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Inequalities for Semistable Families of Arithmetic Varieties

By

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Abstract

In this paper, we will consider a generalization of Bogomolov's inequality and Cornalba-Harris-Bost's inequality to the case of semistable families of arithmetic varieties under the idea that geometric semistability implies a certain kind of arithmetic positivity. The first one is an arithmetic analogue of the relative Bogomolov's inequality in [22]. We also establish the arithmetic Riemann-Roch formulae for stable curves over regular arithmetic varieties and generically finite morphisms of arithmetic varieties.

Introduction

In this paper, we will consider a generalization of Bogomolov's inequality and Cornalba-Harris-Bost's inequality to the case of semistable families of arithmetic varieties. An underlying idea of these inequalities as in [4], [5], [8], [17], [18], [19], [20], [21], [24], and [27] is that geometric semistability implies a certain kind of arithmetic positivity. The first one is related to the semistability of vector bundles, and the second one involves the Chow (or Hilbert) semistability of cycles.

First of all, let us consider Bogomolov's inequality. Let X and Y be smooth algebraic varieties over an algebraically closed field of characteristic zero, and $f : X \to Y$ a semi-stable curve. Let E be a vector bundle of rank r on X, and y a point of Y. In [22], the second author proved that if f is smooth over y and $E|_{X_{\bar{y}}}$ is semistable, then $\operatorname{dis}_{X/Y}(E) = f_*\left(2rc_2(E) - (r-1)c_1^2(E)\right)$ is weakly positive at y.

In the first half of this paper, we would like to consider an arithmetic analogue of the above result. Let us fix regular arithmetic varieties X and Y, and a semistable curve $f : X \to Y$. Since we have a good dictionary for translation from a geometric case to an arithmetic case, it looks like routine works. There are, however, two technical difficulties to work over the standard dictionary.

The first one is how to define a push-forward of arithmetic cycles in our situation. If $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth, then, according to Gillet-Soulé's arithmetic intersection theory [9], we can get the push-forward $f_* : \widehat{\mathrm{CH}}^{p+1}(X) \to \widehat{\mathrm{CH}}^{p}(Y)$.

We would not like to restrict ourselves to the case where $f_{\mathbb{Q}}$ is smooth because in the geometric case, the weak positivity of $\mathrm{dis}_{X/Y}(E)$ gives wonderful applications to analyses of the boundary of the moduli space of stable curves. Thus the usual push-forward for arithmetic cycles is insufficient for our purpose. A difficulty in defining the push-forward arises from a fact: if $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is not smooth, then $(f_{\mathbb{C}})_*(\eta)$ is not necessarily C^∞ even for a C^∞ form η. This suggests us that we need to extend the usual arithmetic Chow groups defined by Gillet-Soulé [9]. For this purpose, we will introduce an arithmetic L^1-cycle of codimension p, namely, a pair (Z, g) such that Z is a cycle of codimension p, g is a current of type $(p-1, p-1)$, and g and $dd^c(g) + \delta_{Z(\mathbb{C})}$ are represented by locally integrable forms. Thus, dividing by the usual arithmetical rational equivalence, an arithmetic Chow group, denoted by $\widehat{\mathrm{CH}}^p_{L^1}$, consisting of arithmetic L^1-cycles of codimension p will be defined (cf. Section 2.2). In this way, we have the natural push-forward

$$f_* : \widehat{\mathrm{CH}}^{p+1}_{L^1}(X) \to \widehat{\mathrm{CH}}^p_{L^1}(Y)$$

as desired (cf. Proposition 2.2.2).

The second difficulty is the existence of a suitable Riemann-Roch formula in our situation. As before, if $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth, we have the arithmetic Riemann-Roch theorem due to Gillet-Soulé [11]. If we ignore Noether's formula, then, under the assumption that $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth, their Riemann-Roch formula can be written in the following form:

$$\begin{aligned}\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right)\\ = f_*\left(\frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E})\right)\end{aligned}$$

where $\overline{E} = (E, h)$ is a Hermitian vector bundle on X and $\overline{\omega}_{X/Y}$ is the dualizing sheaf of $f : X \to Y$ with a Hermitian metric. If we consider a general case where $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is not necessarily smooth, the right hand side in the above equation is well defined and sits in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$. On the other hand, the left hand side is rather complicated. If we admit singular fibers of $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$, then the Quillen metric $h_Q^{\overline{E}}$ is no longer C^∞. According to [1], it extends to a generalized metric. Thus, we may define $\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right)$ (cf. Section 3.2). In general, this cycle is not an L^1-cycle. However, using Bismut-Bost's formula [1], we can see that

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right)$$

is an element of $\widehat{\mathrm{CH}}^1_{L^1}(Y)$. Thus, we have a way to establish a Riemann-Roch formula in the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$. Actually, we will prove the above formula in our situation (cf. Theorem 5.2.1). The idea of comparing two sides in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$ is the tricky Lemma 2.5.1.

Let us go back to our problem. First of all, we need to define an arithmetic analogue of weak positivity. Let α be an element of $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$, S a subset of $Y(\mathbb{C})$, and y a closed point of $Y_{\mathbb{Q}}$. We say α is semi-ample at y with respect to S if there are an arithmetic L^1-cycle (E, f) and a positive integer n such that (1) $dd^c(f) + \delta_{E(\mathbb{C})}$ is C^∞ around each $z \in S$, (2) E is effective, (3) $y \notin \mathrm{Supp}(E)$, (4) $f(z) \geq 0$ for all $z \in S$, and (5) $n\alpha$ coincides with the class of (E, f) in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$. Moreover, α is said to be weakly positive at y with respect to S if it is the limit of semi-ample cycles at y with respect to S (for details, see Section 3.5). For example, if $Y = \mathrm{Spec}(O_K)$, y is the generic point, and $S = Y(\mathbb{C})$, then, α is weakly positive at y with respect to S if and only if $\widehat{\deg}(\alpha) \geq 0$, where K is a number field and O_K is the ring of integers in K (cf. Proposition 3.6.1).

Let (E, h) be a Hermitian vector bundle of rank r on X, and $\widehat{\mathrm{dis}}_{X/Y}(E, h)$ the arithmetic discriminant divisor of (E, h) with respect to $f : X \to Y$, that is, the element of $\widehat{\mathrm{CH}}^1_{L^1}(Y)$ given by $f_*\left(2r\widehat{c}_2(E, h) - (r-1)\widehat{c}_1(E, h)^2\right)$. We assume that f is smooth over y and $E|_{X_{\bar{y}}}$ is poly-stable. In the case where $\dim X = 2$ and $Y = \mathrm{Spec}(O_K)$, Miyaoka [17], Moriwaki [18], [19], [20], and Soulé [24] proved that $\widehat{\deg}\left(\widehat{\mathrm{dis}}_{X/Y}(E, h)\right) \geq 0$, consequently, $\widehat{\mathrm{dis}}_{X/Y}(E, h)$ is weakly positive at y with respect to $Y(\mathbb{C})$. One of the main theorems of this paper is the following generalization.

Theorem A (cf. Theorem 8.1)**.** *Under the above assumptions,* $\widehat{\mathrm{dis}}_{X/Y}(E, h)$ *is weakly positive at* y *with respect to any subsets* S *of* $Y(\mathbb{C})$ *with the following properties*: (1) S *is finite, and* (2) $f_{\mathbb{C}}^{-1}(z)$ *is smooth and* $E_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)}$ *is poly-stable for all* $z \in S$*. In particular, if the residue field of* x *is* K*, and the canonical morphism* $\mathrm{Spec}(K) \to X$ *induced by* x *extends to* $\tilde{x} : \mathrm{Spec}(O_K) \to X$*, then* $\widehat{\deg}\left(\tilde{x}^*\left(\widehat{\mathrm{dis}}_{X/Y}(E, h)\right)\right) \geq 0$.

Next, let us consider Cornalba-Harris-Bost's inequality. Motivated by the work of Cornalba and Harris [6] in the geometric case, Bost [4, Theorem I] proved that, roughly speaking, if $X(\overline{\mathbb{Q}}) \subset \mathbb{P}^{r-1}(\overline{\mathbb{Q}})$ gives rise to an $\mathrm{SL}_r(\overline{\mathbb{Q}})$ semi-stable Chow point, then the height of X has a certain kind of positivity. We call this result Cornalba-Harris-Bost's inequality. Zhang [27] then gave precision to it and also showed the converse of Bost's result. Further, Gasbarri [8] considered a wide range of actions instead of the $\mathrm{SL}_r(\overline{\mathbb{Q}})$-action.

In the second half of this paper, we would like to consider a relative version of Cornalba-Harris-Bost's inequality. First, let us fix terminology. Let V be a set, ϕ a non-negative function on V, and S a finite subset of V. We define the geometric mean $\mathrm{g.m.}(\phi; S)$ of ϕ over S to be

$$\mathrm{g.m.}(\phi; S) = \left(\prod_{s \in S} \phi(s)\right)^{1/\#(S)}.$$

Then, the following is our solution.

Theorem B (cf. Theorem 10.1.4). *Let Y be a regular projective arithmetic variety, and $\overline{E} = (E, h)$ a Hermitian vector bundle of rank r. Let $\pi : \mathbb{P}(E) = \mathrm{Proj}(\bigoplus_{n\geq 0} \mathrm{Sym}^n(E^\vee)) \to Y$ be the projection and $\overline{\mathcal{O}_E(1)}$ the tautological line bundle with the quotient metric induced from $f^*(h)$. Let X be an effective cycle in $\mathbb{P}(E)$ such that X is flat over Y with the relative dimension d and degree δ on the generic fiber. For each irreducible component X_i of X_{red}, let $\tilde{X}_i \to X_i$ be a proper birational morphism such that $(\tilde{X}_i)_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$. Let Y_0 be the maximal open set of Y such that the induced morphism $\tilde{X}_i \to Y$ is smooth over Y_0 for every i. Let (B, h_B) be a line bundle equipped with a generalized metric on Y given by the equality*:

$$\widehat{c}_1(B, h_B) = r\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_E(1)})^{d+1} \cdot (X, g_X)\right) + \delta(d+1)\widehat{c}_1(\overline{E}).$$

(*Here we postpone defining g_X, i.e., a suitable compactification of X in the arithmetic sense.*) *Then, h_B is C^∞ over Y_0. Moreover, there are a positive integer $e = e(r, d, \delta)$, a positive integer $l = l(r, d, \delta)$, a positive constant $C = C(r, d, \delta)$, and sections $s_1, \ldots, s_l \in H^0(Y, B^{\otimes e})$ with the following properties.*

(i) *e, l, and C depend only on r, d, and δ.*

(ii) *For a closed point y of $Y_{\mathbb{Q}}$, if X_y is Chow semistable, then $s_i(y) \neq 0$ for some i.*

(iii) *For all i and all closed points y of $(Y_0)_{\mathbb{Q}}$,*

$$\text{g.m.}\left(\left(h_B^{\otimes e}\right)(s_i, s_i);\ O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)\right) \leq C,$$

where $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)$ is the orbit of y by the Galois action in $Y_0(\overline{\mathbb{Q}})$.

Compared with the geometric analogue (cf. Remark 10.1.5), a difficult part of this theorem is the estimate of the geometric mean of the norm over the Galois orbits of closed points. We will do this by reducing it to the absolute case. For this purpose, we have to associate X with a 'nice' Green current g_X. How do we do? One way is to fix a Kähler metric $\mu \in A^{1,1}(\mathbb{P}(E)_{\mathbb{R}})$ and to attach a μ-normalized Green current for X, namely, a Green current g such that $dd^c g + \delta_X = H(\delta_Y)$ and $H(g_Y) = 0$, where $H : D^{p,p}(\mathbb{P}(E)_{\mathbb{R}}) \to H^{p,p}(\mathbb{P}(E)_{\mathbb{R}})$ is the harmonic projection (cf. [5, 2.3.2]). This construction however is not suitable for our purpose because it does not behave well when restricted on fibers.

Thus we are led to define an Ω-normalized Green form which is given, roughly speaking, by attaching a Green form fiberwisely (Here $\Omega = c_1(\overline{\mathcal{O}_E(1)})$). Precisely, an Ω-normalized Green form g_X for X is characterized by the following three conditions; (i) g_X is an L^1-form on $\mathbb{P}(E)$, (ii) $dd^c([g_X]) + \delta_X = \sum_{i=0}^{d}\left[\pi^*(\gamma_i) \wedge \Omega^i\right]$, where γ_i is a d-closed L^1-form of type $(d-i, d-i)$ on Y $(i = 0, \ldots, d)$, (iii) $\pi_*(g_X \wedge \Omega^{r-d}) = 0$ (cf. Proposition 9.1.1). Then we can show that it has a desired property when restricted on fibers (cf. Remark 9.1.4).

Suppose now X is regular. Let $i : X \to \mathbb{P}(E)$ be the inclusion map and $f : X \to Y$ the restriction of π. If we set $\overline{L} = i^*(\overline{\mathcal{O}_E(1)})$, then $\pi_*(\widehat{c}_1(\overline{\mathcal{O}_E(1)})^{d+1} \cdot (X, g_X)) = f_*(\widehat{c}_1(\overline{L})^{d+1})$ (cf. Proposition 9.3.1). Since $f_*(\widehat{c}_1(\overline{L})^{d+1})$ is in general only an element of $\widehat{\mathrm{CH}}^1_{L^1}(Y)$, the above equality explains why we need to consider (X, g_X) in the enlarged arithmetic Chow group $\widehat{\mathrm{CH}}_{L^1}^{r-d-1}(\mathbb{P}(E))$. Moreover, a similar equality when X is not necessarily regular shows that $\pi_*(\widehat{c}_1(\overline{\mathcal{O}_E(1)}) \cdot (X, g_X))$ is independent of the choice of an Ω-normalized Green form g_X for X (cf. Proposition 9.3.1).

Suppose now $Y = \mathrm{Spec}(O_K)$, y is the generic point, and X_y is Chow semistable, where K is a number field. In this case, there exists a generic resolution of X smooth over y. Then Theorem B tells us that

$$r\widehat{\deg}(\widehat{c}_1(\overline{L})^{d+1}) + \delta(d+1)\widehat{\deg}(\overline{E}) + [K : \mathbb{Q}]\alpha(r, d, \delta) \geq 0$$

for some constant $\alpha(r, d, \delta)$ depending only on r, d and δ, which is nothing but Theorem I of Bost [4].

We can also think a wide range of actions like [8]. Namely, let $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ be a morphism of group schemes such that there is an integer k with $\rho(tI_r) = t^k I_R$ for any t, and that ρ commutes with the transposed morphism. For a Hermitian vector bundle $\overline{E}$, we then get the associated Hermitian vector bundle $\overline{E}^\rho$ (cf. Section 9.2). If X is a flat cycle on $\mathbb{P}(E^\rho)$ and y is a closed point of $Y_{\mathbb{Q}}$, then $\mathrm{SL}_r(\overline{\mathbb{Q}})$ acts on a Chow form Φ_{X_y}. The stability of Φ_{X_y} under this action yields a similar inequality (cf. Theorem 10.1.4).

Finally, in Section 10.2 we make a comparison between the relative Bogomolov's inequality (Theorem 8.1) and the relative Cornalba-Harris-Bost's inequality (Theorem 10.1.4).

1. Locally integrable forms and their push-forward

1.1. Locally integrable forms

Let M be an n-dimensional orientable differential manifold. We assume that M has a countable basis of open sets. Let ω be a C^∞ volume element of M, and $C^0_c(M)$ the set of all complex valued continuous functions on M with compact supports. Then, there is a unique Radon measure μ_ω defined on the topological σ-algebra of M such that

$$L\!\!\int_M f d\mu_\omega = \int_M f\omega$$

for all $f \in C^0_c(M)$, where $L\!\int_M f d\mu_\omega$ is the Lebesgue integral arising from the measure μ_ω.

Let f be a complex valued function on M. We say f is *locally integrable*, denoted by $f \in L^1_{\mathrm{loc}}(M)$, if f is measurable and, for any compact set K,

$$L\!\!\int_K |f| d\mu_\omega < \infty.$$

Let ω' be another C^∞ volume form on M. Then, there is a positive C^∞ function a on M with $\omega' = a\omega$. Thus,

$$L\!\!\int_K |f| d\mu_{\omega'} = L\!\!\int_K |f| a d\mu_\omega,$$

which shows us that local integrability does not depend on the choice of the volume form ω. Moreover, it is easy to see that, for a measurable complex valued function f on M, the following are equivalent.

(a) f is locally integrable.

(b) For each point $x \in M$, there is an open neighborhood U of x such that the closure of U is compact and $L\int_U |f| d\mu_\omega < \infty$.

Let Ω_M^p be a C^∞ vector bundle consisting of C^∞ complex valued p-forms. Let $\pi_p : \Omega_M^p \to M$ be the canonical map. We denote $C^\infty(M, \Omega_M^p)$ (resp. $C_c^\infty(M, \Omega_M^p)$) by $A^p(M)$ (resp. $A_c^p(M)$). Let α be a section of $\pi_p : \Omega_M^p \to M$. We say α is *locally integrable*, or simply an *L^1-form* if, at any point of M, all coefficients of α in terms of local coordinates are locally integrable functions. The set of all locally integrable p-forms is denoted by $L^1_{\mathrm{loc}}(M, \Omega_M^p)$. For an maximal form α on M, there is a unique function g on M with $\alpha = g\omega$. We denote this function g by $c_\omega(\alpha)$.

Let us define the Lebesgue integral of locally integrable n-forms with compact support. Let α be an element of $L^1_{\mathrm{loc}}(M, \Omega_M^n)$ such that the support of α is compact. Then $c_\omega(\alpha) \in L^1_{\mathrm{loc}}(M)$ and $\mathrm{supp}(c_\omega(\alpha))$ is compact. Thus, $L\int_M c_\omega(\alpha) d\mu_\omega$ exists. Let ω' be another C^∞ volume element of M. Then, there is a positive C^∞ function a on M with $\omega' = a\omega$. Here $ac_{\omega'}(\alpha) = c_\omega(\alpha)$. Thus,

$$L\int_M c_{\omega'}(\alpha) d\mu_{\omega'} = L\int_M c_{\omega'}(\alpha) a d\mu_\omega = L\int_M c_\omega(\alpha) a d\mu_\omega.$$

Hence, $L\int_M c_\omega(\alpha) d\mu_\omega$ does not depend on the choice of the volume form ω. Thus, the Lebesgue integral of α is defined by

$$L\int_M \alpha = L\int_M c_\omega(\alpha) d\mu_\omega.$$

Moreover, we denote by $D^p(M)$ the space of currents of type p on M. Then, there is the natural homomorphism

$$[\ ] : L^1_{\mathrm{loc}}(M, \Omega_M^p) \to D^p(M)$$

given by $[\alpha](\phi) = L\int_M \alpha \wedge \phi$ for $\phi \in A_c^{n-p}(M)$. It is well known that the kernel of $[\ ]$ is $\{\alpha \in L^1_{\mathrm{loc}}(M, \Omega_M^p) \mid \alpha = 0 \text{ (a.e.)}\}$. A topology on $D^p(M)$ is defined in the following way. For an sequence $\{T_n\}_{n=1}^\infty$ in $D^p(M)$, $T_n \to T$ as $n \to \infty$ if and only if $T_n(\phi) \to T(\phi)$ as $n \to \infty$ for each $\phi \in A_c^{n-p}(M)$. For an element $T \in D^n(M)$, by abuse of notation, we denote by $c_\omega(T)$ a unique distribution g on M given by $T = g\omega$.

