned}
\tag{2.7}
$$

Again from integrating by parts and the Bochner-Lichnerowicz formula

$$
\frac{1}{2}\Delta_0|\overset{0}{\nabla}f|^2=|\overset{0}{\nabla}{}^2f|^2+\langle\overset{0}{\nabla}f,\overset{0}{\nabla}\Delta_0 f\rangle+Ric_0(\overset{0}{\nabla}f,\overset{0}{\nabla}f),
$$

we have ([Ch2])

$$
\begin{aligned}
&\int_M f^{\frac{\alpha-4}{\gamma}-3}\Delta_0 f|\overset{0}{\nabla}f|^2d\mu_0 \\
&=\frac{2}{3}\frac{\gamma}{2\gamma+4-\alpha}\int_M f^{\frac{\alpha-4}{\gamma}-2}(\Delta_0 f)^2d\mu_0+\frac{3\gamma+4-\alpha}{3\gamma}\int_M f^{\frac{\alpha-4}{\gamma}-4}|\overset{0}{\nabla}f|^4d\mu_0 \\
&-\frac{2}{3}\frac{\gamma}{2\gamma+4-\alpha}\int_M f^{\frac{\alpha-4}{\gamma}-2}|\overset{0}{\nabla}{}^2f|^2d\mu_0-\frac{2}{3}\frac{\gamma}{2\gamma+4-\alpha}\int_M f^{\frac{\alpha-4}{\gamma}-2}Ric(\overset{0}{\nabla}f,\overset{0}{\nabla}f)d\mu_0 .
\end{aligned}
$$

From this and (2.7), one obtains

$$
\frac{\gamma^2}{\alpha-3}\int_M e^{\alpha\lambda}\Delta R\,d\mu_0
$$

$$
\begin{aligned}
&=\gamma\int_M R_0 f^{\frac{\alpha-4}{\gamma}-1}\Delta_0 f\,d\mu_0+(\alpha-2-\gamma)\int_M R_0 f^{\frac{\alpha-4}{\gamma}-2}|\overset{0}{\nabla} f|^2 d\mu_0\\
&-\frac{4}{3}\frac{2\alpha-4\gamma-3}{\alpha-2\gamma-4}\int_M f^{\frac{\alpha-4}{\gamma}-2}|\overset{0}{\nabla}{}^2 f|^2 d\mu_0-\frac{4}{3}\frac{2\alpha-4\gamma-3}{\alpha-2\gamma-4}\int_M f^{\frac{\alpha-4}{\gamma}-2}Ric_0(\overset{0}{\nabla} f,\overset{0}{\nabla} f)d\mu_0\\
&+4\left[\frac{2\alpha-4\gamma-3}{3(\alpha-2\gamma-4)}-1\right]\int_M f^{\frac{\alpha-4}{\gamma}-2}(\Delta_0 f)^2 d\mu_0\\
&+\frac{4}{3}\gamma^{-2}(\alpha^2-(2\gamma+7)\alpha+3\gamma^2+8\gamma+9)\int_M f^{\frac{\alpha-4}{\gamma}-4}|\overset{0}{\nabla} f|^4 d\mu_0 .
\end{aligned}
$$

(i) If we choose

$$\alpha=4;\ \gamma=-1.$$

Then

$$\alpha^2-(2\gamma+7)\alpha+3\gamma^2+8\gamma+9=0$$

and

$$
\begin{aligned}
&\int_M e^{4\lambda}\Delta R\,d\mu_0\\
&=-\int_M R_0 f^{-1}\Delta_0 f\,d\mu_0+3\int_M R_0 f^{-2}|\overset{0}{\nabla} f|^2 d\mu_0\\
&-6\int_M f^{-2}|\overset{0}{\nabla}{}^2 f|^2 d\mu_0+2\int_M f^{-2}(\Delta_0 f)^2 d\mu_0\\
&-6\int_M f^{-2}Ric_0(\overset{0}{\nabla} f,\overset{0}{\nabla} f)d\mu_0 .
\end{aligned}
$$