Proposition 1.1.1. *Let T be a current of type p on M. Then, the following are equivalent.*

(1) *T is represented by an L^1-form.*
(2) *For any $\phi \in A^{n-p}(M)$, $c_\omega(T \wedge \phi)$ is represented by a locally integrable function.*

Proof. (1) $\Longrightarrow$ (2): Let $\phi \in A^{n-p}(M)$. Then, by our assumption, for any point $x \in M$, there are an open neighborhood U of x, C^∞ functions $a_1, \ldots, a_r$ on U, and locally integrable functions $b_1, \ldots, b_r$ on U such that

$$c_\omega(T \wedge \phi)|_U = \sum_{i=1}^{r} [a_i b_i].$$

Thus, if K is a compact set in U, then

$$\begin{aligned} L\!\!\int_K \left| \sum_{i=1}^{r} a_i b_i \right| d\mu_\omega &\leq L\!\!\int_K \sum_{i=1}^{r} |a_i||b_i| d\mu_\omega \\ &\leq \max_i \sup_{x \in K} \{|a_i(x)|\} \sum_{i=1}^{r} L\!\!\int_K |b_i| d\mu_\omega < \infty. \end{aligned}$$

Thus, we get (2).

(2) $\Longrightarrow$ (1): Before starting the proof, we would like to claim the following fact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of M such that A is at most a countable set. Let λ_α be a locally integrable form U_α with $\lambda_\alpha = \lambda_\beta$ (a.e.) on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in A$. Then, there is a locally integrable form λ on M such that $\lambda = \lambda_\alpha$ (a.e.) on U_α for all $\alpha \in A$. Indeed, let us fix a map $a : M \to A$ with $x \in U_{a(x)}$ and define a form λ by $\lambda(x) = \lambda_{a(x)}(x)$. Then, λ is our desired form because for each $\alpha \in A$,

$$\{x \in U_\alpha \mid \lambda(x) \neq \lambda_\alpha(x)\} \subseteq \bigcup_{\beta \in A \setminus \{\alpha\}} \{x \in U_\alpha \cap U_\beta \mid \lambda_\beta(x) \neq \lambda_\alpha(x)\}$$

and the right hand side has measure zero.

Let U be an open neighborhood of a point $x \in M$ and $(x_1, \ldots, x_n)$ a local coordinate of U such that $dx_1 \wedge \cdots \wedge dx_n$ coincides with the orientation by ω. Then, there is a positive C^∞ function a on U with $\omega = a dx_1 \wedge \cdots \wedge dx_n$ over U. We set

$$T = \sum_{i_1 < \cdots < i_p} T_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

for some distributions $T_{i_1 \cdots i_p}$. We need to show that $T_{i_1 \cdots i_p}$ is represented by a locally integrable function. Since M has a countable basis of open sets, by the above claim, it is sufficient to check that $T_{i_1 \cdots i_p}$ is represented by an integral function on every compact set K in U. Let f be a non-negative C^∞ function on M such that $f = 1$ on K and $\mathrm{supp}(f) \subset U$. Choose $i_{p+1}, \ldots, i_n$ such that $\{i_1, \dots, i_n\} = \{1, \dots, n\}$. Here we set $\phi = fadx_{i_{p+1}} \wedge \cdots \wedge dx_{i_n}$. Then, $\phi \in A^{n-p}(M)$ and

$$T \wedge \phi = \epsilon T_{i_1 \cdots i_p} fadx_1 \wedge \cdots \wedge dx_n = \epsilon T_{i_1 \cdots i_p} f\omega,$$

where $\epsilon = 1$ or -1 depending on the orientation of $\{x_{i_1}, \dots, x_{i_n}\}$. By our assumption, there is a locally integrable function h on M with $c_\omega(T \wedge \phi) = [h]$. Thus, $[\epsilon h] = T_{i_1 \cdots i_p} f$. Therefore, $T_{i_1 \cdots i_p}$ is represented by ϵh on K because $f = 1$ on K. Thus, we get (2). $\square$

1.2. Push-forward of L^1-forms as current

First of all, we recall the push-forward of currents. Let $f : M \to N$ be a proper morphism of orientable manifolds with the relative dimension $d = \dim M - \dim N$. Then,

$$f_* : D^p(M) \to D^{p-d}(N)$$

is defined by $(f_*(T))(\phi) = T(f^*(\phi))$ for $\phi \in A_c^{\dim N - p + d}(N)$. It is easy to see that f_* is a continuous homomorphism. Let us begin with the following lemma.

Lemma 1.2.1. *Let F be an orientable compact differential manifold and Y an orientable differential manifold. Let ω_F (resp. ω_Y) be a C^∞ volume element of F (resp. Y). Let $p : F \times Y \to Y$ be the projection to the second factor. Then, we have the following.*

(1) *If g is a continuous function on $F \times Y$, then $\int_F g\omega_F$ is a continuous function on Y.*

(2) *If α is a continuous maximal form on $F \times Y$, then $p_*([\alpha])$ is represented by a unique continuous from. This continuous form is denoted by $\int_p \alpha$.*

(3) *For a continuous function g on $F \times Y$,*

$$\left| c_{\omega_Y}\left(\int_p g\omega_F \wedge \omega_Y\right)\right| \le c_{\omega_Y}\left(\int_p |g|\omega_F \wedge \omega_Y\right).$$

Proof. (1) This is standard.

(2) Since $\omega_F \wedge \omega_Y$ is a volume form on $F \times Y$, there is a continuous function g on $F \times Y$ with $\alpha = g\omega_F \wedge \omega_Y$. Thus, it is sufficient to show that

$$p_*([\alpha]) = \left[\left(\int_F g\omega_F\right)\omega_Y\right].$$

Indeed, by Fubini's theorem, for $\phi \in A_c^0(Y)$,

$$p_*([\alpha])(\phi) = \int_{F \times Y} \phi\alpha = \int_Y \left(\int_F g\omega_F\right)\phi\omega_Y = \left[\left(\int_F g\omega_F\right)\omega_Y\right](\phi).$$

(3) This is obvious because

$$\left|\int_F g\omega_F\right| \le \int_F |g|\omega_F. \qquad \square$$

Corollary 1.2.2. *Let $f : X \to Y$ be a proper, surjective and smooth morphism of connected complex manifolds. Let ω_X and ω_Y be volume elements of X and Y respectively. Then,*

(1) *For a continuous maximal form α on X, $f_*([\alpha])$ is represented by a unique continuous form. We denote this continuous form by $\int_f \alpha$.*

(2) *For any continuous functions g on X,*

$$\left| c_{\omega_Y}\left(\int_f g\omega_X\right)\right| \leq c_{\omega_Y}\left(\int_f |g|\omega_X\right).$$

Proof. (1) This is a local question on Y. Thus, we may assume that there are a compact complex manifold F and a differomorphism $h : X \to F \times Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow[h]{\sim} & F \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & = & Y, \end{array}$$

where $p : F \times Y \to Y$ is the natural projection. Hence, (1) is a consequence of (2) of Lemma 1.2.1.

(2) First, we claim that if the above inequality holds for some special volume elements ω_X and ω_Y, then the same inequality holds for any volume elements. Let ω'_X and ω'_Y be another volume elements of X and Y respectively. We set $\omega'_X = a\omega_X$ and $\omega'_Y = b\omega_Y$. Then, a and b are positive C^∞ functions. Let g be any continuous function on X. Then, by our assumption,

$$\left| c_{\omega_Y}\left(\int_f g\omega'_X\right)\right| = \left| c_{\omega_Y}\left(\int_f ga\omega_X\right)\right| \leq c_{\omega_Y}\left(\int_f |g|a\omega_X\right) = c_{\omega_Y}\left(\int_f |g|\omega'_X\right).$$

On the other hand, for any maximal forms α on Y,

$$c_{\omega_Y}(\alpha) = bc_{\omega'_Y}(\alpha).$$

Thus, we get our claim.

Hence, as in the proof of (1), using the differomorphism h and (3) of Lemma 1.2.1, we can see (2). □

Remark 1.2.3. In the situation of Corollary 1.2.2, if α is a C^∞-form on X, then $f_*([\alpha])$ is represented by a unique C^∞-form.

Proposition 1.2.4. *Let $f : X \to Y$ be a proper and surjective morphism of connected complex manifolds. Let U be a non-empty Zariski open set of Y such that f is smooth over U. Let α be a compactly supported continuous maximal form on X. If we set*

$$\lambda = \begin{cases} \displaystyle\int_{f^{-1}(U)\to U} \alpha & \text{on } U, \\ 0 & \text{on } Y \setminus U, \end{cases}$$

then λ is integrable. Moreover, $f_*([\alpha]) = [\lambda]$.

Proof. Let ω_X and ω_Y be volume forms of X and Y respectively. Let h be a function on Y with $\lambda = h\omega_Y$. Then, h is continuous on U by Corollary 1.2.2. Moreover, let g be a continuous function on X with $\alpha = g\omega_X$. We need to show that h is an integrable function. First note that $\int_X |g|\omega_X < \infty$ because g is a compactly supported continuous function. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open sets such that $\overline{U_n} \subset U$, $\overline{U_n}$ is compact, $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots$, and $\bigcup_{n=1}^{\infty} U_n = U$. Here we set

$$h_n(y) = \begin{cases} |h(y)| & \text{if } y \in U_n \\ 0 & \text{otherwise.} \end{cases}$$

Then, $0 \leq h_1 \leq h_2 \leq \cdots \leq h_n \leq \cdots$ and $\lim_{n\to\infty} h_n(y) = |h(y)|$. By Corollary 1.2.2,

$$|\,h|_U| \leq c_{\omega_Y}\left(\int_{f^{-1}(U)\to U} |g|\omega_X\right).$$

Thus,

$$\begin{aligned}\int_{U_n} |h|\omega_Y &\leq \int_{U_n} c_{\omega_Y}\left(\int_{f^{-1}(U_n)\to U_n} |g|\omega_X\right)\omega_Y = \int_{U_n}\int_{f^{-1}(U_n)\to U_n} |g|\omega_X \\ &= \int_{f^{-1}(U_n)} |g|\omega_X \leq \int_X |g|\omega_X.\end{aligned}$$

Therefore,

$$L\!\!\int_Y h_n d\mu_{\omega_Y} = \int_{U_n} |h|\omega_Y \leq \int_X |g|\omega_X < \infty.$$

Thus, by Fatou's theorem,

$$L\!\!\int_Y |h| d\mu_{\omega_Y} = \lim_{n\to\infty} L\!\!\int_Y h_n d\mu_{\omega_Y} \leq L\!\!\int_X |g|\omega_X < \infty.$$

Hence, h is integral.

Let ϕ be any element of $A_c^0(Y)$. Then, since $\lim_{n\to\infty} \mu_{\omega_Y}(Y \setminus U_n) = 0$ and $h\phi$ is integrable, by the absolute continuity of Lebesgue integral,

$$\lim_{n\to\infty} L\!\!\int_{Y\setminus U_n} h\phi d\mu_{\omega_Y} = 0.$$

Thus,

$$\begin{aligned}L\!\!\int_Y \lambda\phi &= \lim_{n\to\infty}\left(L\!\!\int_{U_n} h\phi d\mu_{\omega_Y} + L\!\!\int_{Y\setminus U_n} h\phi d\mu_{\omega_Y}\right) \\ &= \lim_{n\to\infty} L\!\!\int_{U_n} h\phi d\mu_{\omega_Y} = \lim_{n\to\infty}\int_{U_n} h\phi\omega_Y = \lim_{n\to\infty}\int_{U_n} \lambda\phi.\end{aligned}$$

In the same way,

$$\int_X \alpha f^*(\phi) = \lim_{n\to\infty} \int_{f^{-1}(U_n)} \alpha f^*(\phi).$$

On the other hand, we have

$$\int_{U_n} \lambda\phi = \int_{f^{-1}(U_n)} \alpha f^*(\phi).$$

Hence

$$\begin{aligned} f_*([\alpha])(\phi) = [\alpha](f^*(\phi)) &= \int_X \alpha \wedge f^*(\phi) = \lim_{n\to\infty} \int_{f^{-1}(U_n)} \alpha f^*(\phi) \\ &= \lim_{n\to\infty} \int_{U_n} \lambda\phi = L\!\!\int_Y \lambda\phi = [\lambda](\phi) \end{aligned}$$

Therefore, $f_*([\alpha]) = [\lambda]$. □

Let X be an equi-dimensional complex manifold, i.e., every connected component has the same dimension. We denote by $A^{p,q}(X)$ the space of C^∞ complex valued (p,q)-forms on X. Let $A_c^{p,q}(X)$ be the subspace of compactly supported forms. Let $D^{p,q}(X)$ be the space of currents on X of type (p,q). As before, there is a natural homomorphism

$$[\ ] : L^1_{\mathrm{loc}}(\Omega_X^{p,q}) \to D^{p,q}(X).$$

Then, as a corollary of Proposition 1.2.4, we have the following main result of this section.

Proposition 1.2.5. *Let $f : X \to Y$ be a proper morphism of equi-dimensional complex manifolds. We assume that every connected component of X maps surjectively to a connected component of Y. Let α be an L^1-form of type $(p+d, q+d)$ on X, where $d = \dim X - \dim Y$. Then there is a $\lambda \in L^1_{\mathrm{loc}}(\Omega_Y^{p,q})$ with $f_*([\alpha]) = [\lambda]$.*

Proof. Clearly we may assume that Y is connected. Since f is proper, there are finitely many connected components of X, say, $X_1, \ldots, X_e$. If we set $\alpha_i = \alpha|_{X_i}$ and $f_i = f|_{X_i}$ for each i, then $f_*([\alpha]) = (f_1)_*([\alpha_1]) + \cdots + (f_e)_*([\alpha_e])$. Thus, we may assume that X is connected. Further, since $f_*([\alpha \wedge f^*(\phi)]) = f_*([\alpha]) \wedge \phi$ for all $\phi \in A^{\dim Y - p, \dim Y - q}(Y)$, we may assume that α is a maximal form by Proposition 1.1.1.

Let g be a locally integrable function on X with $\alpha = g\omega_X$. Since the question is local with respect to Y, we may assume that g is integrable. Thus, since $C_c^0(Y)$ is dense on $L^1(Y)$ (cf. [23, Theorem 3.14]), there is a sequence $\{g_n\}_{n=1}^\infty$ of compactly supported continuous functions on X such that

$$\lim_{n\to\infty} L\!\!\int_X |g_n - g| d\mu_{\omega_X} = 0.$$

By Proposition 1.2.4, for each n, there is an integrable function h_n on Y such that $f_*([g_n\omega_X]) = [h_n\omega_Y]$. Moreover, by (2) of Corollary 1.2.2,

$$|h_n - h_m| \leq c_{\omega_Y}\left(\int_{f^{-1}(U)\to U} |g_n - g_m|\omega_X\right)$$

over U. Thus, we can see

$$L\!\!\int_Y |h_n - h_m| d\mu_{\omega_Y} \leq L\!\!\int_X |g_n - g_m| d\mu_{\omega_X}$$

for all n, m. Hence, $\{h_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^1(Y)$. Therefore, by the completeness of $L^1(Y)$, there is an integrable function h on Y with $h = \lim_{n\to\infty} h_n$ in $L^1(Y)$. Then, for any $\phi \in A_c^{0,0}(Y)$,

$$\lim_{n\to\infty} L\!\!\int_Y h_n\phi\omega_Y = L\!\!\int_Y h\phi\omega_Y \quad \text{and} \quad \lim_{n\to\infty} L\!\!\int_X g_n f^*(\phi)\omega_X = L\!\!\int_X g f^*(\phi)\omega_X.$$

Thus,

$$\begin{aligned} f_*([\alpha])(\phi) &= L\!\!\int_X g f^*(\phi)\omega_X = \lim_{n\to\infty} L\!\!\int_X g_n f^*(\phi)\omega_X \\ &= \lim_{n\to\infty} L\!\!\int_Y h_n\phi\omega_Y = L\!\!\int_Y h\phi\omega_Y = [h\omega_Y](\phi). \end{aligned}$$

Therefore, $f_*([\alpha]) = [h\omega_Y]$. □

2. Variants of arithmetic Chow groups

2.1. Notation for arithmetic varieties

An *arithmetic variety* X is an integral scheme which is flat and quasi-projective over $\mathrm{Spec}(\mathbb{Z})$, and has the smooth generic fiber $X_{\mathbb{Q}}$.

Let us consider the $\mathbb{C}$-scheme $X \otimes_{\mathbb{Z}} \mathbb{C}$. We denote the underlying analytic space of $X \otimes_{\mathbb{Z}} \mathbb{C}$ by $X(\mathbb{C})$. We may view $X(\mathbb{C})$ as the set of all $\mathbb{C}$-valued points of X. Let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be the anti-holomorphic involution given by the complex conjugation. For an arithmetic variety X, every (p,p)-form α on $X(\mathbb{C})$ is always assumed to be compatible with F_∞, i.e., $F_\infty^*(\alpha) = (-1)^p\alpha$.

Let E be a locally free sheaf on X of finite rank, and $\pi : \boldsymbol{E} \to X$ the vector bundle associated with E, i.e., $\boldsymbol{E} = \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(E)\right)$. As before, we have the analytic space $\boldsymbol{E}(\mathbb{C})$ and the anti-holomorphic involution $F_\infty : \boldsymbol{E}(\mathbb{C}) \to \boldsymbol{E}(\mathbb{C})$. Then, $\pi_{\mathbb{C}} : \boldsymbol{E}(\mathbb{C}) \to X(\mathbb{C})$ is a holomorphic vector bundle on $X(\mathbb{C})$, and the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{E}(\mathbb{C}) & \xrightarrow{F_\infty} & \boldsymbol{E}(\mathbb{C}) \\ {\scriptstyle \pi_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbb{C}}} \\ X(\mathbb{C}) & \xrightarrow[F_\infty]{} & X(\mathbb{C}) \end{array}$$

Here note that $F_\infty : \boldsymbol{E}(\mathbb{C}) \to \boldsymbol{E}(\mathbb{C})$ is anti-complex linear at each fiber. Let h be a C^∞ Hermitian metric of $\boldsymbol{E}(\mathbb{C})$. We can think h as a C^∞ function on $\boldsymbol{E}(\mathbb{C}) \times_{X(\mathbb{C})} \boldsymbol{E}(\mathbb{C})$. For simplicity, we denote by $F_\infty^*(h)$ the C^∞ function $\left(F_\infty \times_{X(\mathbb{C})} F_\infty\right)^*(h)$ on $\boldsymbol{E}(\mathbb{C}) \times_{X(\mathbb{C})} \boldsymbol{E}(\mathbb{C})$. Then, $\overline{F_\infty^*(h)}$ is a C^∞ Hermitian metric of $\boldsymbol{E}(\mathbb{C})$. We say *$h$ is invariant under F_∞* if $F_\infty^*(h) = \overline{h}$. Moreover, the pair (E, h) is called *a Hermitian vector bundle on X* if h is invariant under F_∞. Note that even if h is not invariant under F_∞, $h + \overline{F_\infty^*(h)}$ is an invariant metric.

2.2. Variants of arithmetic cycles

Let X be an arithmetic variety. We would like to define three types of arithmetic cycles, namely, arithmetic A-cycles, arithmetic L^1-cycles, and arithmetic D-cycles. In the following definition, g is compatible with F_∞ as mentioned in Section 2.1.

(a) (arithmetic A-cycle on X of codimension p) : a pair (Z, g) such that Z is a cycle on X of codimension p and g is represented by a Green form ϕ of $Z(\mathbb{C})$, namely, ϕ is a C^∞ form on $X(\mathbb{C}) \setminus \operatorname{Supp}(Z(\mathbb{C}))$ of logarithmic type along $\operatorname{Supp}(Z(\mathbb{C}))$ with $dd^c([\phi]) + \delta_{Z(\mathbb{C})} \in A^{p,p}(X(\mathbb{C}))$.

(b) (arithmetic L^1-cycle on X of codimension p) : a pair (Z, g) such that Z is a cycle on X of codimension p and, there are $\phi \in L^1_{\mathrm{loc}}(\Omega^{p-1,p-1}_{X(\mathbb{C})})$ and $\omega \in L^1_{\mathrm{loc}}(\Omega^{p,p}_{X(\mathbb{C})})$ with $g = [\phi]$ and $dd^c(g) + \delta_{Z(\mathbb{C})} = [\omega]$.

(c) (arithmetic D-cycle on X of codimension p) : a pair (Z, g) such that Z is a cycle on X of codimension p and $g \in D^{p-1,p-1}(X(\mathbb{C}))$.

The set of all arithmetic A-cycles (resp. L^1-cycles, D-cycles) of codimension p is denoted by $\widehat{Z}^p_A(X)$ (resp. $\widehat{Z}^p_{L^1}(X)$, $\widehat{Z}^p_D(X)$).

Let $\widehat{R}^p(X)$ be the subgroup of $\widehat{Z}^p(X)$ generated by the following elements:

(i) $((f), -[\log|f|^2])$, where f is a rational function on some subvariety Y of codimension $p-1$ and $[\log|f|^2]$ is the current defined by

$$[\log|f|^2](\gamma) = L\int_{Y(\mathbb{C})} (\log|f|^2)\gamma.$$

(ii) $(0, \partial(\alpha) + \bar{\partial}(\beta))$, where $\alpha \in D^{p-2,p-1}(X(\mathbb{C}))$, $\beta \in D^{p-1,p-2}(X(\mathbb{C}))$.

Here we define

$$\begin{cases} \widehat{\mathrm{CH}}^p_A(X) = \widehat{Z}^p_A(X)/\widehat{R}^p(X) \cap \widehat{Z}^p_A(X), \\ \widehat{\mathrm{CH}}^p_{L^1}(X) = \widehat{Z}^p_{L^1}(X)/\widehat{R}^p(X) \cap \widehat{Z}^p_{L^1}(X), \\ \widehat{\mathrm{CH}}^p_D(X) = \widehat{Z}^p_D(X)/\widehat{R}^p(X). \end{cases}$$

Proposition 2.2.1. *The natural homomorphism $\widehat{\mathrm{CH}}^p_A(X) \to \widehat{\mathrm{CH}}^p(X)$ is an isomorphism.*

Proof. Let $(Z, g) \in \widehat{Z}^p(X)$. By [9, Theorem 1.3.5], there is a Green form g_Z of $Z(\mathbb{C})$. Then, $dd^c(g - [g_Z]) \in A^{p,p}(X(\mathbb{C}))$. Hence, by [9, Theorem 1.2.2], there are $a \in A^{d,d}(X(\mathbb{C}))$ and $v \in \mathrm{Image}(\partial) + \mathrm{Image}(\bar{\partial})$ with $g - [g_Z] = [a] + v$. Since $g - [g_Z]$ is compatible with F_∞, replacing a and v by $(1/2)(a + (-1)^p F_\infty^*(a))$ and $(1/2)(v + (-1)^p F_\infty^*(v))$ respectively, we may assume that a and v are compatible with F_∞. Here, $g_Z + a$ is a Green form of Z. Thus, $(Z, [g_Z + a]) \in \widehat{Z}_A^p(X)$. Moreover, since $(Z, g) - (Z, [g_Z + a]) \in \widehat{R}^p(X)$, our proposition follows. □

Let $f : X \to Y$ be a proper morphism of arithmetic varieties with $d = \dim X - \dim Y$. Then, we have a homomorphism

$$f_* : \widehat{Z}_D^{p+d}(X) \to \widehat{Z}_D^p(Y)$$

defined by $f_*(Z, g) = (f_*(Z), f_*(g))$. In the same way as in the proof of [9, Theorem 3.6.1], we can see $f_*(\widehat{R}^{p+d}(X)) \subseteq \widehat{R}^p(Y)$. Thus, the above homomorphism induces

$$f_* : \widehat{\mathrm{CH}}_D^{p+d}(X) \to \widehat{\mathrm{CH}}_D^p(Y).$$

Then we have the following.

Proposition 2.2.2. *If f is surjective, then $f_* : \widehat{\mathrm{CH}}_D^{p+d}(X) \to \widehat{\mathrm{CH}}_D^p(Y)$ gives rise to*

$$f_* : \widehat{\mathrm{CH}}_{L^1}^{p+d}(X) \to \widehat{\mathrm{CH}}_{L^1}^p(Y).$$

In particular, we have the homomorphism $f_* : \widehat{\mathrm{CH}}^{p+d}(X) \to \widehat{\mathrm{CH}}_{L^1}^p(Y)$.

Proof. Clearly we may assume that $p \geq 1$. It is sufficient to show that if $(Z, g) \in \widehat{Z}_{L^1}^{p+d}(X)$, then $(f_*(Z), f_*(g)) \in \widehat{Z}_{L^1}^p(Y)$. By the definition of L^1-arithmetic cycles, g and $dd^c(g) + \delta_{Z(\mathbb{C})}$ are represented by L^1-forms. Thus, by Proposition 1.2.5, there is an $\omega \in L^1_{\mathrm{loc}}(\Omega^{p,p}_{Y(\mathbb{C})})$ with

$$f_*\left(dd^c(g) + \delta_{Z(\mathbb{C})}\right) = [\omega].$$

On the other hand,

$$f_*\left(dd^c(g) + \delta_{Z(\mathbb{C})}\right) = dd^c(f_*(g)) + \delta_{f_*(Z(\mathbb{C}))}.$$

Moreover, by Proposition 1.2.5, $f_*(g)$ is represented by an L^1-form on $Y(\mathbb{C})$. Thus, $(f_*(Z), f_*(g))$ is an element of $\widehat{Z}_{L^1}^p(Y)$. □

2.3. Scalar product for arithmetic L^1-cycles and arithmetic D-cycles

Let X be a regular arithmetic variety. The purpose of this subsection is to give a scalar product on $\widehat{\mathrm{CH}}_D^*(X)_\mathbb{Q} = \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}_D^p(X)_\mathbb{Q}$ by the arithmetic

Chow ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} = \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$. Roughly speaking, the scalar product is defined by

$$(Y, f) \cdot (Z, g) = (Y \cap Z, f \wedge \delta_Z + \omega((Y, f)) \wedge g)$$

for $(Y, f) \in \widehat{Z}^p(X)$ and $(Z, g) \in \widehat{Z}^q_D(X)$. This definition, however, works only under the assumption that Y and Z intersect properly. Usually, by using Chow's moving lemma, we can avoid the above assumption. This is rather complicated, so that in this paper we try to use the standard arithmetic intersection theory to define the scalar product.

Let $x \in \widehat{\mathrm{CH}}^p(X)$, $(Z, g) \in \widehat{Z}^q_D(X)$, and g_Z a Green current for Z. First we shall check that

$$x \cdot [(Z, g_Z)] + [(0, \omega(x) \wedge (g - g_Z))]$$

in $\widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}}$ does not depend on the choice of g_Z. For, let g'_Z be another Green current for Z. Then, there are $\eta \in A^{p-1,p-1}(X(\mathbb{C}))$, and $v \in \mathrm{Image}(\partial) + \mathrm{Image}(\bar{\partial})$ with $g'_Z = g_Z + [\eta] + v$. Then, since $[(0, [\eta] + v)] \in \widehat{\mathrm{CH}}^p(X)$,

$$\begin{aligned} x \cdot [(Z, g'_Z)] + [(0, \omega(x) \wedge (g - g'_Z))] &= x \cdot [(Z, g_Z)] + x \cdot [(0, [\eta] + v)] \\ &\quad + [(0, \omega(x) \wedge (g - g_Z - [\eta] - v))] \\ &= x \cdot [(Z, g_Z)] + [(0, \omega(x) \wedge ([\eta] + v))] \\ &\quad + [(0, \omega(x) \wedge (g - g_Z - [\eta] - v))] \\ &= x \cdot [(Z, g_Z)] + [(0, \omega(x) \wedge (g - g_Z))]. \end{aligned}$$

Thus, we have the bilinear homomorphism

$$\widehat{\mathrm{CH}}^p(X) \times \widehat{Z}^q_D(X) \to \widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}}$$

given by $x \cdot (Z, g) = x \cdot [(Z, g_Z)] + [(0, \omega(x) \wedge (g - g_Z))]$. Moreover, if $(Z, g) \in \widehat{R}^q(X)$, then, by [9, Theorem 4.2.3], $x \cdot (Z, g) = 0$ in $\widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}}$. Thus, the above induces

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q_D(X) \to \widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}}, \tag{2.3.1}$$

which may give rises to a natural scalar product of $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$ over the arithmetic Chow ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$. To see that this is actually a scalar product, we need to check that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

for all $x \in \widehat{\mathrm{CH}}^p(X)$, $y \in \widehat{\mathrm{CH}}^q(X)$ and $z \in \widehat{\mathrm{CH}}^r_D(X)$. If $z \in \widehat{\mathrm{CH}}^r(X)$, then this is nothing more than the associativity of the product of the arithmetic Chow ring (cf. [9, Theorem 4.2.3]). Thus, we may assume that $z = [(0, g)]$ for some $g \in D^{r-1,r-1}(X(\mathbb{C}))$. Then, since

$$(x \cdot y) \cdot z = [(0, \omega(x \cdot y) \wedge g)] = [(0, \omega(x) \wedge \omega(y) \wedge g)]$$

and

$$x \cdot (y \cdot z) = x \cdot [(0, \omega(y) \wedge g)] = [(0, \omega(x) \wedge \omega(y) \wedge g)],$$

we can see $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Therefore, we get the natural scalar product.

Moreover, (2.3.1) induces

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q_{L^1}(X) \to \widehat{\mathrm{CH}}^{p+q}_{L^1}(X)_{\mathbb{Q}}. \tag{2.3.2}$$

Indeed, if $(Z, g) \in \widehat{Z}^q_{L^1}(X)$ and g_Z is a Green form of Z, then,

$$x \cdot [(Z, g)] = x \cdot [(Z, g_Z)] + [(0, \omega(x) \wedge (g - g_Z))].$$

Thus, in order to see that $x \cdot [(Z, g)] \in \widehat{\mathrm{CH}}^{p+q}_{L^1}(X)_{\mathbb{Q}}$, it is sufficient to check that

$$\begin{cases} \omega(x) \wedge (g - g_Z) \in L^1_{\mathrm{loc}}(\Omega^{p+q-1,p+q-1}_{X(\mathbb{C})}), \\ dd^c\,(\omega(x) \wedge (g - g_Z)) \in L^1_{\mathrm{loc}}(\Omega^{p+q,p+q}_{X(\mathbb{C})}). \end{cases}$$

The first assertion is obvious because g and g_Z are L^1-forms. Further, we can easily see the second assertion because

$$dd^c\,(\omega(x) \wedge (g - g_Z)) = \pm\omega(x) \wedge dd^c(g - g_Z) = \pm\omega(x) \wedge (\omega(g) - \omega(g_Z)).$$

Gathering all observations, we can conclude the following proposition, which is a generalization of [9, Theorem 4.2.3].

Proposition 2.3.3. *$\widehat{\mathrm{CH}}^*_{L^1}(X)_{\mathbb{Q}}$ and $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$ has a natural module structure over the arithmetic Chow ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$.*

Moreover, we have the following projection formula.

Proposition 2.3.4. *Let $f : X \to Y$ be a proper morphism of regular arithmetic varieties such that $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth. Then, for any $\alpha \in \widehat{\mathrm{CH}}^p(Y)$ and $\beta \in \widehat{\mathrm{CH}}^q_{L^1}(X)$,*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

in $\widehat{\mathrm{CH}}^{p+q-(\dim X - \dim Y)}_{L^1}(Y)_{\mathbb{Q}}$.

Proof. If $\alpha \in \widehat{\mathrm{CH}}^p(Y)$ and $\beta \in \widehat{\mathrm{CH}}^q(X)$, then this is well known (cf. [9]). Thus, we may assume that $\beta = (0, [\phi]) \in \widehat{Z}^q_{L^1}(Y)$. Then

$$\begin{aligned} f_*(f^*(\alpha) \cdot \beta) &= f_*((0, \omega(f^*(\alpha)) \wedge [\phi]) \\ &= (0, [f_*\,(\omega(f^*(\alpha) \wedge \phi))]). \end{aligned}$$

On the other hand,

$$\alpha \cdot f_*(\beta) = \alpha \cdot (0, [f_*(\phi)]) = (0, \omega(\alpha) \wedge [f_*(\phi)]).$$

Since $f_*(\omega(f^*(\alpha))) = \omega(\alpha)$, we have proven the projection formula. $\square$

2.4. Scalar product, revisited (singular case)

Let X be an arithmetic variety. Here X is not necessarily regular. Let Rat_X be the sheaf of rational functions on X. We denote $H^0(X, \mathrm{Rat}_X^\times / \mathcal{O}_X^\times)$ by $\mathrm{Div}(X)$. An element of $\mathrm{Div}(X)$ is called *a Cartier divisor on* X. For a Cartier divisor D on X, we can assign a divisor $[D] \in Z^1(X)$ in a natural way. This gives rise to the homomorphism

$$c_X : \mathrm{Div}(X) \to Z^1(X).$$

Note that c_X is neither injective nor surjective in general. An exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathrm{Rat}_X^\times \to \mathrm{Rat}_X^\times / \mathcal{O}_X^\times \to 1$$

induces to the homomorphism $\mathrm{Div}(X) \to H^1(X, \mathcal{O}_X^\times)$. For a Cartier divisor D on X, the image of D by the above homomorphism induces the line bundle on X. We denote this line bundle by $\mathcal{O}_X(D)$.

Here we set

$$\begin{aligned}&\widehat{\mathrm{Div}}(X)\\ &\quad = \{(D, g) \mid D \in \mathrm{Div}(X) \text{ and } g \text{ is a Green function for } D(\mathbb{C}) \text{ on } X(\mathbb{C})\}.\end{aligned}$$

Similarly, we can define $\widehat{\mathrm{Div}}_{L^1}(X)$ and $\widehat{\mathrm{Div}}_D(X)$. The homomorphism c_X : $\mathrm{Div}(X) \to Z^1(X)$ gives rise to the homomorphism $\hat{c}_X : \widehat{\mathrm{Div}}(X) \to \widehat{Z}^1(X)$. Then, we define $\widehat{\mathrm{Pic}}(X)$, $\widehat{\mathrm{Pic}}_{L^1}(X)$, and $\widehat{\mathrm{Pic}}_D(X)$ as follows.

$$\begin{cases} \widehat{\mathrm{Pic}}(X) = \widehat{\mathrm{Div}}(X)/\hat{c}_X^{-1}(\widehat{R}^1(X)), \\ \widehat{\mathrm{Pic}}_{L^1}(X) = \widehat{\mathrm{Div}}_{L^1}(X)/\hat{c}_X^{-1}(\widehat{R}^1(X)), \\ \widehat{\mathrm{Pic}}_D(X) = \widehat{\mathrm{Div}}_D(X)/\hat{c}_X^{-1}(\widehat{R}^1(X)). \end{cases}$$

Note that if X is regular, then

$$\widehat{\mathrm{Pic}}(X) = \widehat{\mathrm{CH}}^1(X), \quad \widehat{\mathrm{Pic}}_{L^1}(X) = \widehat{\mathrm{CH}}^1_{L^1}(X) \quad \text{and} \quad \widehat{\mathrm{Pic}}_D(X) = \widehat{\mathrm{CH}}^1_D(X).$$

Let (E, h) be a Hermitian vector bundle on X. Then, by virtue of [11, Theorem 4], we have a cap product of $\widehat{\mathrm{ch}}(E, h)$ on $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$, i.e., a homomorphism $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$ given by $x \mapsto \widehat{\mathrm{ch}}(E, h) \cap x$ for $x \in \widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$. In the same way as before, we can see that the above homomorphism extends to

$$\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}} \quad \text{and} \quad \widehat{\mathrm{CH}}^*_{L^1}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^*_{L^1}(X)_{\mathbb{Q}}$$

as follows. If $(Z, g) \in \widehat{Z}^p_D(X)$ and g_Z is a Green current of Z, then

$$\widehat{\mathrm{ch}}(E, h) \cap (Z, g) = \widehat{\mathrm{ch}}(E, h) \cap (Z, g_Z) + a(\mathrm{ch}(E, h) \wedge (g - g_Z)).$$

In particular, we have intersection pairings

$$\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \otimes \widehat{\mathrm{CH}}_D^p(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_D^{p+1}(X)_{\mathbb{Q}}$$
$$\text{and} \quad \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \otimes \widehat{\mathrm{CH}}_{L^1}^p(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_{L^1}^{p+1}(X)_{\mathbb{Q}}.$$

For simplicity, the cap product "$\cap$" is denoted by the dot "$\cdot$". Note that

$$\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \otimes \widehat{\mathrm{CH}}_D^p(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_D^{p+1}(X)_{\mathbb{Q}}$$

is actually defined by

$$(D, g) \cdot (Z, f) = (D \cdot Z, g \wedge \delta_Z + \omega(g) \wedge f)$$

if D and Z intersect properly. Then, we have the following projection formula.

Proposition 2.4.1. *Let $f : X \to Y$ be a proper morphism of arithmetic varieties. Let (L, h) be a Hermitian line bundle on Y, and $z \in \widehat{\mathrm{CH}}_D^p(X)$. Then*

$$f_*(\widehat{c}_1(f^*L, f^*h) \cdot z) = \widehat{c}_1(L, h) \cdot f_*(z).$$

Proof. Let (Z, g) be a representative of z. Clearly, we may assume that Z is reduced and irreducible. We set $T = f(Z)$ and $\pi = f|_Z : Z \to T$. Let s be a rational section of $L|_T$. Then, $\pi^*(s)$ gives rise to a rational section of $f^*(L)|_Z = \pi^*(L|_T)$. Thus, $\widehat{c}_1(f^*L, f^*h) \cdot z$ can be represented by

$$(\mathrm{div}(\pi^*(s)), [-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))] + c_1(f^*L, f^*h) \wedge g),$$

where $[-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))]$ is the current given by

$$[-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))](\phi) = \int_{Z(\mathbb{C})} (-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s)))\,\phi.$$

If we set

$$\deg(\pi) = \begin{cases} 0 & \text{if } \dim T < \dim Z \\ \deg(Z \to T) & \text{if } \dim T = \dim Z, \end{cases}$$

then

$$\begin{aligned} \int_{Z(\mathbb{C})} (-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s)))\, f^*(\psi) &= \int_{Z(\mathbb{C})} \pi^*((-\log(h|_T)(s, s))\,\psi) \\ &= \deg(\pi) \int_{T(\mathbb{C})} (-\log(h|_T)(s, s))\,\psi \end{aligned}$$

for a C^∞-form ψ on $Y(\mathbb{C})$. Thus, we have

$$f_*[-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))] = \deg(\pi)[-\log(h|_T)(s, s)].$$

Therefore,

$$\begin{aligned} f_*(\widehat{c}_1(f^*L, f^*h) \cdot z) &= (\deg(\pi) \operatorname{div}(s), \deg(\pi) \left[-\log \left(h|_T \right) (s,s) \right] \\ &\qquad + c_1(L,h) \wedge f_*(g)) \\ &= \widehat{c}_1(L,h) \cdot (\deg(\pi) T, f_*(g)) = \widehat{c}_1(L,h) \cdot f_*(z). \end{aligned}$$

Hence, we get our proposition. □

Let Z be a quasi-projective integral scheme over $\mathbb{Z}$. Then, by virtue of Hironaka's resolution of singularities [14], there is a proper birational morphism $\mu : Z' \to Z$ of quasi-projective integral schemes over $\mathbb{Z}$ such that $Z'_{\mathbb{Q}}$ is non-singular. The above $\mu : Z' \to Z$ is called a *generic resolution of singularities of* Z. As a corollary of the above projection formula, we have the following proposition.

Proposition 2.4.2. *Let X be a arithmetic variety, and $\overline{L}_1 = (L_1, h_1), \ldots, \overline{L}_n = (L_n, h_n)$ be Hermitian line bundles on X. Let (Z, g) be an arithmetic D-cycle on X, and $Z = a_1 Z_1 + \cdots + a_r Z_r$ the irreducible decomposition as cycles. For each i, let $\tau_i : Z'_i \to Z_i$ be a proper birational morphism of quasi-projective integral schemes. We assume that if Z_i is horizontal with respect to $X \to \operatorname{Spec}(\mathbb{Z})$, then τ_i is a generic resolution of singularities of Z_i. Then, we have*

$$\begin{aligned} &\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_n) \cdot (Z, g) \\ &\quad = \sum_{i=1}^{r} a_i \mu_{i*} \left(\widehat{c}_1(\mu_i^* \overline{L}_1) \cdots \widehat{c}_1(\mu_i^* \overline{L}_n) \right) + a(c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n) \wedge g) \end{aligned}$$

in $\widehat{\operatorname{CH}}^*_D(X)_{\mathbb{Q}}$, where μ_i is the composition of $Z'_i \xrightarrow{\tau_i} Z_i \hookrightarrow X$ for each i.