But

$$3|\overset{0}{\nabla}{}^2 f|^2\geq(\Delta_0 f)^2.$$

One has, for $f=e^{-\lambda}$

$$
\begin{aligned}
\int_M e^{4\lambda}\Delta R\,d\mu_0&\leq-\int_M R_0 f^{-1}\Delta_0 f\,d\mu_0+3\int_M R_0 f^{-2}|\overset{0}{\nabla} f|^2 d\mu_0\\
&-6\int_M f^{-2}Ric_0(\overset{0}{\nabla} f,\overset{0}{\nabla} f)d\mu_0\\
&\leq-\int_M R_0(-\Delta_0\lambda+|\overset{0}{\nabla}\lambda|^2)d\mu_0+3\int_M R_0|\overset{0}{\nabla}\lambda|^2 d\mu_0
\end{aligned}
$$

$$+C(g_0)\int_M |\overset{0}{\nabla}\lambda|^2 d\mu_0$$

$$\leq \int_M R_0\Delta_0\lambda d\mu_0 + C\int_M |\overset{0}{\nabla}\lambda|^2 d\mu_0 .$$

But

$$R = e^{-2\lambda}(R_0 - 4\Delta_0\lambda - 2|\overset{0}{\nabla}\lambda|^2).$$

It follows that

$$\int_M R_0\Delta_0\lambda d\mu_0 = \int R_0(-\frac{1}{4}e^{2\lambda}R + \frac{1}{4}R_0 - \frac{1}{2}|\overset{0}{\nabla}\lambda|^2)d\mu_0$$

$$= \frac{1}{4}\int R_0^2 d\mu_0 - \frac{1}{4}\int e^{2\lambda}RR_0 d\mu_0 - \frac{1}{2}\int R_0|\overset{0}{\nabla}\lambda|^2 d\mu_0$$

$$\leq C + C\int e^{\lambda}d\mu_0 + C\int R^2 d\mu + C\int |\overset{0}{\nabla}\lambda|^2 d\mu_0$$

$$\leq C + C\int |\overset{0}{\nabla}\lambda|^2 d\mu_0 .$$

On the other hand

$$2\int |\overset{0}{\nabla}\lambda|^2 d\mu_0 = \int R_0 d\mu_0 - \int e^{2\lambda}R d\mu_0$$

$$\leq C + C\int e^{\lambda}d\mu_0 + C\int R^2 d\mu$$

$$\leq C.$$

All these imply

$$\frac{1}{4}\frac{d}{dt}\int_M e^{4\lambda}d\mu_0 \leq \int_M e^{4\lambda}(\Delta R + \frac{1}{8}R^2)d\mu_0 \leq C + \frac{1}{8}\int_M e^{4\lambda}R^2 d\mu_0 .$$

From this estimate and Lemma 2.3, we can conclude the estimate in (i).
(ii) If we choose

$$\gamma = \frac{1}{2}, \quad \text{and so} \quad f = e^{\frac{1}{2}\lambda}.$$

Then

$$-\frac{4}{3}\frac{2\alpha-4\gamma-3}{\alpha-2\gamma-4}=-\frac{4}{3}\frac{2\alpha-5}{\alpha-5}<0$$

and

$$\frac{4}{3}\gamma^{-2}(\alpha^2-(2\gamma+7)\alpha+3\gamma^2+8\gamma+9)=\frac{16}{3}(\alpha^2-8\alpha+\frac{55}{4})<0,$$

for $5<\alpha<11/2$. But $\Delta_0 f=\Delta_0 e^{\frac{1}{2}\lambda}=\frac{1}{8}e^{\frac{1}{2}\lambda}(R_0-e^{2\lambda}R)$, and from Young's inequality, all these imply

$$\int_M e^{\alpha\lambda}\Delta R d\mu_0\le C_1'\int_M e^{(\alpha-4)\lambda}d\mu_0+C_2'\int_M e^{(\alpha-4)\lambda}|\overset{0}{\nabla}\lambda|^2 d\mu_0$$
$$+C_3'\left(\int_M e^{(\alpha-2)\lambda}Rd\mu_0+\int_M e^{\alpha\lambda}R^2 d\mu_0\right)-C_4'\int_M e^{(\alpha-4)\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0,$$

where $C_i'=C_i'(g_0)$. Again Young's inequality implies

$$\begin{aligned}\frac{1}{\alpha}\frac{d}{dt}\int_M e^{\alpha\lambda}d\mu_0&\le\int_M e^{\alpha\lambda}(\Delta R+\frac{1}{8}R^2)d\mu_0\\&\le C\int_M e^{(\alpha-4)\lambda}d\mu_0+C\left(\int_M e^{(\alpha-2)\lambda}Rd\mu_0+\int_M e^{\alpha\lambda}R^2 d\mu_0\right)\\&\quad-C\int_M e^{(\alpha-4)\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0\\&\le C+C\int_M e^{\alpha\lambda}R^2 d\mu_0-C\int_M e^{(\alpha-4)\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0\end{aligned}\tag{2.8}$$

for $5<\alpha<11/2$. From this estimate and Lemma 2.3, we can also conclude the estimate in (ii).

This completes the proof.

Now, from the comparison theorem of ordinary differential equations, we have the following:

Corollary 2.5. *Under the flow* (1.3),

(*i*) *For any background metric* g_0, *then*

$$\int_M e^{4\lambda}d\mu_0\le\bar{e}(t_0)e^{I_1(t-t_0)},$$

for $\bar{e}(t_0)=\int e^{4\lambda}d\mu_0|_{t=t_0}$.

(ii) *For any background metric* g_0, *we have, for* $5<\alpha<\frac{11}{2}$

$$\int_M e^{\alpha\lambda} d\mu_0 \leq \underline{e}(t_0) e^{K(t-t_0)},$$

for $\underline{e}(t_0) = \int e^{\alpha\lambda} d\mu_0|_{t=t_0}$.

Proof. (i) Let $\bar{e}(t) = \int_M e^{4\lambda} d\mu_0$, then from (i) of Theorem 2.4

$$\frac{d}{dt}\bar{e}(t) \leq I_1 \bar{e}(t).$$

Consider $m(t) = \bar{e}(t) - \bar{e}(t_0)$, then $m(t_0) = 0$ and

$$\frac{d}{dt} m(t) \leq I_1 m(t) + I_1 \bar{e}(t_0).$$

From comparison theorem of ordinary differential equation, we get

$$m(t) \leq \bar{e}(t_0)(e^{I_1(t-t_0)} - 1).$$

This implies (i).
(ii) The same method as in (i) also implies the inequality in (ii).
This implies the corollary.

Now we will give the proof of Lemma 2.3. Given $x \in M$, inspired by [CY], we define the mass

$$m(x) = \text{mass of } x = \lim_{\rho \to 0} \limsup_{t \to T} \int_{B(x,\rho)} e^{3\lambda} d\mu_0 .$$

Remark 2.2. *A point $x \in M$ will have large mass $m(x)$ if e^λ concentrates at x. However, if $m(x)$ is small enough, e^λ will be bounded in a small neighborhood of x.*

Indeed, from Proposition 2.1 of [G]. We have

Lemma 2.6. *The same assumptions as in Lemma 2.3, Given $x \in M$, either*
(i)

$$m(x) = 0$$

or
(ii)

$$m(x) \geq \frac{A_0}{\beta^6}$$

where A_0 is the Sobolev constant with respect to g_0.

Then

Lemma 2.7. *The same assumptions as in Lemma 2.3, either*

(i)

$$\max_M \lambda \le C\left(\int d\mu, \int R^2 d\mu\right) \tag{2.9}$$

or

(ii) *there is a nonempty finite set* $\Sigma=\{p_1,\cdots,p_k\}$ *and a subsequence* $\{t_j\}$ *such that, given a compact set* $K\subset\subset\widetilde{M}=M-\Sigma$,

$$\max_K \lambda \le C\left(K, \int d\mu, \int R^2 d\mu\right). \tag{2.10}$$

Moreover, $w=\limsup_{t\to T}\lambda$ *which is defined on* $\widetilde{M}$ *and*

$$w\le C(V,\beta).$$

Proof of Lemma 2.3. It is trivial for $0\le\alpha\le 3$, for simplicity, we do for $\alpha=4$. We may assume $\Sigma=\{p\}$, then

$$m(p)\ge\frac{A_0}{\beta^6}.$$

Now

$$\begin{aligned}\int_M e^{4\lambda}R^2 d\mu_0 &= \int_M e^{\lambda}R^2 d\mu\\ &=\int_{M\backslash B(p,\rho)} e^{\lambda}R^2 d\mu+\int_{B(p,\rho)} e^{\lambda}R^2 d\mu\\ &\le C\left(\int_M R^2 d\mu\right)+\int_{B(p,\rho)} e^{\lambda}R^2 d\mu,\end{aligned}$$

where ρ will be determined later.