Proof. We prove this proposition by induction on n. First, let us consider the case $n = 1$. Clearly we may assume that Z is integral, i.e., $Z = Z_1$. Let h_1 be the Hermitian metric of $\overline{L}_1$, and s a rational section of $L_1|_Z$. Then,

$$\begin{aligned} &\left(\operatorname{div}(s), -\log(h_1|_Z)(s,s) + c_1(\overline{L}_1) \wedge g \right) \\ &\qquad = (\operatorname{div}(s), -\log(h_1|_Z)(s,s)) + a(c_1(\overline{L}_1) \wedge g) \end{aligned}$$

is a representative of $\widehat{c}_1(\overline{L}_1) \cdot (Z, g)$. Moreover,

$$(\operatorname{div}(\tau_1^*(s)), -\log \tau_1^*(h_1|_Z)(\tau_1^*(s), \tau_1^*(s)))$$

is a representative of $\widehat{c}_1(\mu_1^* \overline{L}_1)$. Hence, we have our assertion in the case $n = 1$ because

$$(\mu_{1*}(\operatorname{div}(\tau_1^*(s)), -\log \tau_1^*(h_1|_Z)(\tau_1^*(s), \tau_1^*(s))) = (\operatorname{div}(s), -\log(h_1|_Z)(s,s)).$$

Thus, we may assume that $n > 1$. Therefore, using Proposition 2.4.1 and hypothesis of induction,

$$\begin{aligned}
\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_n)\cdot(Z,g) &= \widehat{c}_1(\overline{L}_1)\cdot\left(\widehat{c}_1(\overline{L}_2)\cdots\widehat{c}_1(\overline{L}_n)\cdot(Z,g)\right)\\
&= \sum_{i=1}^{r} a_i\widehat{c}_1(\overline{L}_1)\mu_{i*}\left(\widehat{c}_1(\mu_i^*\overline{L}_2)\cdots\widehat{c}_1(\mu_i^*\overline{L}_n)\right)\\
&\quad + \widehat{c}_1(\overline{L}_1)a(c_1(\overline{L}_2)\wedge\cdots\wedge c_1(\overline{L}_n)\wedge g)\\
&= \sum_{i=1}^{r} a_i\mu_{i*}\left(\widehat{c}_1(\mu_i^*\overline{L}_1)\cdots\widehat{c}_1(\mu_i^*\overline{L}_n)\right)\\
&\quad + a(c_1(\overline{L}_1)\wedge\cdots\wedge c_1(\overline{L}_n)\wedge g).
\end{aligned}$$

□

2.5. Injectivity of i^*

Let X be an arithmetic variety, U a non-empty Zariski open set of X, and $i : U \to X$ the inclusion map. Then, there is a natural homomorphism

$$i^* : \widehat{Z}^1_{L^1}(X) \to \widehat{Z}^1_{L^1}(U)$$

given by $i^*(D,g) = (\left.D\right|_U, \left.g\right|_{U(\mathbb{C})})$. Since $i^*\left(\widehat{(f)}\right) = \widehat{(\left.f\right|_U)}$ for any non-zero rational functions f on X, the above induces the homomorphism

$$i^* : \widehat{\mathrm{CH}}^1_{L^1}(X) \to \widehat{\mathrm{CH}}^1_{L^1}(U).$$

Then, we have the following useful lemma.

Lemma 2.5.1. *If $X \setminus U$ does not contain any irreducible components of fibers of $X \to \mathrm{Spec}(\mathbb{Z})$, then*

$$i^* : \widehat{\mathrm{CH}}^1_{L^1}(X) \to \widehat{\mathrm{CH}}^1_{L^1}(U).$$

is injective. In particular, $i^* : \widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^1_{L^1}(U)_{\mathbb{Q}}$ is injective.

Proof. Suppose that $i^*(\alpha) = 0$ for some $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X)$. Let $(D,g) \in \widehat{Z}^1_{L^1}(X)$ be a representative of α. Since $i^*(\alpha) = 0$, there is a non-zero rational function f on X with

$$(\left.D\right|_U, \left.g\right|_{U(\mathbb{C})}) = (\left.(f)\right|_U, -\left.[\log|f|^2]\right|_{U(\mathbb{C})}).$$

Pick up $\phi \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$ with $g = [\phi]$. Then, the above implies that $\left.[\phi]\right|_{U(\mathbb{C})} = -\left.[\log|f|^2]\right|_{U(\mathbb{C})}$. Thus, $\phi = -\log|f|^2$ (a.e.). Therefore, we have

$$g = [\phi] = -[\log|f|^2]. \tag{2.5.1.1}$$

Here, $dd^c(g) + \delta_{D(\mathbb{C})} = [h]$ for some $h \in L^1_{\mathrm{loc}}(\Omega^{1,1}_{X(\mathbb{C})})$ and $dd^c(-[\log|f|^2]) + \delta_{(f)(\mathbb{C})} = 0$. Thus, by (2.5.1.1), $\delta_{D(\mathbb{C})} - \delta_{(f)(\mathbb{C})} = [h]$. This shows us that $h = 0$ (a.e.) over $X(\mathbb{C}) \setminus (\mathrm{Supp}(D(\mathbb{C})) \cup \mathrm{Supp}((f)(\mathbb{C})))$. Hence $h = 0$ (a.e.) on $X(\mathbb{C})$. Therefore, we have $D(\mathbb{C}) = (f)(\mathbb{C})$, which implies $D = (f)$ on $X_{\mathbb{Q}}$. Thus, $D - (f)$ is a linear combination of irreducible divisors lying on finite fibers. On the other hand, $D = (f)$ on U and $X \setminus U$ does not contain any irreducible components of fibers. Therefore, $D = (f)$. Hence $\alpha = 0$ because $(D, g) = \widehat{(f)}$. □

3. Weakly positive arithmetic divisors

3.1. Generalized metrics

Let X be a smooth algebraic scheme over $\mathbb{C}$ and L a line bundle on X. We say h is *a generalized metric on* L if there is a C^∞ Hermitian metric h_0 of L over X and $\varphi \in L^1_{\mathrm{loc}}(X)$ with $h = e^{\varphi} h_0$.

To see when a metric of a line bundle defined over a dense Zariski open set extends to a generalized metric, the following lemma is useful.

Lemma 3.1.1. *Let X be a smooth algebraic variety over $\mathbb{C}$ and L a line bundle on X. Let U be a non-empty Zariski open set of X and h a C^∞ Hermitian metric of L over U. We fix a non-zero rational section s of L. Then, h extends to a generalized metric of L on X if and only if* $\log h(s,s) \in L^1_{\mathrm{loc}}(X)$.

Proof. If h extends to a generalized metric of L on X, then $\log h(s,s) \in L^1_{\mathrm{loc}}(X)$ by the definition of generalized metrics. Conversely, we assume that $\log h(s,s) \in L^1_{\mathrm{loc}}(X)$. Let h_0 be a C^∞ Hermitian metric of L over X. Here we consider the function ϕ given by

$$\phi = \frac{h(s,s)}{h_0(s,s)}.$$

Let $y \in U$ and ω be a local frame of L around y. If we set $s = f\omega$ for some meromorphic function f around y, then

$$\phi = \frac{h(s,s)}{h_0(s,s)} = \frac{|f|^2 h(\omega,\omega)}{|f|^2 h_0(\omega,\omega)} = \frac{h(\omega,\omega)}{h_0(\omega,\omega)}.$$

This shows us that ϕ is a positive C^∞ function on U and $h = \phi h_0$ over U. On the other hand,

$$\log\phi = \log h(s,s) - \log h_0(s,s).$$

Here since $\log h(s,s), \log h_0(s,s) \in L^1_{\mathrm{loc}}(X)$, we have $\log\phi \in L^1_{\mathrm{loc}}(X)$. Thus, if we set $\varphi = \log\phi$, then $\varphi \in L^1_{\mathrm{loc}}(X)$ and $h = e^{\varphi} h_0$. □

3.2. Arithmetic D-divisors and generalized metrics

Let X be an arithmetic variety, L a line bundle on X, and h a generalized metric of L on $X(\mathbb{C})$ with $F_\infty^*(h) = \overline{h}$ (a.e.). We would like to define $\widehat{c}_1(L,h)$ as an element of $\widehat{\mathrm{CH}}_D^1(X)$. Let s, s' be two non-zero rational sections of L, and f a non-zero rational function on X with $s' = fs$. Then, it is easy to see that

$$(\mathrm{div}(s'), [-\log h(s',s')]) = (\mathrm{div}(s), [-\log h(s,s)]) + \widehat{(f)}$$

in $\widehat{Z}_D^1(X)$. Thus, we may define $\widehat{c}_1(L,h)$ as the class of $(\mathrm{div}(s), [-\log h(s,s)])$ in $\widehat{\mathrm{CH}}_D^1(X)$.

Let us consider the homomorphism

$$\omega : \widehat{Z}_D^p(X) \to D^{p,p}(X(\mathbb{C}))$$

given by $\omega(Z,g) = dd^c(g) + \delta_{Z(\mathbb{C})}$. Since $\omega\left(\widehat{R}^p(X)\right) = 0$, the above homomorphism induces the homomorphism $\widehat{\mathrm{CH}}_D^p(X) \to D^{p,p}(X(\mathbb{C}))$. Hence, we get the homomorphism $\widehat{\mathrm{CH}}_D^p(X)_{\mathbb{Q}} \to D^{p,p}(X(\mathbb{C}))$ because $D^{p,p}(X(\mathbb{C}))$ has no torsion. By abuse of notation, we denote this homomorphism by ω.

Proposition 3.2.1. *Let X be an arithmetic variety, $(Z,[\phi]) \in \widehat{\mathrm{Div}}_D(X)$ with $\phi \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$, and 1 a rational section of $\mathcal{O}_X(Z)$ with $\mathrm{div}(1) = Z$. Then, there is a unique generalized metric h of $\mathcal{O}_X(Z)$ such that $F_\infty^*(h) = \overline{h}$ (a.e.) and $[-\log h(1,1)] = [\phi]$. (Here uniqueness of h means that if h' is another generalized metric with the same property, then $h = h'$ (a.e.).) Moreover, $\omega(Z,[\phi])$ is C^∞ around $x \in X(\mathbb{C})$ if and only if h is C^∞ around x. We denote this line bundle $(\mathcal{O}_X(Z), h)$ with the generalized metric h by $\mathcal{O}_Z((Z,[\phi]))$. With this notation, for $(Z_1,[\phi_1]), (Z_2,[\phi_2]) \in \widehat{\mathrm{Div}}_D(X)$ with $\phi_1, \phi_2 \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$, if $(Z_1,[\phi_1]) \sim (Z_2,[\phi_2])$, then $\mathcal{O}_X((Z_1,[\phi_1]))$ is isometric to $\mathcal{O}_X((Z_2,[\phi_2]))$ at every point around which $\omega(Z_1,[\phi_1]) = \omega(Z_2,[\phi_2])$ is a C^∞ form.*

Proof. First, let us see uniqueness. Let h and h' be generalized metrics of $\mathcal{O}_X(Z)$ with $[-\log h(1,1)] = [-\log h'(1,1)] = [\phi]$. Take $a \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$ with $h' = e^a h$. Then, by our assumption, $a = 0$ (a.e.). Thus, $h = h'$ (a.e.).

Pick up an arbitrary point $x \in X(\mathbb{C})$. Let s be a local basis of $\mathcal{O}_X(Z)$ around x. Then, there is a non-zero rational rational function f on $X(\mathbb{C})$ with $1 = fs$. Let us consider

$$\exp(-\phi - \log|f|^2)$$

around x. Let s' be a another local basis of $\mathcal{O}_X(Z)$ around x. We set $s' = us$ and $1 = f's'$. Then,

$$\exp(-\phi - \log|f'|^2) = \exp(-\phi - \log|f/u|^2) = |u|^2 \exp(-\phi - \log|f|^2),$$

which means that if we define the generalized metric h by

$$h(s,s) = \exp(-\phi - \log|f|^2),$$

then h is patched globally, and h is a generalized metric by Lemma 3.1.1. Moreover,

$$-\log h(1,1) = -\log h(fs, fs) = -\log\left(|f|^2 h(s,s)\right) = \phi.$$

Here, since $F_\infty^*(\phi) = \phi$ (a.e.), we can see $F_\infty^*(h) = \overline{h}$ (a.e.). Thus, we can construct our desired metric.

We set $b = \omega(Z, [\phi]) \in D^{1,1}(X(\mathbb{C}))$. Then, since $1 = fs$ around x, we have $Z = (f)$ around x. Thus, since $dd^c([\phi]) + \delta_{Z(\mathbb{C})} = b$ and $dd^c(-[\log|f|^2]) + \delta_{(f)} = 0$,

$$dd^c(-[\phi + \log|f|^2]) = \delta_{Z(\mathbb{C})} - b - \delta_{(f)} = -b$$

around x. Therefore,

$$\begin{aligned} &h \text{ is } C^\infty \text{ around } x \\ &\quad\Longleftrightarrow -\phi - \log|f|^2 \text{ is } C^\infty \text{ around } x \\ &\quad\Longleftrightarrow dd^c(-[\phi + \log|f|^2]) \text{ is } C^\infty \text{ around } x \qquad (\because \text{[9, Theorem 1.2.2]}) \\ &\quad\Longleftrightarrow b \text{ is } C^\infty \text{ around } x \end{aligned}$$

Finally, let us consider the last assertion. By our assumption, there is a rational function z on X such that

$$(Z_1, [\phi_1]) = (Z_2, [\phi_2]) + \widehat{(z)}.$$

Then, $Z_1 = Z_2 + (z)$ and $\phi_1 = \phi_2 - \log|z|^2$. Let us consider the homomorphism $\alpha : \mathcal{O}_X(Z_1) \to \mathcal{O}_X(Z_2)$ defined by $\alpha(s) = zs$. Then, α is an isomorphism. Let 1 be the unit in the rational function field of X. Then, 1 gives rise to canonical rational sections of $\mathcal{O}_X(Z_1)$ and $\mathcal{O}_X(Z_2)$. Let x be a point of $X(\mathbb{C})$ such that $\omega(Z_1, [\phi_1])$ is C^∞ around x, and s a local basis of $\mathcal{O}_X(Z_1)$ around x. Then, $\alpha(s) = zs$ is a local basis of $\mathcal{O}_X(Z_2)$ around x. Choose a rational function f with $1 = fs$. Then, $1 = z^{-1}f\alpha(s)$. Thus, if h_1 and h_2 are metrics of $\mathcal{O}_X((Z_1, [\phi_1]))$ and $\mathcal{O}_X((Z_2, [\phi_2]))$ respectively, then

$$h_1(s,s) = \exp(-\phi_1 - \log|f|^2) = \exp(-\phi_2 - \log|z^{-1}f|^2) = h_2(\alpha(s), \alpha(s))$$

Hence, α is an isometry. □

3.3. Semi-ampleness and small sections

Let X be an arithmetic variety and S a subset of $X(\mathbb{C})$. We set

$$\widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}} = \{\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}} \mid \omega(\alpha) \text{ is } C^\infty \text{ around } z \text{ for all } z \in S\}.$$

In the same way, we can define $\widehat{\mathrm{CH}}^1_{L^1}(X; S)$, $\widehat{Z}^1_{L^1}(X; S)$, $\widehat{Z}^1_{L^1}(X; S)_{\mathbb{Q}}$, $\widehat{\mathrm{Div}}_{L^1}(X; S)$, $\widehat{\mathrm{Div}}_{L^1}(X; S)_{\mathbb{Q}}$, $\widehat{\mathrm{Pic}}_{L^1}(X; S)$ and $\widehat{\mathrm{Pic}}_{L^1}(X; S)_{\mathbb{Q}}$. Let x be a closed point of

$X_{\mathbb{Q}}$. An element α of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ is said to be *semi-ample at x with respect to S* if there are a positive integer n and $(E,g) \in \widehat{Z}^1_{L^1}(X;S)$ with the following properties:

(a) E is effective and $x \notin \mathrm{Supp}(E)$.

(b) $g(z) \geq 0$ for each $z \in S$. (Note that $g(z)$ might be ∞.)

(c) $n\alpha$ coincides with (E,g) in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$.

We remark that $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ by the condition (c). Moreover, it is easy to see that if α_1 and α_2 are semi-ample at x with respect to S, so is $\lambda\alpha_1 + \mu\alpha_2$ for all non-negative rational numbers λ and μ.

In terms of the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X(\overline{\mathbb{Q}})$, we can consider the orbit $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$ of x. If we fix an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, we have an injection $X(\overline{\mathbb{Q}}) \to X(\mathbb{C})$. It is easy to see that the image of $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$ does not depend on the choice of the embedding $\overline{\mathbb{Q}} \to \mathbb{C}$. By abuse of notation, we denote this image by $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$. Then, $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$ is one of the examples of S.

Let U be a Zariski open set of X, and F a coherent $\mathcal{O}_X$-module such that F is locally free over U. Let h_F be a C^∞ Hermitian metric of F over $U(\mathbb{C})$. We assume that $S \subseteq U(\mathbb{C})$. For a closed point x of $U_{\mathbb{Q}}$, we say (F, h_F) is *generated by small sections at x with respect to S* if there are sections $s_1, \ldots, s_n \in H^0(X, F)$ such that F_x is generated by $s_1, \ldots, s_n$, and that $h_F(s_i, s_i)(z) \leq 1$ for all $1 \leq i \leq n$ and $z \in S$.

Proposition 3.3.1. *We assume that $S \subseteq U(\mathbb{C})$. For an element (Z,g) of $\widehat{\mathrm{Div}}_{L^1}(X;S)$, (Z,g) is semi-ample at x with respect to S if and only if there is a positive integer n such that $\mathcal{O}_X(n(Z,g))$ is generated by small sections at x with respect to S.*

Proof. First, we assume that (Z,g) is semi-ample at x with respect to S. Then, there is $(E,f) \in \widehat{Z}^1_{L^1}(X;S)$ and a positive integer n such that $n(Z,g) \sim (E,f)$, E is effective, $x \notin \mathrm{Supp}(E)$, and $f(z) \geq 0$ for each $z \in S$. Note that E is a Cartier divisor. Then, by Proposition 3.2.1, $\mathcal{O}_X(n(Z,g)) \simeq \mathcal{O}_X((E,f))$. Moreover, if h is the metric of $\mathcal{O}_X((E,f))$ and 1 is the canonical section of $\mathcal{O}_X(E)$ with $\mathrm{div}(1) = E$, then $-\log(h(1,1)) = f$. Here $f(z) \geq 0$ for each $z \in S$. Thus, $h(1,1)(z) \leq 1$ for each $z \in S$. Therefore, $\mathcal{O}_X((E,f))$ is generated by small sections at x with respect to S.