On the other hand, since

$$\lim_{\rho\to 0}\limsup_{t\to T}\int_{B(x,\rho)} e^{3\lambda}d\mu_0\ge\frac{A_0}{\beta^6},$$

but

$$\int_M R^2 d\mu\le\beta^2,$$

It follows that

$$(R_0-4\Delta_0\lambda-2|\overset{0}{\nabla}\lambda|^2)^2<\frac{2\beta^8}{A_0}e^{4\lambda}$$

at p for t_j sufficed close to T. Otherwise, for small enough $\rho'>0$

$$\beta^2 \geq \int_{B(p,\rho')} R^2 d\mu = \int_{B(p,\rho')} e^{-\lambda}(R_0 - 4\Delta_0\lambda - 2|\overset{0}{\nabla}\lambda|^2)^2 d\mu_0 \geq \frac{2\beta^8}{A_0}\int_{B(p,\rho')} e^{3\lambda} d\mu_0 \geq 2\beta^2.$$

This leads to a contradiction. Then for small enough ρ again, one obtains

$$\int_{B(p,\rho)} e^{\lambda} R^2 d\mu = \int_{B(p,\rho)} (R_0 - 4\Delta_0\lambda - 2|\overset{0}{\nabla}\lambda|^2)^2 d\mu_0 \leq C(\beta, A_0)\int_{B(p,\rho)} e^{4\lambda} d\mu_0 .$$

This completes the proof of Lemma 2.3.

3. A priori estimates and long time existence

In this section, following [Ch1], [G] and Theorem 2.4, we will have the C^0-bound as in Lemma 3.2. Then, based on [Ch2] and [Chru], one can get the bounds on all $W_{k,2}$ norms as in Lemma 3.3. All these together will imply the long-time existence of solutions of (1.3).

Let (M, g_0) be a closed 3-manifold with the background metric g_0 at $t=0$. One has ([G])

Lemma 3.1. *Let (M, g_0) be a closed 3-manifold with $Q<0$. For $g \in [g_0]$, say $g = e^{2\lambda} g_0$. If $\int d\mu \leq V$ and $\int_M R^2 d\mu \leq \beta^2$, for some positive constants V, β, then there exists $H = H(V,\beta)$ such that*

(*) $$\lambda \geq -H.$$

(*) holds also, up to the conformal group, for (M, g_0) is conformal equivalent to the standard sphere. That is, there exist conformal transformations φ_t of M such that, if $\varphi_t^* g_t = e^{2\tilde{\lambda}} g_0$, then (*) holds for $\tilde{\lambda}$.

Now we are ready to prove the Theorems 1.1, 1.3 and 1.4.

Since

$$R = e^{-2\lambda} R_0 - e^{-2\lambda}(4\Delta_0\lambda + 2|\overset{0}{\nabla}\lambda|^2),$$

and

$$-\Delta_0 e^{\lambda} \leq \frac{1}{4}|e^{2\lambda}R - R_0| e^{\lambda}.$$

That is, for $f = e^{\lambda} > 0$, $b = \frac{1}{4}|e^{2\lambda}R - R_0| \geq 0$, we have

(3.1) $$-\Delta_0 f \leq bf.$$

Then, from Corollary 2.5

$$b \in L^q$$

for some $q > \frac{3}{2}$. More precisely,

$$\begin{aligned}
\int_M b^{\frac{8}{5}} d\mu_0 &\le C \int |e^{2\lambda} R|^{\frac{8}{5}} d\mu_0 + C(R_0) \\
&= C \int e^{\frac{1}{5}\lambda} |R|^{\frac{8}{5}} d\mu + C(R_0) \\
&\le C \left(\int (e^{\frac{1}{5}\lambda})^5 d\mu \right)^{\frac{1}{5}} \left(\int (|R|^{\frac{8}{5}})^{\frac{5}{4}} d\mu \right)^{\frac{4}{5}} + C(R_0) \qquad (3.2) \\
&\le C \left(\int e^{\lambda} d\mu \right)^{\frac{1}{5}} \left(\int |R|^2 d\mu \right)^{\frac{4}{5}} + C(R_0) \\
&\le C(g_0, V, \beta, T).
\end{aligned}$$

But

$$\int_M f^2 d\mu_0 = \int_M e^{2\lambda} d\mu_0 \le C. \qquad (3.3)$$

All together with (3.1), (3.2), (3.3), and Moser iteration ([Ch1, Theorem 3.3], [G]), this leads

$$\sup e\lambda \le C$$

on M.

Then we have the C^0-bound of solution of (1.3) as the following:

Lemma 3.2. *Suppose λ satisfies the hypotheses of Theorem* 1.1, *there exists a constant* $C = C(H, K, \lambda_0, g_0, T)$, *such that*

$$\|\lambda\|_{L^\infty(M)} \le C,$$

$\forall t \in [0,T)$. *Moreover, we have*

$$\|\lambda(t)\|_{W_{2,2}} \le C.$$

for $t \in [0,T)$.

Proof. Since

$$\int_M R^2 d\mu \le C,$$

so

$$\int_M e^{-\lambda}(R_0 - e^{-\frac{\lambda}{2}}\Delta_0 e^{\frac{\lambda}{2}})^2 d\mu_0 \leq C.$$

But $\|\lambda\|_{L^\infty} \leq C$, it follows

$$\int_M (\Delta_0 e^{\frac{\lambda}{2}})^2 d\mu_0 \leq C.$$

This implies

$$\|e^{\frac{\lambda}{2}}\|_{W_{2,2}} \leq C,$$

and from Sobolev imbedding theorem $W_{2,2} \subset W_{1,6}$ for $n=3$, we have

$$\|\lambda\|_{W_{2,2}} \leq C.$$

This completes the proof.

For higher order estimates, it is straightforward, we refer to [Chru] and [Ch2] for details.

Lemma 3.3. ([Chru, Proposition 4.1.]). *The same assumptions as in the previous lemma. There exists a constant* $C = C(\|\lambda_0\|_{W_{2,2}}, g_0, T)$, $l \geq 2$ *such that*

$$\|\overset{0}{\nabla}{}^l \lambda(p,t)\|_{L_2} \leq C,$$

$\forall t \in [0, T)$.

Then Theorem 1.1 follows.

Furthermore, by applying Lemma 3.1, we prove the Theorem 1.3 and 1.4.

4. Asymptotic convergence to an extremal metric

In the previous sections, we show the following bound:

$$\int e^{4\lambda} d\mu_0 \leq C e^{C_t}$$

and the C^0-bound

$$\sup_{p \in M_t} |\lambda(p,t)| \leq C(T), \quad 0 \leq t < T. \tag{4.1}$$

Then we have the long time existence of solution of (1.3).

However, at the previous steps, the $C(T)$ as in (4.1) may blow up as $t\to\infty$, and then a solution of

$$\frac{\partial\lambda}{\partial t}=\Delta R+\frac{1}{8}R^2-\frac{1}{8}r$$

need not converge to a solution of

$$\Delta R+\frac{1}{8}R^2-\frac{1}{8}r=0.$$

In this section, we will show the uniformly bound on $C(T)$ and $\|\lambda\|_{W_{2,2}}$. Then there exists a subsequence of solutions of (1.3) converges to an extremal metric.

From now on, the constant C will denote the universal constant which is independent of t, for $t\in[0,\infty]$ and may vary from line to line.

Lemma 4.1. *The same assumptions as in Theorem* 1.2, *then*

$$\int_M e^{-2\lambda}(\Delta_0 e^{\frac{\lambda}{2}})^2 d\mu_0\leq C.$$

Proof. Since $\int_M R^2 d\mu\leq C$, we have

$$\int_M e^{-\lambda}(R_0-e^{-\frac{\lambda}{2}}\Delta_0 e^{\frac{\lambda}{2}})^2 d\mu_0\leq C.$$

But $\lambda\geq -C$, this completes the proof.