Next we assume that $\mathcal{O}_X(n(Z,g))$ is generated by small sections at x with respect to S for some positive integer n. Then, there is a section s of $\mathcal{O}_X(nZ)$ such that $h(s,s)(z) \leq 1$ for each $z \in S$. Thus, if we set $E = \mathrm{div}(s)$ and $f = -\log h(s,s)$, then we can see (Z,g) is semi-ample at x with respect to S. □

Proposition 3.3.2. *Let U be a Zariski open set of X, and L a line bundle on X. Let h be a C^∞ Hermitian metric of L over $U(\mathbb{C})$. Fix a closed point x of $U_{\mathbb{Q}}$. If X is projective over $\mathbb{Z}$, then the followings are equivalent.*

(1) *(L,h) is generated by small sections at x with respect to $U(\mathbb{C})$.*

(2) *(L, h) is generated by small sections at x with respect to any finite subsets S of $U(\mathbb{C})$.*

Proof. Clearly, (1) implies (2). So we assume (2). First of all, we can easily find $z_1, \dots, z_n \in U(\mathbb{C})$ such that, for any $s \in H^0(X(\mathbb{C}), L_{\mathbb{C}})$, if $s(z_1) = \cdots = s(z_n) = 0$, then $s = 0$. Thus, if we set

$$\|s\| = \sqrt{h(s,s)(z_1)} + \cdots + \sqrt{h(s,s)(z_n)}$$

for each $s \in H^0(X(\mathbb{C}), L_{\mathbb{C}})$, then $\| \ \|$ gives rise to a norm on $H^0(X(\mathbb{C}), L_{\mathbb{C}})$. Here we set

$$B_z = \{s \in H^0(X, L) \mid h(s,s)(z) \le 1\}$$

for each $z \in U(\mathbb{C})$. Then, since $H^0(X, L)$ is a discrete subgroup of $H^0(X(\mathbb{C}), L_{\mathbb{C}})$, $\bigcap_{i=1}^n B_{z_i}$ is a finite set. Thus, adding finite points $z_{n+1}, \dots, z_N \in U(\mathbb{C})$ to $z_1, \dots, z_n$ if necessary, we have

$$\bigcap_{z \in U(\mathbb{C})} B_z = \bigcap_{i=1}^N B_{z_i}.$$

By our assumption, there is a section $s \in H^0(X, L)$ such that $s(x) \neq 0$ and $h(s,s)(z_i) \le 1$ for all $i = 1, \dots, N$. Then, $s \in \bigcap_{i=1}^N B_{z_i} = \bigcap_{z \in U(\mathbb{C})} B_z$. Thus, we get (2). □

3.4. Restriction to arithmetic curves

Let X be an arithmetic variety, S a subset of $X(\mathbb{C})$, x a closed point of $X_{\mathbb{Q}}$, K the residue field of x, and O_K the ring of integers in K. We assume that the orbit of x by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is contained in S, namely, $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x) \subseteq S$, and that the canonical morphism $\mathrm{Spec}(K) \to X$ induced by x extends to $\tilde{x} : \mathrm{Spec}(O_K) \to X$.

Proposition 3.4.1. *There is a natural homomorphism*

$$\tilde{x}^* : \widehat{\mathrm{Pic}}_{L^1}(X; S)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$$

such that the restriction of $\tilde{x}^*$ to $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ coincides with the usual pull-back homomorphism.

Proof. Let $\alpha \in \widehat{\mathrm{Pic}}_{L^1}(X; S)_{\mathbb{Q}}$. Choose $(Z, g) \in \widehat{\mathrm{Div}}_{L^1}(X; S)$ and a positive integer e such that the class of $(1/e)(Z, g)$ in $\widehat{\mathrm{Pic}}_{L^1}(X; S)_{\mathbb{Q}}$ coincides with α. We would like to define $\tilde{x}^*(\alpha)$ by

$$(1/e)\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z, g)))\right).$$

For this purpose, we need to check that the above does not depend on the choice (Z,g) and e. Let (Z',g') and e' be another L^1-cycle of codimension 1 and positive integer such that the class of $(1/e')(Z',g')$ is α. Then, there is a positive integer d such that $de'(Z,g) \sim de(Z',g')$. Thus, by Proposition 3.2.1, $\mathcal{O}_Z(de'(Z,g))$ is isometric to $\mathcal{O}_Z(de(Z',g'))$. Hence,

$$\begin{aligned} de'\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z,g)))\right) &= \widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X(de'(Z,g)))\right) \\ &= \widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X(de(Z',g')))\right) \\ &= de\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z',g')))\right). \end{aligned}$$

Therefore,

$$(1/e)\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z,g)))\right) = (1/e')\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z',g')))\right).$$

Thus, we can define $\tilde{x}^*$. □

3.5. Weak positivity of arithmetic L^1-divisors

Let X be an arithmetic variety, S a subset of $X(\mathbb{C})$, and x a closed point of $X_{\mathbb{Q}}$. Let $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ and $\{\alpha_n\}_{n=1}^{\infty}$ a sequence of elements of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. We say α is the limit of $\{\alpha_n\}_{n=1}^{\infty}$ as $n \to \infty$, denoted by $\alpha = \lim_{n\to\infty} \alpha_n$, if there are (1) $Z_1,\dots,Z_{l_1} \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$, (2) $g_1,\dots,g_{l_2} \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$ with $a(g_j) \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ for all j, (3) sequences $\{a_n^1\}_{n=1}^{\infty},\dots,\{a_n^{l_1}\}_{n=1}^{\infty}$ of rational numbers, and (4) sequences $\{b_n^1\}_{n=1}^{\infty},\dots,\{b_n^{l_2}\}_{n=1}^{\infty}$ of real numbers with the following properties:

(a) l_1 and l_2 does not depend on n.

(b) $\lim_{n\to\infty} a_n^i = \lim_{n\to\infty} b_n^j = 0$ for all $1 \le i \le l_1$ and $1 \le j \le l_2$.

(c) $\alpha = \alpha_n + \sum_{i=1}^{l_1} a_n^i Z_i + \sum_{j=1}^{l_2} a(b_n^j g_j)$ in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ for all n.

It is easy to see that if $\alpha = \lim_{n\to\infty}\alpha_n$ and $\beta = \lim_{n\to\infty}\beta_n$ in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$, then $\alpha+\beta = \lim_{n\to\infty}(\alpha_n+\beta_n)$.

An element α of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ is said to be *weakly positive at x with respect to S* if it is the limit of semi-ample $\mathbb{Q}$-cycles at x with respect to S, i.e., there is a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of elements of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ such that α_n's are semi-ample at x with respect to S and $\alpha = \lim_{n\to\infty}\alpha_n$.

Let K be the residue field of x and O_K the ring of integers in K. We assume that $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x) \subseteq S$, and the canonical morphism $\mathrm{Spec}(K) \to X$ induced by x extends to $\tilde{x} : \mathrm{Spec}(O_K) \to X$. Then, we have the following proposition.

Proposition 3.5.1. *If X is regular and an element α of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ is weakly positive at x with respect to S, then $\widehat{\deg}(\tilde{x}^*(\alpha)) \ge 0$.*

Proof. Take $Z_1, \ldots, Z_{l_1}$, $g_1, \ldots, g_{l_2}$, $\{a_n^1\}_{n=1}^\infty, \ldots, \{a_n^{l_1}\}_{n=1}^\infty$, $\{b_n^1\}_{n=1}^\infty, \ldots, \{b_n^{l_2}\}_{n=1}^\infty$, and $\{\alpha_n\}_{n=1}^\infty$ as in the previous definition of weak positive arithmetic divisors. Then,

$$\widehat{\deg}(\tilde{x}^*(\alpha)) = \widehat{\deg}(\tilde{x}^*(\alpha_n)) + \sum_{i=1}^{l_1} a_n^i \widehat{\deg}(\tilde{x}^*(Z_i)) + \sum_{j=1}^{l_2} b_n^j \widehat{\deg}(\tilde{x}^* a(g_j)).$$

Thus, since $\lim_{n\to\infty} a_n^i = \lim_{n\to\infty} b_n^j = 0$ for all $1 \le i \le l_1$ and $1 \le j \le l_2$ and $\widehat{\deg}(\tilde{x}^*(\alpha_n)) \ge 0$ for all n, we have $\widehat{\deg}(\tilde{x}^*(\alpha)) \ge 0$. $\square$

3.6. Characterization of weak positivity

Let X be a regular arithmetic variety, S a subset of $X(\mathbb{C})$, and x a closed point of $X_{\mathbb{Q}}$. For an element $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$, we say α is *ample at x with respect to S* if there are $(A, f) \in \widehat{Z}^1_{L^1}(X; S)$ and a positive integer n such that A is an effective and ample Cartier divisor on X, $x \notin \mathrm{Supp}(A)$, $f(z) > 0$ for all $z \in S$, and $n\alpha$ is equal to (A, f) in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$.

First, let us consider the case where $X = \mathrm{Spec}(O_K)$.

Proposition 3.6.1. *We assume that $X = \mathrm{Spec}(O_K)$, x is the generic of X, and $S = X(\mathbb{C})$. For an element $\alpha \in \widehat{\mathrm{CH}}^1(X; S)_{\mathbb{Q}}$, we have the following.*

(1) *α is ample at x with respect to S if and only if $\widehat{\deg}(\alpha) > 0$.*

(2) *α is weakly positive at x with respect to S if and only if $\widehat{\deg}(\alpha) \ge 0$.*

Proof. (1) Clearly, if α is ample at x with respect to S, then $\widehat{\deg}(\alpha) > 0$. Conversely, we assume that $\widehat{\deg}(\alpha) > 0$. We take a positive integer e and a Hermitian line bundle (L, h) on X such that $\widehat{c}_1(L, h) = e\alpha$. Then, $\widehat{\deg}(L, h) > 0$. Thus, by virtue of Riemann-Roch formula and Minkowski's theorem, there are a positive integer n and a non-zero element s of $L^{\otimes n}$ with $(h^{\otimes n})(s, s)(z) < 1$ for all $z \in S$. Thus, α is ample at x with respect to S.

(2) First, we assume that α is weakly positive at x with respect to S. Then, by Proposition 3.5.1, $\widehat{\deg}(\alpha) \ge 0$. Next, we assume that $\widehat{\deg}(\alpha) \ge 0$. Let β be an element of $\widehat{\mathrm{CH}}^1(X; S)_{\mathbb{Q}}$ such that β is ample at x with respect to S. Then, for any positive integer n, $\widehat{\deg}(\alpha + (1/n)\beta) > 0$. Thus, $\alpha + (1/n)\beta$ is ample at x with respect to S by (1). Hence, α is weakly positive at x with respect to S. $\square$

Before starting a general case, let us consider the following lemma.

Lemma 3.6.2. *We assume that S is compact. Let α be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$ such that α is ample at x with respect to S. Then, we have the following.*

(1) *Let β be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then, there is a positive number ϵ_0 such that $\alpha+\epsilon\beta$ is semi-ample at x with respect to S for all rational numbers ϵ with $|\epsilon| \leq \epsilon_0$.*

(2) *Let g be a locally integrable function on $X(\mathbb{C})$ with $a(g) \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then, there is a positive number ϵ_0 such that $\alpha + a(\epsilon g)$ is semi-ample at x with respect to S for all real numbers ϵ with $|\epsilon| \leq \epsilon_0$.*

Proof. (1) First, we claim that there is a positive number t_0 such that $t\alpha + \beta$ is semi-ample at x with respect to S for all rational numbers $t \geq t_0$.

Let us choose $(A, f) \in \widehat{Z}^1_{L^1}(X;S)$ and a positive integer n_0 such that A is an effective and ample Cartier divisor on X, $x \notin \mathrm{Supp}(A)$, $f(z) > 0$ for all $z \in S$, and $n_0\alpha$ is equal to (A, f) in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$. Moreover, we choose $(D, g) \in \widehat{Z}^1_{L^1}(X;S)$ and a positive integer e such that $e\beta$ is equal to (D, g) in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$. Since A is ample, there is a positive integer n_1 such that $\mathcal{O}_X(n_1A) \otimes \mathcal{O}_X(D)$ is generated by global sections at x. Thus, there are $(Z, h) \in \widehat{Z}^1_{L^1}(X;S)_{\mathbb{Q}}$ such that Z is effective, $x \notin \mathrm{Supp}(Z)$ and $(Z, h) \sim n_1(A, f) + (D, g)$.

We would like to find a positive integer n_2 with $n_2f(z) + h(z) \geq 0$ for all $z \in S$. Let U be an open set of $X(\mathbb{C})$ such that $S \subseteq U$, and $\omega(A, f)$ and $\omega(Z, h)$ are C^∞ over U. We set $\phi = \exp(-f)$ and $\psi = \exp(-h)$. Then, ϕ and ψ are continuous on U, and $0 \leq \phi < 1$ on S. Since $n_2f + h = -\log(\phi^{n_2}\psi)$, it is sufficient to find a positive integer n_2 with $\phi^{n_2}\psi \leq 1$ on S. If we set $a = \sup_{z\in S} \phi(z)$ and $b = \sup_{z\in S} \psi(z)$, then $0 \leq a < 1$ and $0 \leq b$ because S is compact. Thus, there is a positive integer n_2 with $a^{n_2}b \leq 1$. Therefore, $\phi^{n_1}\psi \leq 1$ on S.

Here we set $t_0 = (n_1+n_2)n_0e^{-1}$. In order to see that $t\alpha+\beta$ is semi-ample at x with respect to S for $t \geq t_0$, it is sufficient to show that $(n_1+n_2)n_0\alpha+e\beta$ is semi-ample at x with respect to S because $et \geq (n_1 + n_2)n_0$. Here

$$\begin{aligned}(n_1 + n_2)n_0\alpha + e\beta &\sim n_2(A, f) + (n_1(A, f) + (D, g)) \\ &\sim n_2(A, f) + (Z, h) \\ &= (n_2A + Z, n_2f + h),\end{aligned}$$

$x \notin \mathrm{Supp}(n_2A+Z)$, and $(n_2f+h)(z) \geq 0$ for all $z \in S$. Thus, $(n_1+n_2)n_0\alpha+e\beta$ is semi-ample at x with respect to S. Hence, we get our claim.

In the same way, we can find a positive number t_1 such that $t\alpha-\beta$ is semi-ample with respect to S for all $t \geq t_1$. Thus, if we set $\epsilon_0 = \min\{1/t_0, 1/t_1\}$, then we have (1).

(2) In the same way as in the proof of (1), we can find a positive number ϵ_0 such that $(f + \epsilon n_0g)(z) \geq 0$ for all $z \in S$ and all real number ϵ with $|\epsilon| \leq \epsilon_0$. Thus we have (2) because $n_0(\alpha + a(\epsilon g)) \sim (A, f + \epsilon n_0g)$. $\square$

Proposition 3.6.3. *We assume that S is compact. Let β be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then the following are equivalent.*

(1) *β is weakly positive at x with respect to S.*

(2) *$\beta+\alpha$ is semi-ample at x with respect to S for any ample $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ at x with respect to S.*

Proof. (1) $\Longrightarrow$ (2): Since β is weakly positive at x with respect to S, there is a sequence of $\{\beta_n\}$ such that $\beta_n \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$, β_n's are semi-ample at x with respect to S, and $\lim_{n\to\infty}\beta_n = \beta$. Take $Z_1, \ldots, Z_{l_1}$, $g_1, \ldots, g_{l_2}$, $\{a_n^1\}_{n=1}^{\infty}, \ldots, \{a_n^{l_1}\}_{n=1}^{\infty}$, and $\{b_n^1\}_{n=1}^{\infty}, \ldots, \{b_n^{l_2}\}_{n=1}^{\infty}$ as in the definition of the limit in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then, by Lemma 3.6.2, there is a positive number ϵ_0 such that $\alpha+\epsilon Z_i$'s are semi-ample at x with respect to S for all rational numbers ϵ with $|\epsilon| \leq \epsilon_0$, and $\alpha + a(\epsilon g_j)$'s are semi-ample at x with respect to S for all real numbers ϵ with $|\epsilon| \leq \epsilon_0$. We choose n such that $(l_1+l_2)|a_n^i| \leq \epsilon_0$ and $(l_1+l_2)|b_n^j| \leq \epsilon_0$ for all i and j. Then,

$$\beta + \alpha = \beta_n + \sum_{i=1}^{l_1} \frac{\alpha + (l_1+l_2)a_n^i Z_i}{l_1+l_2} + \sum_{j=1}^{l_2} \frac{\alpha + a((l_1+l_2)b_n^j g_j)}{l_1+l_2}.$$

Here, $\alpha + (l_1+l_2)a_n^i Z_i$ and $\alpha + a((l_1+l_2)b_n^j g_j)$ are semi-ample x with respect to S. Thus, we get the direction (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1): Let α be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$ such that α is ample at x with respect to S. We set $\beta_n = \beta + (1/n)\alpha$. Then, by our assumption, β_n is semi-ample at x with respect to S. Further, $\beta = \lim_{n\to\infty}\beta_n$. □

3.7. Small sections via generically finite morphisms

Let $g : V \to U$ be a proper and étale morphism of complex manifolds. Let (E,h) be a Hermitian vector bundle on V. Then, a Hermitian metric $g_*(h)$ of $g_*(E)$ is defined by

$$g_*(h)(s,t)(y) = \sum_{x \in g^{-1}(y)} h(s,t)(x)$$

for any $y \in U$ and $s,t \in g_*(E)_y$.

Proposition 3.7.1. *Let X be a scheme such that every connected component of X is a arithmetic variety. Let Y be a regular arithmetic variety, and $g : X \to Y$ a proper and generically finite morphism such that every connected component of X maps surjectively to Y. Let U be a Zariski open set of Y such that g is étale over U. Let S be a subset of $U(\mathbb{C})$ and y a closed point of $U_{\mathbb{Q}}$. Then, we have the following.*

(1) *Let $\phi : E \to Q$ be a homomorphism of coherent $\mathcal{O}_X$-modules such that ϕ is surjective over $g^{-1}(U)$, and E and Q are locally free over $g^{-1}(U)$. Let h_E be a C^∞ Hermitian metric of E over $g^{-1}(U)(\mathbb{C})$, and h_Q the quotient metric of Q induced by h_E. If $(g_*(E), g_*(h_E))$ is generated by small sections at y with respect to S, then $(g_*(Q), g_*(h_Q))$ is generated by small sections at y with respect to S.*

(2) *Let E_1 and E_2 be coherent $\mathcal{O}_X$-modules such that E_1 and E_2 are locally free over $g^{-1}(U)$. Let h_1 and h_2 be C^∞ Hermitian metrics of E_1 and E_2 over $g^{-1}(U)(\mathbb{C})$. If $(g_*(E_1), g_*(h_1))$ and $(g_*(E_2), g_*(h_2))$ are generated by small sections at y with respect to S, then so is $(g_*(E_1 \otimes E_2), g_*(h_1 \otimes h_2))$.*

(3) *Let E be a coherent $\mathcal{O}_X$-module such that E is locally free over $g^{-1}(U)$. Let h_E be a C^∞ Hermitian metric of E over $g^{-1}(U)(\mathbb{C})$. If $(g_*(E), g_*(h_E))$ is generated by small sections at y with respect to S, then $(g_*(\mathrm{Sym}^n(E)), g_*(\mathrm{Sym}^n(h_E)))$ is generated by small sections at y with respect to S. (For the definition of $\mathrm{Sym}^n(h_E)$, see Section 7.1.)*

(4) *Let F be a coherent $\mathcal{O}_Y$-module such that F is locally free over U. Let h_F be a C^∞ Hermitian metric of F over $U(\mathbb{C})$. Since $\det(F)|_U$ is canonically isomorphic to $\det(F|_U)$, $\det(h_F)$ gives rise to a C^∞ Hermitian metric of $\det(F)$ over $U(\mathbb{C})$. If (F, h_F) is generated by small sections at y with respect to S, then so is $(\det(F), \det(h_F))$.*

Proof. (1) By our assumption, $g_*(\phi) : g_*(E) \to g_*(Q)$ is surjective over U. Let $s_1, \ldots, s_l \in H^0(Y, g_*(E)) = H^0(X, E)$ such that $g_*(E)_y$ is generated by $s_1, \ldots, s_l$, and that $g_*(h_E)(s_i, s_i)(z) \le 1$ for all i and $z \in S$. Then, $g_*(Q)_y$ is generated by $g_*(\phi)(s_1), \ldots, g_*(\phi)(s_l)$. Moreover, by the definition of the quotient metric h_Q,

$$
\begin{aligned}
&g_*(h_Q)(g_*(\phi)(s_i), g_*(\phi)(s_i))(z) \\
&\qquad = \sum_{x \in g^{-1}(z)} h_Q(\phi(s_i), \phi(s_i))(x) \le \sum_{x \in g^{-1}(z)} h_E(s_i, s_i)(x) \le 1
\end{aligned}
$$

for all $z \in S$. Hence, $g_*(Q)$ is generated by small sections at y with respect to S.

(2) Since g is étale over U, $\alpha : g_*(E_1) \otimes g_*(E_2) \to g_*(E_1 \otimes E_2)$ is surjective over U. By our assumption, there are $s_1, \ldots, s_l \in H^0(Y, g_*(E_1))$ and $t_1, \ldots, t_m \in H^0(Y, g_*(E_2))$ such that $g_*(E_1)_y$ (resp. $g_*(E_2)_y$) is generated by $s_1, \ldots, s_l$ (resp. $t_1, \ldots, t_m$), and that $g_*(h_1)(s_i, s_i)(z) \le 1$ and $g_*(h_2)(t_j, t_j)(z) \le 1$ for all i, j and $z \in S$. Then, $g_*(E_1 \otimes E_2)_y$ is generated by $\{\alpha(s_i \otimes t_j)\}_{i,j}$. Moreover,

$$
\begin{aligned}
g_*(h_1 \otimes h_2)&(\alpha(s_i \otimes t_j), \alpha(s_i \otimes t_j))(z) \\
&= \sum_{x \in g^{-1}(z)} (h_1 \otimes h_2)(s_i \otimes t_j, s_i \otimes t_j)(x) \\
&= \sum_{x \in g^{-1}(z)} h_1(s_i, s_i)(x) h_2(t_j, t_j)(x) \\
&\le \left(\sum_{x \in g^{-1}(z)} h_1(s_i, s_i)(x) \right) \left(\sum_{x \in g^{-1}(z)} h_2(t_j, t_j)(x) \right) \\
&\le 1
\end{aligned}
$$

for all $z \in S$. Thus, we get (2).

(3) This is a consequence of (1) and (2).

(4) Let r be the rank of F. Since F is generated by small sections at y with respect to S, there are $s_1, \dots, s_r \in H^0(Y, F)$ such that $F \otimes \kappa(y)$ is generated by $s_1, \dots, s_r$ and $h(s_i, s_i)(z) \le 1$ for all i and $z \in S$. Let us consider an exact sequence:

$$0 \to F_{tor} \to F \to F/F_{tor} \to 0.$$

Then, $\det(F) = \det(F/F_{tor}) \otimes \det(F_{tor})$. Noting that $F_{tor} = 0$ on U, let g be a Hermitian metric of $\det(F/F_{tor})$ over $U(\mathbb{C})$ given by $\det(h_F)$. Then, there is a Hermitian metric k of $\det(F_{tor})$ over $U(\mathbb{C})$ such that $(\det(F), \det(h_F)) = (\det(F/F_{tor}), g) \otimes (\det(F_{tor}), k)$ over $U(\mathbb{C})$. If we identify $\det(F_{tor})$ with $\mathcal{O}_Y$ over U, k is nothing more than the canonical metric of $\mathcal{O}_Y$ over $U(\mathbb{C})$.

Let us fix a locally free sheaf P on Y and a surjective homomorphism $P \to F_{tor}$. Let P' be the kernel of $P \to F_{tor}$. Here $\left(\bigwedge^{\operatorname{rk} P'} P'\right)^*$ is an invertible sheaf on Y because Y is regular. Thus we may identify $\det(F_{tor})$ with

$$\bigwedge^{\operatorname{rk} P} P \otimes \left(\bigwedge^{\operatorname{rk} P'} P'\right)^* .$$

Further, a homomorphism $\bigwedge^{\operatorname{rk} P'} P' \to \bigwedge^{\operatorname{rk} P} P$ induced by $P' \hookrightarrow P$ gives rise to a non-zero section t of $\det(F_{tor})$ because

$$\mathcal{H}om\left(\bigwedge^{\operatorname{rk} P'} P', \bigwedge^{\operatorname{rk} P} P\right) = \mathcal{H}om\left(\bigwedge^{\operatorname{rk} P'} P', \mathcal{O}_Y\right) \otimes \bigwedge^{\operatorname{rk} P} P.$$

Here $F_{tor} = 0$ on U. Thus, $\det(F_{tor})$ is canonically isomorphic to $\mathcal{O}_Y$ over U. Since $P' = P$ over U, under the above isomorphism, t goes to the determinant of $P' \xrightarrow{\mathrm{id}} P$, namely $1 \in \mathcal{O}_Y$ over U. Thus, $k(t,t)(z) = 1$ for each $z \in S$.

Let $\overline{s}_i$ be the image of s_i in F/F_{tor}. Then, $\overline{s}_1 \wedge \cdots \wedge \overline{s}_r$ gives rise to a section s of $\det(F/F_{tor})$. Thus, $s \otimes t$ is a section of $\det(F)$. By our construction, $(s \otimes t)(y) \neq 0$. Moreover, using Hadamard's inequality,

$$\begin{aligned}\det(h_F)(s \otimes t, s \otimes t)(z) &= g(s,s)(z) \cdot k(t,t)(z) = \det\left(h(s_i, s_j)(z)\right) \\ &\le h(s_1, s_1)(z) \cdots h(s_r, s_r)(z) \le 1\end{aligned}$$

for each $z \in S$. Thus, we get (4). $\square$

4. Arithmetic Riemann-Roch for generically finite morphisms

4.1. Quillen metric for generically finite morphisms

Before starting Proposition 4.1.1, we recall the tensor product of two matrices, which we will use in the proof. For an $r \times r$ matrix $A = (a_{ij})$ and an $n \times n$ matrix $B = (b_{kl})$, consider the following $rn \times rn$ matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1r}B \\ a_{21}B & a_{22}B & \cdots & a_{2r}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}B & a_{r2}B & \cdots & a_{rr}B \end{pmatrix}.$$

This matrix, denoted by $A \otimes B$, is called *the tensor product of* A *and* B. Then for $r \times r$ matrices A, A' and $n \times n$ matrices B, B', we immediately see

$$\begin{aligned} (A \otimes B)(A' \otimes B') &= AA' \otimes BB', \\ \det(A \otimes B) &= (\det A)^n (\det B)^r. \end{aligned}$$

Let X be a smooth algebraic scheme over $\mathbb{C}$, Y a smooth algebraic variety over $\mathbb{C}$, and $f : X \to Y$ a proper and generically finite morphism. We assume that every connected component of X maps surjectively to Y. Let W be the maximal open set of Y such that f is étale over there. Let (E, h) be a Hermitian vector bundle on X such that on every connected component of X, E has the same rank r.

Proposition 4.1.1. *With notation and assumptions being as above, the Quillen metric* $h_Q^{\overline{E}}$ *on* $\det Rf_*(E)$ *over* W *extends to a generalized metric on* $\det Rf_*(E)$ *over* Y.

Proof. Let n be the degree of f. Since f is étale over W, $f_*(E)$ is a locally free sheaf of rank rn and $R^i f_*(E) = 0$ for $i \geq 1$ over there. Thus

$$\det Rf_*(E)|_W = \bigwedge^{rn} f_*(E)|_W.$$

If $y \in W$ is a complex point and $X_y = \{x_1, x_2, \cdots, x_n\}$ the fiber of f over y, then we have

$$\det Rf_*(E)_y = \det H^0(X_y, E).$$

The Quillen metric on $\det Rf_*(E)$ over W is defined as follows. On $H^0(X_y, E)$ the L^2-metric is defined by the formula:

$$h_{L^2}(s, t) = \sum_{\alpha=1}^{n} h(s, t)(x_\alpha),$$

where $s, t \in H^0(X_y, E)$. This metric naturally induces the L^2-metric on $\det H^0(X_y, E)$. Since X_y is zero-dimensional, there is no need for zeta function regularization to obtain the Quillen metric. Thus the Quillen metric $h_Q^{\overline{E}}$ on $\det Rf_*(E)|_W$ is defined by the family of Hermitian line bundles $\{\det H^0(X_y, E)\}_{y \in W}$ with the induced L^2-metrics pointwisely.

To see that the Quillen metric over W extends to a generalized metric over Y, let $s_1, s_2, \cdots, s_r$ be rational sections of E such that at the generic point of every connected component of X, they form a basis of E. Also let $\omega_1, \omega_2, \cdots, \omega_n$ be rational sections of $f_*(\mathcal{O}_X)$ such that at the generic point they form a basis of $f_*(\mathcal{O}_X)$. Since

$$\det Rf_*(E) = \left(\bigwedge^{rn} (f_*(E)) \right)^{**}. \tag{4.1.1.1}$$

over Y, we can regard $\bigwedge_{ik} s_i\omega_k = s_1\omega_1 \wedge s_1\omega_2 \wedge \cdots \wedge s_1\omega_n \wedge \cdots \wedge s_r\omega_n$ as a non-zero rational section of $\det Rf_*(E)$. Shrinking W, we can find a non-empty Zariski open set W_0 of W such that s_i's and ω_j's has no poles or zeros over $f^{-1}(W_0)$.

To proceed with our argument, we need the following lemma. $\square$

Lemma 4.1.2. *Let L be the total quotient field of X, and K the function field of Y. Then,*

$$\log h_Q^{\overline{E}} \left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k \right) = r \log \left| \det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j)) \right| + f_* \log \det(h(s_i, s_j))$$

over W_0.

Proof. Let $y \in W_0$ be a complex point, and $\{x_1, x_2, \ldots, x_n\}$ the fiber of $f^{-1}(y)$ over y. Then,

$$\begin{aligned}
&\log h_Q^{\overline{E}} \left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k \right)(y) \\
&= \log \det \left(\sum_{\alpha=1}^{n} h(s_i\omega_k, s_j\omega_l)(x_\alpha) \right)_{ij,kl} \\
&= \log \det \left(\sum_{\alpha=1}^{n} \omega_k(x_\alpha) h(s_i, s_j)(x_\alpha) \overline{\omega_l(x_\alpha)} \right)_{ij,kl} \\
&= \log \det \left((I_r \otimes \Omega) \begin{pmatrix} H(x_1) & & 0 \\ & \ddots & \\ 0 & & H(x_n) \end{pmatrix} \overline{{}^t(I_r \otimes \Omega)} \right) \\
&= \log \det \left\{ |\det(\Omega)|^{2r} \prod_{\alpha=1}^{n} \det \left(h(s_i, s_j)(x_\alpha) \right)_{ij} \right\}
\end{aligned}$$

$$= r \log \det |\det(\Omega)|^2 + \sum_{\alpha=1}^{n} \log \det\left(h(s_i, s_j)\right)(x_\alpha),$$

where $\Omega = (\omega_k(x_\alpha))_{k\alpha}$ and $H(x_\alpha) = (h(s_i, s_j)(x_\alpha))_{ij}$. On the other hand, we have

$$\sum_{\alpha=1}^{n} \log \det\left(h(s_i, s_j)\right)(x_\alpha) = \left(f_* \log \det\left(h(s_i, s_j)\right)\right)(y).$$

Moreover, using the following Lemma 4.1.3, we have

$$\begin{aligned} |\det(\Omega)|^2 &= |\det(\Omega^t\Omega)| \\ &= \left|\det\left(\sum_{\alpha=1}^{n} \omega_k(x_\alpha)\omega_l(x_\alpha)\right)_{kl}\right| = |\det\left(\mathrm{Tr}_{L/K}(\omega_k \cdot \omega_l)\right)_{kl}|. \end{aligned}$$

Thus we get the lemma. □

Lemma 4.1.3. *Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a finite étale morphism of regular affine schemes. Let $\mathfrak{m}$ be the maximal ideal of A and $\mathfrak{n}_1, \mathfrak{n}_2, \cdots, \mathfrak{n}_n$ the prime ideals lying over $\mathfrak{m}$. Assume that $\kappa(\mathfrak{m})$ is algebraically closed and hence $\kappa(\mathfrak{n}_i)$ is (naturally) isomorphic to $\kappa(\mathfrak{m})$ for each $1 \le i \le n$. Let b be an element of B and $b(\mathfrak{n}_i)$ the value of b in $\kappa(\mathfrak{n}_i) \cong \kappa(\mathfrak{m})$. Then*

$$\mathrm{Tr}_{B/A}(b)(\mathfrak{m}) = \sum_{i=1}^{n} b(\mathfrak{n}_i)$$

in $\kappa(\mathfrak{m})$, where $\mathrm{Tr}_{B/A}(b)(\mathfrak{m})$ is the value of $\mathrm{Tr}_{B/A}(b)$ in $\kappa(\mathfrak{m})$.

Proof. It is easy to see that every $\mathfrak{n}_i$ is the maximal ideal and that $\mathfrak{m}B = \mathfrak{n}_1\mathfrak{n}_2\cdots\mathfrak{n}_n$. Let $\widehat{A}$ be the completion of A with respect to $\mathfrak{m}$, $\widehat{B}$ the completion of B with respect to $\mathfrak{m}B$, and $\widehat{B_i}$ the completion of B with respect to $\mathfrak{n}_i$ for each $1 \le i \le n$. Then by Chinese remainder theorem, $\widehat{B} = \prod_{i=1}^{n} \widehat{B_i}$ as an $\widehat{A}$-algebra. Note that $\widehat{A}/\mathfrak{m}\widehat{A} = \kappa(\mathfrak{m})$ and $\widehat{B_i}/\mathfrak{n}_i\widehat{B_i} = \kappa(\mathfrak{n}_i)$. Since $\widehat{A} \to \widehat{B_i}$ is étale and $\kappa(\mathfrak{m}) \cong \kappa(\mathfrak{n}_i)$, we have $\widehat{A} \cong \widehat{B_i}$. Let $e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 1) \in \prod_{i=1}^{n} \widehat{B_i} = \widehat{B}$ be a free basis of $\widehat{B}$ over $\widehat{A}$. We put $be_i = b_i e_i$ with $b_i \in \widehat{B_i} \cong \widehat{A}$ for each $1 \le i \le n$. Then $b_i \equiv b(\mathfrak{n}_i) \pmod{\mathfrak{n}_i}$. Now the lemma follows from

$$\mathrm{Tr}_{B/A}(b) = \mathrm{Tr}_{\widehat{B}/\widehat{A}}(b) = \sum_{i=1}^{n} b_i$$

in $\widehat{A}$. □

Let us go back to the proof of Proposition 4.1.1. Since $\det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j))\big|_{W_0}$ extends to a rational function $\det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j))$ on Y,

$$\log\left|\det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j))\right| \in L^1_{\mathrm{loc}}(Y).$$

Moreover, by Proposition 1.2.5, $f_* \log\det(h(s_i, s_j)) \in L^1_{\mathrm{loc}}(Y)$. Thus, by Lemma 4.1.2,

$$\log h_Q^{\overline{E}} \left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k \right)\Bigg|_{W_0}$$

extends to a locally integrable function on Y. Hence by Lemma 3.1.1 the Quillen metric over W extends to a generalized metric over Y. $\square$

Remark 4.1.4. In the above situation, Let W' be a open set of Y such that f is flat and finite over there. Then the Quillen metric extends to a continuous function over W' by the same formula as in (4.1.2)

4.2. Riemann-Roch for generically finite morphisms

In this subsection, we formulate the arithmetic Riemann-Roch theorem for generically finite morphisms.

Theorem 4.2.1. *Let X be a scheme such that every connected component of X is an arithmetic variety. Let Y be a regular arithmetic variety, and $f : X \to Y$ a proper and generically finite morphism such that every connected component of X maps surjectively to Y. Let (E, h) a Hermitian vector bundle on X such that on each connected component of X, E has the same rank r. Then,*

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - r\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$$

and

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - r\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) = f_*\left(\widehat{c}_1(E, h)\right)$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$, where $h_Q^{\overline{E}}$ and $h_Q^{\overline{\mathcal{O}}_X}$ are the Quillen metric of $\det Rf_*(E)$ *and* $\det Rf_*(\mathcal{O}_X)$ *respectively.*

Proof. Let $X = \coprod_{\alpha\in A} X_\alpha$ be the decomposition into connected components of X. Since f is proper, A is a finite set. We set $f_\alpha = f|_{X_\alpha}$ and $(E_\alpha, h_\alpha) = (E, h)|_{X_\alpha}$. Then

$$Rf_*(E) = \bigoplus_{\alpha\in A} R(f_\alpha)_*(E_\alpha),$$
$$Rf_*(\mathcal{O}_X) = \bigoplus_{\alpha\in A} R(f_\alpha)_*(\mathcal{O}_{X_\alpha}),$$
$$\widehat{c}_1(E, h) = \sum_{\alpha\in A} \widehat{c}_1(E_\alpha, h_\alpha).$$

Hence we have the following:

$$\begin{cases} \widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) = \sum_{\alpha\in A} \widehat{c}_1\left(\det R(f_\alpha)_*(E_\alpha), h_Q^{\overline{E}_\alpha}\right), \\ \widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) = \sum_{\alpha\in A} \widehat{c}_1\left(\det R(f_\alpha)_*(\mathcal{O}_{X_\alpha}), h_Q^{\overline{\mathcal{O}}_{X_\alpha}}\right), \\ f_*\left(\widehat{c}_1(E,h)\right) = \sum_{\alpha\in A} f_*\left(\widehat{c}_1(E_\alpha, h_\alpha)\right). \end{cases}$$

Thus, we may assume that X is connected, i.e., X is an arithmetic variety.

Let $K = K(Y)$ and $L = K(X)$ be the function fields of Y and X respectively. Let n be the degree of f and $\omega_1, \omega_2, \cdots, \omega_n$ rational functions on X such that at the generic point they form a basis of K-vector space L. Further, let $s_1, s_2, \ldots, s_r$ be rational sections of E such that at the generic point they form a basis of L-vector space E_L. Then $s_1\omega_1 \wedge s_1\omega_2 \wedge \cdots \wedge s_1\omega_n \wedge \cdots \wedge s_r\omega_n$, $s_1 \wedge \cdots \wedge s_r$ and $\omega_1 \wedge \cdots \wedge \omega_n$ are non-zero rational sections of $\det f_*(E)$, $\det(E)$ and $\det f_*(\mathcal{O}_X)$ respectively. Here we shall prove the following equality in $\widehat{Z}^1_D(Y)$:

$$\begin{aligned} (4.2.1.1) \quad & \left(\operatorname{div}\left(\bigwedge_{ik} s_i\omega_k\right), \left[-\log h_Q^{\overline{E}}\left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k\right)\right]\right) \\ & \quad - r\left(\operatorname{div}\left(\bigwedge_k \omega_k\right), \left[-\log h_Q^{\overline{\mathcal{O}}_X}\left(\bigwedge_k \omega_k, \bigwedge_k \omega_k\right)\right]\right) \\ & \qquad = f_*\left(\operatorname{div}\left(\bigwedge_i s_i\right), \left[-\log\det h\left(\bigwedge_i s_i, \bigwedge_i s_i\right)\right]\right), \end{aligned}$$

where $\bigwedge_{ik} s_i\omega_k = s_1\omega_1 \wedge s_1\omega_2 \wedge \cdots \wedge s_1\omega_n \wedge \cdots \wedge s_r\omega_n$, $\bigwedge_k \omega_k = \omega_1 \wedge \cdots \wedge \omega_n$ and $\bigwedge_i s_i = s_1 \wedge \cdots \wedge s_r$.

First we shall show the equality of divisors. Let Y_0 be the maximal Zariski open set of X such that f is flat over Y_0. Then, $\operatorname{codim}_Y(Y \setminus Y_0) \geq 2$ by [13, III, Proposition 9.7]. Since f is generically finite, f is in fact finite over Y_0. Then $Z^1(Y) = Z^1(Y_0)$ and thus it suffices to prove the equality of divisors over Y_0. Since it suffices to prove it locally, let $U = \operatorname{Spec}(A)$ be an affine open set of Y_0 and $f^{-1}(U) = \operatorname{Spec}(B)$ the open set of $X_0 = f^{-1}(Y_0)$. Shrinking U if necessary, we may assume that B is a free A-module of rank n and that E is a free B-module of rank r. Let $d_1, d_2, \cdots, d_n$ be a basis of B over A, and $e_1, e_2, \cdots, e_r$ be a basis of E over B. Note that K and L are the quotient fields of A and B respectively. In the following we freely identify a rational function (or section) by the corresponding element at the generic point. In this sense, we set

$$\omega_k = \sum_{l=1}^{n} a^{kl} d_l \quad (k = 1, 2, \cdots, n)$$

$$s_i = \sum_{j=1}^{r} \sigma_{ij} e_j \quad (i = 1, 2, \cdots, r),$$

where $a^{kl} \in K\,(1 \le k, l \le n)$ and $\sigma_{ij} \in L\,(1 \le i, j \le r)$.