Lemma 4.2. *The same assumptions as in Theorem* 1.2, *we have*

$$\int_M e^{\lambda}|\overset{0}{\nabla}\lambda|^2 d\mu_0\leq C.$$

Proof. We compute

$$\begin{aligned}\int_M e^{\lambda}|\overset{0}{\nabla}\lambda|^2 d\mu_0&=4\int_M |\overset{0}{\nabla} e^{\frac{\lambda}{2}}|^2 d\mu_0\\ &=-4\int_M e^{\frac{\lambda}{2}}\Delta_0 e^{\frac{\lambda}{2}} d\mu_0\\ &\leq C\int_M e^{3\lambda}d\mu_0+C\int_M e^{-2\lambda}(\Delta_0 e^{\frac{\lambda}{2}})^2 d\mu_0\\ &\leq C.\end{aligned}$$

Lemma 4.3. *Under the fiow* (1.3), *if*

$$\lambda \geq -H$$

for all $0 \leq t \leq \infty$. *Then*

$$\int_M |\overset{0}{\nabla}\lambda|^3 d\mu_0 \leq C(K, g_0)$$

and

$$\int_M e^{4\lambda} d\mu_0 \leq C(K, g_0)$$

for all $0 \leq t \leq \infty$.

Proof. As in Lemma 2.7, we have two cases. For the case (i) of Lemma 2.7, we are done.

Case (ii) of Lemma 2.7: From

$$2\int_M |\overset{0}{\nabla}\lambda|^3 d\mu_0 \leq \int_M e^{\lambda}|\overset{0}{\nabla}\lambda|^2 d\mu_0 + \int_M e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0,$$

and the previous lemma, in order to show the fist part of Lemma 4.3, it suffices to control

$$\int_M e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 \leq C.$$

By passing a subsequence $\{t_j\}$, we may assume $\Sigma = \{p\} \neq \phi$, then from Lemma 2.7, $\lambda \leq C$ on $M \backslash B_\rho$, for $B_\rho = B(p, \rho)$. On the other hand, from (ii) of Theorem 2.4, we have

$$\frac{d}{dt}\int_M e^{\alpha\lambda} d\mu_0 \leq I_2 + I_3 \int_M e^{\alpha\lambda} d\mu_0 - I_4 \int_M e^{(\alpha-4)\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0, \quad 5 < \alpha < \frac{11}{2}.$$

This implies

$$\begin{aligned} \frac{d}{dt}\int_M e^{\alpha\lambda} d\mu_0 &\leq I_2 + I_3 \int_{M\backslash B_\rho} e^{\alpha\lambda} d\mu_0 + I_3 \int_{B_\rho} e^{\alpha\lambda} d\mu_0 - I_4 \int_{B_\rho} e^{(\alpha-4)\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 \\ &\leq C(\alpha) + I_3 \int_{B_\rho} e^{\alpha\lambda} d\mu_0 - I_4 \int_{B_\rho} e^{(\alpha-4)\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0. \end{aligned}$$

Thus, from the same method as in Lemma 2.3, for small enough $\rho > 0$,

$$|\overset{0}{\nabla}\lambda|^4 < \frac{2I_3}{I_4} e^{4\lambda}$$

on B_ρ. Otherwise

$$\frac{d}{dt}\int_M e^{\alpha\lambda}d\mu_0 \le C - I_3\int_{B_\rho} e^{\alpha\lambda}d\mu_0 .$$

It follows that we have the uniformly bound of $\int_{B_\rho} e^{\alpha\lambda}d\mu_0$, $5<\alpha<\frac{11}{2}$. This leads to a contradiction for $\Sigma \neq \phi$.

Hence

$$\begin{aligned}
\int_M e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 &= \int_{M\backslash B_\rho} e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 + \int_{B_\rho} e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 \\
&\le \int_{M\backslash B_\rho} e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 + C\int_{B_\rho} e^{3\lambda}d\mu_0 \qquad (4.2)\\
&\le C + \int_{M\backslash B_\rho} e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 .
\end{aligned}$$

But, since $\lambda \le C$ on $M\backslash B_\rho$, we have

$$\int_{M\backslash B_\rho} (\Delta_0 e^{\frac{\lambda}{2}})^2 d\mu_0 \le C$$

and

$$\int_M (e^{\frac{\lambda}{2}})^2 d\mu_0 \le C.$$

One has

$$e^{\frac{\lambda}{2}} \in W_{2,2}$$

on $M\backslash B_\rho$ and then standard Sobolev imbedding theorem implies

$$e^{\frac{\lambda}{2}} \in W_{1,6}$$

on $M\backslash B_\rho$. That is

$$\int_{M\backslash B_\rho} e^{2\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 = \int_{M\backslash B_\rho} |\overset{0}{\nabla} e^{\frac{\lambda}{2}}|^4 d\mu_0 \le C.$$

On the other hand, $\lambda \ge -C$. Thus

$$\int_{M\backslash B_\rho} e^{-\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 \le C\int_{M\backslash B_\rho} e^{2\lambda}|\overset{0}{\nabla}\lambda|^4 d\mu_0 \le C.$$

Then, from (4.2)

$$\int_M e^{-\overset{0}{\lambda}}|\nabla\lambda|^4 d\mu_0 \le C + \int_{M\setminus B_\rho} e^{-\overset{0}{\lambda}}|\nabla\lambda|^4 d\mu_0 \le C.$$

This completes the proof of the first part of the Lemma.
From this, $\int_M e^{3\lambda} d\mu_0 \le C$ and $\lambda \ge -C$, one can have

$$\|\lambda - \bar{\lambda}\|_{W_{1,3}} \le C.$$

Hence from Moser's inequality

$$\int_M e^{4(\lambda-\bar{\lambda})} d\mu_0 \le C' \exp(C''\|\lambda-\bar{\lambda}\|_{W_{1,3}}) \le C$$

and then

$$\int_M e^{4\lambda} d\mu_0 \le C$$

for $0 \le t_j \le \infty$. This and the following theorem lead to $\Sigma = \phi$. This is a contradiction.

Then the same methods as in the previous section, we have the uniformly bound on all $W_{k,2}$ norms.

Now we have the following main result of this section:

Theorem 4.4. *The same assumptions as in Theorem* 1.2. *Then there exists a subsequence* $\{t_j\}$ *such that*

$$R \to R_\infty$$

as $t_j \to \infty$ *with*

$$\Delta R_\infty + \frac{1}{8} R_\infty^2 - \frac{1}{8} r_\infty = 0,$$

where r_∞ *is a constant such that* $r = \int_M R^2 d\mu / \int_M d\mu \to r_\infty$ *as* $t_j \to \infty$.

Proof. Since

$$-\frac{1}{8}\frac{d}{dt}\int_M R^2 d\mu = \int_M \left(\Delta R + \frac{1}{8}(R^2 - r)\right)^2 d\mu,$$

then

$$\int_0^\infty \int_M \left(\Delta R + \frac{1}{8}(R^2 - r)\right)^2 d\mu dt < \infty,$$

and then there exists a subsequence $\{t_j\}$ such that

$$\int_M (\Delta R+\frac{1}{8}(R^2-r))^2 d\mu|_{t_j} \to 0 \quad \text{as} \quad t_j \to \infty.$$

Now since $\|\lambda\|_{W^{k,2}} \leq C$ for all $0 \leq t_j \leq \infty$, we have

$$\int_M (\Delta R+\frac{1}{8}(R^2-r))^2 d\mu_0|_{t_j} \to 0 \quad \text{as} \quad t_j \to \infty.$$

On the other hand, from (2.3) and (2.5), we have

$$r \to r_\infty$$

as $t_j \to \infty$, for some constant r_∞.

Then the elliptic estimates and the interpolation inequalities yield

$$R \overset{C^\infty}{\to} R_\infty$$

as $t_j \to \infty$ such that

$$\Delta R_\infty + \frac{1}{8} R_\infty^2 - \frac{1}{8} r_\infty = 0.$$

Then Theorems 1.2, 1.5 and 1.6 follow.

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