For each $\sigma_{ij}(1 \le i, j \le r)$, let $T_{\sigma_{ij}} : L \to L$ be multiplication by σ_{ij}. With respect to a basis $\omega_1, \omega_2, \cdots, \omega_n$ of L over K, $T_{\sigma_{ij}}$ gives rise to the matrix $(c_{ij}^{kl})_{1 \le k, l \le n} \in M_n(K)$ defined by

$$\sigma_{ij} \omega_k = \sum_{l=1}^{n} c_{ij}^{kl} \omega_l \quad (k = 1, 2, \cdots, n).$$

We also denote this matrix by $T_{\sigma_{ij}}$. Then,

$$\begin{aligned}
\bigwedge_{ik} s_i \omega_k &= \bigwedge_{ik} \left(\sum_{j=1}^{r} \sigma_{ij} e_j \right) \omega_k = \bigwedge_{ik} \left(\sum_{j=1}^{r} \sum_{l=1}^{n} c_{ij}^{kl} \right) e_j \omega_l \\
&= \det(c_{ij}^{kl})_{ik,jl} \bigwedge_{ik} e_i \omega_k = \det(c_{ij}^{kl})_{ik,jl} \bigwedge_{ik} e_i \left(\sum_{l=1}^{n} a^{kl} d_l \right) \\
&= \det(c_{ij}^{kl})_{ik,jl} \bigwedge_{ik} \left(\sum_{j=1}^{r} \delta_{ij} a^{kl} \right) e_j \omega_l \\
&= \det(c_{ij}^{kl})_{ik,jl} \det(\delta_{ij} a^{kl})_{ik,jl} \bigwedge_{ik} e_i d_l.
\end{aligned}$$

On the other hand, since the matrices $T_{\sigma_{ij}}$ and $T_{\sigma_{i'j'}}$ commute with each other, we have

$$\begin{aligned}
\det(c_{ij}^{kl})_{ik,jl} &= \det \begin{pmatrix} T_{\sigma_{11}} & T_{\sigma_{12}} & \cdots & T_{\sigma_{1r}} \\ T_{\sigma_{21}} & T_{\sigma_{22}} & \cdots & T_{\sigma_{2r}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\sigma_{r1}} & T_{\sigma_{r2}} & \cdots & T_{\sigma_{rr}} \end{pmatrix} \\
&= \det \left(\sum_{\tau \in \mathfrak{S}_r} \operatorname{sign}(\tau) T_{\sigma_{1\tau(1)}} \cdot \cdots \cdot T_{\sigma_{r\tau(r)}} \right) \\
&= \det(T_{\det(\sigma_{ij})_{ij}}) \\
&= \operatorname{Norm}_{L/K}(\det(\sigma_{ij})_{ij}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\det(\delta_{ij} a^{kl})_{ik,jl} &= \det(I_r \otimes (a^{kl})_{kl}) \\
&= (\det(a^{kl})_{kl})^r.
\end{aligned}$$

From the above three equalities, $\operatorname{div}\left(\bigwedge_{ik} s_i\omega_k\right)$ is given by the rational function

$$\mathrm{Norm}_{L/K}(\det(\sigma_{ij})_{ij})(\det(a^{kl})_{kl})^r.$$

Further

$$\bigwedge_i s_i = (\det(\sigma_{ij})_{ij}) \bigwedge_i e_k \quad \text{and} \quad \bigwedge_k \omega_k = (\det(a^{kl})_{kl}) \bigwedge_k d_k.$$

Hence we have

$$\operatorname{div}\left(\bigwedge_{ik} s_i\omega_k\right) - r\left(\operatorname{div}\left(\bigwedge_k \omega_k\right)\right) = f_*\left(\operatorname{div}\left(\bigwedge_i s_i\right)\right).$$

Next we shall show the equality of currents. Since all the currents in the equality come from locally integrable functions by Propositions 1.2.5 and 4.1.1, it suffices to show the equality over a non-empty Zariski open set of every connected component of $Y(\mathbb{C})$. So let W_0 be a non-empty Zariski open set of a connected component of $Y(\mathbb{C})$ such that $f_{\mathbb{C}}$ is étale and that $s_i\,(1 \le i \le r)$ or $\omega_k\,(1 \le k \le n)$ have no poles or zeroes over there. Then over W_0 all these currents are defined by C^∞ functions. Let $y \in Y(\mathbb{C})$ be a complex point and $x_1, x_2, \cdots, x_n$ be the fiber $f_{\mathbb{C}}^{-1}(y)$ over y. From the proof of Lemma 4.1.2, as C^∞ functions around y,

$$\begin{aligned} &-\log h_Q^{\overline{E}}\left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k\right)(y) \\ &\qquad = -\log\det\left\{|\det(\Omega)|^{2r}\prod_{\alpha=1}^{n}\det\left(h(s_i,s_j)(x_\alpha)\right)_{ij}\right\}, \end{aligned}$$

where $\Omega = (\omega_k(x_\alpha))_{k\alpha}$ and $H(x_\alpha) = (h(s_i,s_j)(x_\alpha))_{ij}$. Also,

$$-\log h_Q^{\overline{\mathcal{O}}_X}\left(\bigwedge_k \omega_k, \bigwedge_k \omega_k\right)(y) = -\log\det|\det(\Omega)|^2.$$

On the other hand, by the definition of the push-forward f_*,

$$\begin{aligned} f_*\left(-\log\det h\left(\bigwedge_i s_i, \bigwedge_i s_i\right)\right)(y) &= \sum_{\alpha=1}^{n} -\log\det h\left(\bigwedge_i s_i, \bigwedge_i s_i\right)(x_\alpha) \\ &= \sum_{\alpha=1}^{n} -\log\det\left(h(s_i,s_j)(x_\alpha)\right)_{ij}. \end{aligned}$$

Hence we have the desired equality of currents by the above three equalities.

Thus we have showed the equality (4.2.1.1). Since the right hand side belongs in fact to $\widehat{Z}^1_{L^1}(Y)$, the left hand side must also belong to $\widehat{Z}^1_{L^1}(Y)$, and thus we have the equality in $\widehat{Z}^1_{L^1}(Y)$. $\square$

5. Arithmetic Riemann-Roch for stable curves

5.1. Bismut-Bost formula

Let X be a smooth algebraic variety over $\mathbb{C}$, L a line bundle on X, and h a generalized metric of L over X. Let s be a rational section of L. Then, by the definition of the generalized metric h, $-\log h(s,s)$ gives rise to a current $-[\log h(s,s)]$. Moreover, it is easy to see that a current

$$dd^c(-[\log h(s,s)]) + \delta_{\mathrm{div}(s)}$$

does not depend on the choice of s. Thus, we define $c_1(L,h)$ to be

$$c_1(L,h) = dd^c(-[\log h(s,s)]) + \delta_{\mathrm{div}(s)}.$$

Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties $\mathbb{C}$ such that every fiber of f is a reduced and connected curve with only ordinary double singularities. We set $\Sigma = \{x \in X \mid f \text{ is not smooth at } x.\}$ and $\Delta = f_*(\Sigma)$. Let $|\Delta|$ be the support of Δ. We fix a Hermitian metric of $\omega_{X/Y}$. Then, in [1], Bismut and Bost proved the following.

Theorem 5.1.1. *Let $\overline{E} = (E,h)$ be a Hermitian vector bundle on X. Then, the Quillen metric $h_Q^{\overline{E}}$ of $\det Rf_*(E)$ on $Y \setminus |\Delta|$ gives rise to a generalized metric of $\det Rf_*(E)$ on Y. Moreover,*

$$c_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) = -f_*\left[\mathrm{td}(\overline{\omega}_{X/Y}{}^{-1})\,\mathrm{ch}(\overline{E})\right]^{(2,2)} - \frac{\mathrm{rk}\,E}{12}\delta_{\Delta}.$$

5.2. Riemann-Roch for stable curves

In this subsection, we prove the arithmetic Riemann-Roch theorem for stable curves.

Theorem 5.2.1. *Let $f : X \to Y$ be a projective morphism of regular arithmetic varieties such that every fiber of $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is a reduced and connected curve with only ordinary double singularities. We fix a Hermitian metric of the dualizing sheaf $\omega_{X/Y}$. Let $\overline{E} = (E,h)$ be a Hermitian vector bundle on X. Then,*

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$$

and

$$\begin{aligned}\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) &- \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) \\ &= f_*\left(\frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E})\right)\end{aligned}$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$, where $h_Q^{\overline{E}}$ and $h_Q^{\overline{\mathcal{O}}_X}$ are the Quillen metric of $\det Rf_*(E)$ and $\det Rf_*(\mathcal{O}_X)$ respectively.

Proof. We prove the theorem in two steps.

Step 1. First, we assume that $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth. In this case, by [11],

$$\begin{aligned}&\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right)\\&\qquad = f_*\left(\widehat{\mathrm{ch}}(E,h)\widehat{\mathrm{td}}(Tf,h_f) - a(\mathrm{ch}(E_{\mathbb{C}})\,\mathrm{td}(Tf_{\mathbb{C}})R(Tf_{\mathbb{C}}))\right)^{(1)}.\end{aligned}$$

in $\widehat{\mathrm{CH}}^1(Y)_{\mathbb{Q}}$. Since

$$\widehat{\mathrm{ch}}(\overline{E}) = \mathrm{rk}(E) + \widehat{c}_1(\overline{E}) + \left(\frac{1}{2}\widehat{c}_1(\overline{E})^2 - \widehat{c}_2(\overline{E})\right) + (\text{higher terms})$$

and

$$\widehat{\mathrm{td}}(Tf,h_f) = 1 - \frac{1}{2}\widehat{c}_1(\overline{\omega}_{X/Y}) + \widehat{\mathrm{td}}_2(Tf,h_f) + (\text{higher terms}),$$

we have

$$\begin{aligned}&\left(\widehat{\mathrm{ch}}(E,h)\widehat{\mathrm{td}}(Tf,h_f)\right)^{(2)}\\&\qquad = \frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E}) + \mathrm{rk}(E)\widehat{\mathrm{td}}_2(Tf,h_f).\end{aligned}$$

On the other hand, since the power series $R(x)$ has no constant term, the $(1,1)$ part of

$$\mathrm{ch}(E_{\mathbb{C}})\,\mathrm{td}(Tf_{\mathbb{C}})R(Tf_{\mathbb{C}})$$

is $\mathrm{rk}(E)R_1(Tf_{\mathbb{C}})$, where $R_1(Tf_{\mathbb{C}})$ is the $(1,1)$ part of $R(Tf_{\mathbb{C}})$. Therefore, we obtain

$$\begin{aligned}(5.2.1.1)\quad \widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) =& f_*\left(\frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E})\right)\\&+ \mathrm{rk}(E)f_*\left(\widehat{\mathrm{td}}_2(Tf,h_f) - a(R_1(Tf_{\mathbb{C}}))\right).\end{aligned}$$

Applying (5.2.1.1) to the case $(E,h) = (\mathcal{O}_X, h_{can})$, we have

$$(5.2.1.2)\qquad \widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) = f_*\left(\widehat{\mathrm{td}}_2(Tf,h_f) - a(R_1(Tf_{\mathbb{C}}))\right).$$

Thus, combining (5.2.1.1) and (5.2.1.2), we have our formula in the case where $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth.

Step 2. Next, we consider the general case. The first assertion is a consequence of Theorem 5.1.1 because using Theorem 5.1.1,

$$\begin{aligned}&c_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)c_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right)\\&\qquad = -f_*\left[\mathrm{td}(\overline{\omega}_{X/Y}{}^{-1})\,\mathrm{ch}(\overline{E})\right]^{(2,2)} + \mathrm{rk}(E)f_*\left[\mathrm{td}(\overline{\omega}_{X/Y}{}^{-1})\,\mathrm{ch}(\overline{\mathcal{O}}_X)\right]^{(2,2)}\end{aligned}$$

belongs to $L^1_{\mathrm{loc}}(\Omega^{1,1}_{Y(\mathbb{C})})$ by Proposition 1.2.5. The second assertion is a consequence of the useful Lemma 2.5.1. In fact, both sides of the second assertion are arithmetic L^1-cycles on Y by the first assertion and the Proposition 2.2.2: If we take $\Delta = \{y \in Y_{\mathbb{Q}} \mid f_{\mathbb{Q}} \text{ is not smooth over } y\}$ and define $\overline{\Delta}$ to be the closure of Δ in Y, then the compliment $U = Y \setminus \overline{\Delta}$ contains no irreducible components of fibers of $Y \to \operatorname{Spec}(\mathbb{Z})$ and $f_{\mathbb{C}}$ is smooth over $U(\mathbb{C})$: The arithmetical linear equivalence of both sides restricted to U is a consequence of Step 1. Thus by Lemma 2.5.1, we also have our formula in the general case. $\square$

6. Asymptotic behavior of analytic torsion

Let M be a compact Kähler manifold of dimension d, $\overline{E} = (E, h_E)$ a flat vector bundle of rank r on M with a flat metric h_E, and $\overline{A} = (A, h_A)$ a Hermitian vector bundle on M. For $0 \le q \le d$, let $\Delta_{q,n}$ be the Laplacian on $A^{0,q}\left(\operatorname{Sym}^n(\overline{E}) \otimes \overline{A}\right)$ and $\Delta'_{q,n}$ the restriction of $\Delta_{q,n}$ to $\operatorname{Image}\partial \oplus \operatorname{Image}\overline{\partial}$. Let $\sigma(\Delta'_{q,n}) = \{0 < \lambda_1 \le \lambda_2 \le \cdots\}$ be the sequence of eigenvalues of $\Delta'_{q,n}$. Here we count each eigenvalue up to its multiplicity. Then, the associated zeta function $\zeta_{q,n}(s)$ is given by

$$\zeta_{q,n}(s) = \operatorname{Tr}\left[(\Delta'_{q,n})^{-s}\right] = \sum_{i=1}^{\infty} \lambda_i^{-s}.$$

It is well known that $\zeta_{q,n}(s)$ converges absolutely for $\Re(s) > d$ and that it has a meromorphic continuation to the whole complex plane without pole at $s = 0$. The analytic torsion $T\left(\operatorname{Sym}^n(\overline{E}) \otimes \overline{A}\right)$ is defined by

$$T\left(\operatorname{Sym}^n(\overline{E}) \otimes \overline{A}\right) = \sum_{q=0}^{d} (-1)^q q \zeta'_{q,n}(0).$$

In the following we closely follow [26, Section 2].

The Theta function associated with $\sigma(\Delta'_{q,n})$ is defined by

$$\Theta_{q,n}(t) = \operatorname{Tr}\left[\exp(-t\Delta'_{q,n})\right] = \sum_{i=1}^{\infty} e^{-\lambda_i t}.$$

By taking Mellin transformation, we have, for $\Re(s) > d$,

$$\zeta_{q,n}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \Theta_{q,n}(t) t^s \frac{dt}{t}.$$

We put

$$\tilde{\zeta}_{q,n}(s) = \frac{1}{\operatorname{rk}(\operatorname{Sym}^n(E))} n^{-d} \frac{1}{\Gamma(s)} \int_0^{\infty} \Theta_{q,n}\left(\frac{t}{n}\right) t^s \frac{dt}{t}.$$

Then we have

$$\frac{1}{\operatorname{rk}(\operatorname{Sym}^n(E))} n^{-d} \zeta_{q,n}(s) = n^{-s} \tilde{\zeta}_{q,n}(s)$$

and thus

$$\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \zeta'_{q,n}(0) = -(\log n)\tilde{\zeta}_{q,n}(0) + \zeta'_{q,n}(0) \tag{6.1}$$

Bismut and Vasserot [3, (14), (19)] showed that $\Theta_{q,n}(t)$ has the following properties (note that these parts of [3] do not depend on the assumption of positivity of a line bundle, as indicated in Vojta [26, Proposition 2.7.3]):

(a) For every $k \in \mathbb{N}$, $0 \le q \le d$ and $n \in \mathbb{N}$, there are real numbers $a^j_{q,n}$ $(-d \le j \le k)$ such that

$$\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \Theta_{q,n}\left(\frac{t}{n}\right) = \sum_{j=-d}^{k} a^j_{q,n} t^j + o(t^k)$$

as $t \downarrow 0$, with $o(t^k)$ uniform with respect to $n \in \mathbb{N}$.

(b) For every $0 \le q \le d$ and $j \ge -d$, there are real numbers a^j_q such that

$$a^j_{q,n} = a^j_q + O\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$.

Also by (b), we can replace the $o(t^k)$ in (a) by $O(t^{k+1})$ and still have the uniformity statement. Thus we can write, for every $k \in \mathbb{N}$,

$$\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \Theta_{q,n}\left(\frac{t}{n}\right) = \sum_{j=-d}^{k} a^k_{q,n} t^j + \rho^k_{q,n}(t)$$

with $\rho^k_{q,n}(t) = o(t^{k+1})$. Then

$$\begin{aligned}
\tilde{\zeta}_{q,n}(s) &= \frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \frac{1}{\Gamma(s)} \int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right) t^s \frac{dt}{t} \\
&\quad + \frac{a^j_{q,n}}{\Gamma(s)} \int_0^1 t^{j+s-1} dt + \sum_{j=-d}^{k} \frac{1}{\Gamma(s)} \int_0^1 \rho^k_{q,n}(t) dt \\
&= \frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \frac{1}{\Gamma(s)} \int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right) t^s \frac{dt}{t} \\
&\quad + \sum_{j=-d}^{k} \frac{a^j_{q,n}}{\Gamma(s)(j+s)} + \frac{1}{\Gamma(s)} \int_0^1 \rho^k_{q,n}(t) t^s \frac{dt}{t}.
\end{aligned}$$

In the last expression, the first integral is holomorphic for all $s \in \mathbb{C}$, while the second integral is holomorphic for $\Re(s) > -k-1$; the middle term is a meromorphic function in the whole complex plane.

Putting $k = 0$ and $s = 0$ in the above equation, we have

$$\tilde{\zeta}_{q,n}(0) = a^0_{q,n}. \tag{6.2}$$

Moreover, by differentiating the above equation when $k = 0$, we have

$$\text{(6.3)} \quad \tilde{\zeta}'_{q,n}(0) = \frac{1}{\operatorname{rk}(\operatorname{Sym}^n(E))} n^{-d} \int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right)\frac{dt}{t} + \sum_{j=-d}^{-1} \frac{a^j_{q,n}}{j} - a^0_{q,n}\Gamma'(1) + \frac{1}{\Gamma(s)}\int_0^1 \rho^0_{q,n}(t)\frac{dt}{t}.$$

We have now the following Proposition.

Proposition 6.4. *There exists a constant c such that for all $n \in \mathbb{N}$,*

$$\zeta'_{q,n}(0) \geq -cn^{d+r-1}\log n$$

Proof. By (6.1), (6.2) and (6.3), we have

$$\begin{aligned}\zeta'_{q,n}(0) = &-\operatorname{rk}(\operatorname{Sym}^n(E))n^d(\log n)a^0_{q,n} \\ &+ \operatorname{rk}(\operatorname{Sym}^n(E))n^d\left(\frac{1}{\operatorname{rk}(\operatorname{Sym}^n(E))}n^{-d}\int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right)\frac{dt}{t}\right. \\ &\left. + \sum_{j=-d}^{-1}\frac{a^j_{q,n}}{j} - a^0_{q,n}\Gamma'(1) + \frac{1}{\Gamma(s)}\int_0^1 \rho^0_{q,n}(t)\frac{dt}{t}\right)\end{aligned}$$

In the first term of the right hand side, $a^0_{q,n}$ is bounded with respect to n by (b). In the second term of the right hand side, the first integral is non-negative; the sum of $a^j_{q,n}$'s is bounded with respect to n by (b); the term $-a^0_{q,n}\Gamma'(1)$ is also bounded with respect to n by (b); the second integral is also bounded with respect to n, for $\rho^0_{q,n}(t) = O(t)$ uniformly with respect to n. Moreover,

$$\operatorname{rk}(\operatorname{Sym}^n(E)) = \binom{n+r-1}{r-1} = O(n^{r-1})$$

as $n \to \infty$. Thus, there is a constant c such that for all $n \in \mathbb{N}$,

$$\zeta'_{q,n}(0) \geq -cn^{d+r-1}\log n. \qquad \square$$

In the following sections, we only need the case of $d = 1$, namely where M is a compact Riemann surface. In this case, the above Proposition 6.4 gives an asymptotic upper bound of analytic torsion.

Corollary 6.5. *Let C be a compact Riemann surface, $\overline{E} = (E, h_E)$ a flat vector bundle of rank r on C with a flat metric h, and $\overline{A} = (A, h_A)$ a Hermitian vector bundle on C. Then, there is a constant c such that for all $n \in \mathbb{N}$,*

$$T\left(\operatorname{Sym}^n(\overline{E}) \otimes \overline{A}\right) \leq cn^r \log n.$$

Proof. Since $\dim C = 1$

$$T\left(\mathrm{Sym}^n(\overline{E}) \otimes \overline{A}\right) = -\zeta'_{1,n}(0).$$

Now the corollary follows from Proposition 6.4. □

7. Formulae for arithmetic Chern classes

7.1. Arithmetic Chern classes of symmetric powers

Let M be a complex manifold and (E, h) a Hermitian vector bundle on M. Since $E^{\otimes n}$ has the natural Hermitian metric $h^{\otimes n}$, we can define a Hermitian metric $\mathrm{Sym}^n(h)$ of $\mathrm{Sym}^n(E)$ to be the quotient metric of $E^{\otimes n}$ in terms of the natural surjective homomorphism $E^{\otimes n} \to \mathrm{Sym}^n(E)$. We denote $(\mathrm{Sym}^n(E), \mathrm{Sym}^n(h))$ by $\mathrm{Sym}^n(E, h)$. If $x \in M$ and $\{e_1, \dots, e_r\}$ is an orthonormal basis of E_x with respect to h_x, then it is easy to see that

$$\begin{aligned}(\mathrm{Sym}^n(h))_x &\left(e_1^{\alpha_1} \cdots e_r^{\alpha_r}, e_1^{\beta_1} \cdots e_r^{\beta_r}\right) \\ &= \begin{cases} \dfrac{\alpha_1! \cdots \alpha_r!}{n!} & \text{if } (\alpha_1, \dots, \alpha_r) = (\beta_1, \dots, \beta_r), \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Then we have the following proposition.

Proposition 7.1.1. *Let X be an arithmetic variety and $\overline{E} = (E, h)$ a Hermitian vector bundle of rank r on X. Then, we have the following.*

(1) $$\widehat{c}_1\left(\mathrm{Sym}^n(\overline{E})\right) = \frac{n}{r}\binom{n+r-1}{r-1}\widehat{c}_1(\overline{E}) + a\left(\sum_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\dots,\alpha_r\geq 0}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)\right).$$

(2) *If X is regular, then*

$$\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(\overline{E})\right) = \binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(\overline{E}) + \frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(\overline{E})^2 + a\left(\frac{n}{r}\sum_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\dots,\alpha_r\geq 0}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right) c_1(\overline{E})\right).$$

Proof. In [24], C. Soulé gives similar formulae in implicit forms. We follow his idea to calculate them.

(1) First of all, we fix notation. We set

$$S_{r,n} = \{(\alpha_1, \dots, \alpha_r) \in (\mathbb{Z}_+)^r \mid \alpha_1 + \cdots + \alpha_r = n\},$$

where $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x \geq 0\}$. For $I = (\alpha_1, \dots, \alpha_r) \in S_{r,n}$ and rational sections $s_1, \dots, s_r$ of E, we denote $s_1^{\alpha_1} \cdots s_r^{\alpha_r}$ by s^I and $\alpha_1! \cdots \alpha_r!$ by $I!$.

Let $s_1, \dots, s_r$ be independent rational sections of E. Then, $\{s^I\}_{I \in S_{r,n}}$ forms independent rational sections of $\mathrm{Sym}^n(E)$. First, let us see that

$$\operatorname{div}\left(\bigwedge_{I \in S_{r,n}} s^I\right) = \frac{n}{r}\binom{n+r-1}{r-1} \operatorname{div}(s_1 \wedge \cdots \wedge s_r). \tag{7.1.1.1}$$

This is a local question. So let $x \in X$ and $\{\omega_1, \dots, \omega_r\}$ be a local basis of E around x. We set $s_i = \sum_{j=1}^r a_{ij}\omega_j$. Then, $s_1 \wedge \cdots \wedge s_r = \det(a_{ij})\omega_1 \wedge \cdots \wedge \omega_r$. Let K be a rational function field of X. Since the characteristic of K is zero, any 1-dimensional representation of $\mathrm{GL}_r(K)$ is a power of the determinant. Thus, there is an integer N with

$$\bigwedge_{I \in S_{r,n}} s^I = \det(a_{ij})^N \bigwedge_{I \in S_{r,n}} \omega^I.$$

Here, by an easy calculation, we can see that

$$N = \frac{n}{r}\binom{n+r-1}{r-1}.$$

Thus, we get (7.1.1.1).

Next, let us see that

$$\begin{aligned} &-\log\det\left(\mathrm{Sym}^n(h)(s^I, s^J)\right)_{I,J \in S_{r,n}} \\ &\qquad = -\frac{n}{r}\binom{n+r-1}{r-1} \log\det(h(s_i, s_j))_{i,j} + \sum_{I \in S_{r,n}} \log\left(\frac{n!}{I!}\right). \end{aligned} \tag{7.1.1.2}$$

Let $x \in X(\mathbb{C})$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of $E \otimes \kappa(x)$. We set $s_i = \sum_{i=1}^r b_{ij}e_j$. Moreover, we set $s^I = \sum_{J \in S_{r,n}} b_{IJ}e^J$. Then, in the same way as before, $\det(b_{IJ}) = \det(b_{ij})^N$. Further, since

$$\mathrm{Sym}^n(h)(s^I, s^J) = \sum_{I', J' \in S_{r,n}} b_{II'}\, \mathrm{Sym}^n(h)(e^{I'}, e^{J'})\overline{b_{J'J}},$$

we have

$$\begin{aligned} \det\left(\mathrm{Sym}^n(h)(s^I, s^J)\right)_{I,J \in S_{r,n}} &= |\det(b_{IJ})|^2 \det\left(\mathrm{Sym}^n(h)(e^I, e^J)\right)_{I,J \in S_{r,n}} \\ &= |\det(b_{ij})|^{2N} \prod_{I \in S_{r,n}} \frac{I!}{n!}. \end{aligned}$$

Thus, we get (7.1.1.2). Therefore, combining (7.1.1.1) and (7.1.1.2), we obtain (1).

(2) First, we recall an elementary fact. Let $\Phi \in \mathbb{R}[X_1, \dots, X_r]$ be a symmetric homogeneous polynomial, and $M_r(\mathbb{C})$ the algebra of complex $r \times r$ matrices. Then, there is a unique polynomial map $\underline{\Phi} : M_r(\mathbb{C}) \to \mathbb{C}$ such that $\underline{\Phi}$ is invariant under conjugation by $\mathrm{GL}_r(\mathbb{C})$ and its value on a diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_r)$ is equal to $\Phi(\lambda_1, \dots, \lambda_r)$.

Let us consider the natural homomorphism

$$\rho_{r,n} : \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^r) \to \mathrm{Aut}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r))$$

as complex Lie groups, which induces a homomorphism

$$\gamma_{r,n} = d(\rho_{r,n})_{\mathrm{id}} : \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r) \to \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r))$$

as complex Lie algebras. Let $\{e_1, \dots, e_r\}$ be the standard basis of $\mathbb{C}^r$. Then, $\{e_I\}_{I \in S_{r,n}}$ forms a basis of $\mathrm{Sym}^n(\mathbb{C}^r)$, where $e_I = e_1^{\alpha_1} \cdots e_r^{\alpha_r}$ for $I = (\alpha_1, \dots, \alpha_r)$. Let us consider the symmetric polynomial

$$\mathrm{ch}_2^{r,n} = \frac{1}{2} \sum_{I \in S_{r,n}} X_I^2$$

in $\mathbb{R}[X_I]_{I \in S_{r,n}}$. Then, by the previous remark, using the basis $\{e_I\}_{I \in S_{r,n}}$, we have a polynomial map

$$\underline{\mathrm{ch}_2^{r,n}} : \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r)) \to \mathbb{C}$$

such that $\underline{\mathrm{ch}_2^{r,n}}$ is invariant under conjugation by $\mathrm{Aut}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r))$ and

$$\underline{\mathrm{ch}_2^{r,n}} \left(\mathrm{diag}(\lambda_I)_{I \in S_{r,n}}\right) = \mathrm{ch}_2^{r,n}(\dots, \lambda_I, \dots).$$

Here we consider a polynomial map given by

$$\theta_{r,n} : \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r) \xrightarrow{\gamma_{r,n}} \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r)) \xrightarrow{\underline{\mathrm{ch}_2^{r,n}}} \mathbb{C}.$$

Since $\gamma_{r,n}(PAP^{-1}) = \rho_{r,n}(P)\gamma_{r,n}(A)\rho_{r,n}(P)^{-1}$ for all $A \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r)$ and $P \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^r)$, $\theta_{r,n}$ is invariant under conjugation by $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^r)$. Let us calculate

$$\theta_{r,n}(\mathrm{diag}(\lambda_1, \dots, \lambda_r)).$$

First of all,

$$\gamma_{r,n}(\mathrm{diag}(\lambda_1, \dots, \lambda_r)) = \mathrm{diag}\left(\dots, (\alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r), \dots\right)_{(\alpha_1, \dots, \alpha_r) \in S_{r,n}}.$$

Thus,

$$\theta_{r,n}(\mathrm{diag}(\lambda_1, \dots, \lambda_r)) = \frac{1}{2} \sum_{(\alpha_1, \dots, \alpha_r) \in S_{r,n}} (\alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r)^2.$$

On the other hand, by easy calculations, we can see that

$$\sum_{(\alpha_1,\dots,\alpha_r)\in S_{r,n}} (\alpha_1\lambda_1+\cdots+\alpha_r\lambda_r)^2 = \binom{n+r}{r+1}\left(\lambda_1^2+\cdots+\lambda_r^2\right)+\binom{n+r-1}{r+1}(\lambda_1+\cdots+\lambda_r)^2 .$$

Therefore, we get

$$\theta_{r,n}(\mathrm{diag}(\lambda_1,\dots,\lambda_r)) = \frac{1}{2}\binom{n+r}{r+1}\left(\lambda_1^2+\cdots+\lambda_r^2\right)+\frac{1}{2}\binom{n+r-1}{r+1}(\lambda_1+\cdots+\lambda_r)^2 .$$

Hence,

$$\theta_{r,n} = \binom{n+r}{r+1}\underline{\mathrm{ch}_2}+\frac{1}{2}\binom{n+r-1}{r+1}\underline{(c_1)^2}, \tag{7.1.1.3}$$

where $\mathrm{ch}_2(X_1,\dots,X_r)=\frac{1}{2}(X_1^2+\cdots+X_r^2)$ and $c_1(X_1,\dots,X_r)=X_1+\cdots+X_r$.

Let M be a complex manifold and $\overline{F}=(F,h_F)$ a Hermitian vector bundle of rank r on M. Let $K_{\overline{F}}$ be the curvature form of $\overline{F}$, and $K_{\mathrm{Sym}^n(\overline{F})}$ the curvature form of $\mathrm{Sym}^n(\overline{F})$. Then,

$$K_{\mathrm{Sym}^n(\overline{F})}=\left(\gamma_{r,n}\otimes\mathrm{id}_{A^{1,1}(M)}\right)(K_{\overline{F}}).$$

Thus, by (7.1.1.3),

$$\mathrm{ch}_2\left(\mathrm{Sym}^n(F,h_F)\right)=\binom{n+r}{r+1}\mathrm{ch}_2(F,h_F)+\frac{1}{2}\binom{n+r-1}{r+1}c_1(F,h_F)^2. \tag{7.1.1.4}$$

Now let $\overline{E}=(E,h)$ be a Hermitian vector bundle on a regular arithmetic variety X. Let h' be another Hermitian metric of E. Then, using the definition of Bott-Chern secondary characteristic classes and (7.1.1.4),

$$\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(E,h)\right)-\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(E,h')\right) = a\left(\binom{n+r}{r+1}\widetilde{\mathrm{ch}_2}(E,h,h')+\frac{1}{2}\binom{n+r-1}{r+1}\widetilde{c_1^2}(E,h,h')\right).$$

Thus,

$$\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(E,h)\right)-\binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(E,h)-\frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(E,h)^2$$

does not depend on the choice of the metric h. Therefore, in order to show (2), by using splitting principle [10, 3.3.2], we may assume that

$$(E,h)=\overline{L}_1\oplus\cdots\oplus\overline{L}_r,$$

where $\overline{L}_i = (L_i, h_i)$'s are Hermitian line bundles. Then,

$$\operatorname{Sym}^n(\overline{E}) = \bigoplus_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\ldots,\alpha_r\geq 0}} \overline{L}_1^{\otimes\alpha_1} \otimes \cdots \otimes \overline{L}_r^{\otimes\alpha_r} \otimes \left(\mathcal{O}_X, \frac{\alpha_1!\cdots\alpha_r!}{n!} h_{can}\right).$$

Therefore, $\widehat{\mathrm{ch}}_2\left(\operatorname{Sym}^n(\overline{E})\right)$ is equal to

$$\sum_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\ldots,\alpha_r\geq 0}} \left\{\widehat{\mathrm{ch}}_2\left(\overline{L}_1^{\otimes\alpha_1} \otimes \cdots \otimes \overline{L}_r^{\otimes\alpha_r}\right) - \log\left(\frac{\alpha_1!\cdots\alpha_r!}{n!}\right) a\left(c_1\left(\overline{L}_1^{\otimes\alpha_1} \otimes \cdots \otimes \overline{L}_r^{\otimes\alpha_r}\right)\right)\right\}.$$

On the other hand, since

$$\begin{aligned}\sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} &\log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)(\alpha_1 X_1 + \cdots + \alpha_r X_r) \\ &= \left(\frac{n}{r}\sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)\right)(X_1 + \cdots + X_r),\end{aligned}$$

we have

$$\begin{aligned}&\widehat{\mathrm{ch}}_2\left(\operatorname{Sym}^n(\overline{E})\right) \\ &\quad = \binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(\overline{E}) + \frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(\overline{E})^2 \\ &\qquad + \sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right) a\left(\alpha_1 c_1(\overline{L_1}) + \cdots + \alpha_r c_1(\overline{L_r})\right) \\ &\quad = \binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(\overline{E}) + \frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(\overline{E})^2 \\ &\qquad + \left(\frac{n}{r}\sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)\right) a(c_1(\overline{E})).\end{aligned}$$

Thus, we get (2). □

7.2. Arithmetic Chern classes of $\overline{E} \otimes \overline{E}^{\vee}$

Here, let us consider arithmetic Chern classes of $\overline{E} \otimes \overline{E}^{\vee}$.

Proposition 7.2.1. *Let X be a regular arithmetic variety and (E, h) a Hermitian vector bundle of rank r on X. Then,*

$$\begin{aligned}\widehat{\mathrm{ch}}_2(E \otimes E^{\vee}, h \otimes h^{\vee}) &= 2r\widehat{\mathrm{ch}}_2(E, h) - \widehat{c}_1(E, h)^2 \\ &= (r-1)\widehat{c}_1(E, h)^2 - 2r\widehat{c}_2(E, h).\end{aligned}$$

Proof. Since $\widehat{\mathrm{ch}}_i(E^\vee, h^\vee) = (-1)^i \widehat{\mathrm{ch}}_i(E, h)$ and $\widehat{\mathrm{ch}}(E \otimes E^\vee, h \otimes h^\vee) = \widehat{\mathrm{ch}}(E, h) \cdot \widehat{\mathrm{ch}}(E^\vee, h^\vee)$, we have

$$\begin{aligned}\widehat{\mathrm{ch}}_2(E \otimes E^\vee, h \otimes h^\vee) &= r\widehat{\mathrm{ch}}_2(E, h) + \widehat{c}_1(E, h) \cdot \widehat{c}_1(E^\vee, h^\vee) + r\widehat{\mathrm{ch}}_2(E^\vee, h^\vee) \\ &= 2r\widehat{\mathrm{ch}}_2(E, h) - \widehat{c}_1(E, h)^2.\end{aligned}$$

The last assertion is derived from the fact

$$\widehat{\mathrm{ch}}_2(E, h) = \frac{1}{2}\widehat{c}_1(E, h)^2 - \widehat{c}_2(E, h). \qquad \square$$

8. The proof of the relative Bogomolov's inequality in the arithmetic case

The purpose of this section is to give the proof of the following theorem.

Theorem 8.1 (Relative Bogomolov's inequality in the arithmetic case)**.** *Let $f : X \to Y$ be a projective morphism of regular arithmetic varieties such that every fiber of $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is a reduced and connected curve with only ordinary double singularities. Let (E, h) be a Hermitian vector bundle of rank r on X, and y a closed point of $Y_{\mathbb{Q}}$. If f is smooth over y and $E|_{X_{\bar{y}}}$ is semi-stable, then*

$$\widehat{\mathrm{dis}}_{X/Y}(E, h) = f_*\left(2r\widehat{c}_2(E, h) - (r-1)\widehat{c}_1(E, h)^2\right)$$

is weakly positive at y with respect to any subsets S of $Y(\mathbb{C})$ with the following properties: (1) *S is finite, and* (2) *$f_{\mathbb{C}}^{-1}(z)$ is smooth and $E_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)}$ is poly-stable for all $z \in S$.*

8.2. Sketch of the proof of the relative Bogomolov's inequality

The proof of the relative Bogomolov's inequality is very long, so that for reader's convenience, we would like to give a rough sketch of the proof of it.

Step 1. Using the Donaldson's Lagrangian, we reduce to the case where the Hermitian metric h of E along $f_{\mathbb{C}}^{-1}(z)$ is Einstein-Hermitian for each $z \in S$.

Step 2. We set

$$\overline{F}_n = \mathrm{Sym}^n\left(\mathcal{E}nd(\overline{E}) \otimes f^*(\overline{H})\right) \otimes \overline{A} \otimes f^*(\overline{H}),$$

where $\overline{A}$ is a Hermitian line bundle on X and $\overline{H}$ is a Hermitian line bundle on Y. Later we will specify these $\overline{A}$ and $\overline{H}$. By virtue of the arithmetic Riemann-Roch for stable curves (cf. Theorem 5.2.1) and formulae of arithmetic Chern classes for symmetric powers (cf. Section 7.1), we can see that

$$\frac{1}{(r^2+1)!}\widehat{\mathrm{dis}}_{X/Y}(\overline{E}) = -\lim_{n\to\infty}\frac{\widehat{c}_1(\det Rf_*(F_n), h_n)}{n^{r^2+1}},$$

where h_n is a generalized metric of $\det Rf_*(F_n)$ such that $\widehat{c}_1(\det Rf_*(F_n), h_n) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$ and h_n coincides with the Quillen metric $h_Q^{\overline{F}_n}$ at each $z \in S$.

Step 3. We assume that A is very ample and $A \otimes \omega_{X/Y}^{-1}$ is ample. We choose an arithmetic variety $B \subset X$ such that $B \in |A^{\otimes 2}|$, $B \to Y$ is étale over y, and $B(\mathbb{C}) \to Y(\mathbb{C})$ is étale over each $z \in S$. (Exactly speaking, B is not realized as an element of $|A^{\otimes 2}|$. For simplicity, we assume it.) We set $\overline{G}_n = \overline{F}_n\big|_B$ and $g = f|_B$. Here we suppose that $g_*(\mathcal{E}nd(\overline{E})\big|_B) \otimes \overline{H}$ and $g_*(\overline{A}\big|_B) \otimes \overline{H}$ are generated by small sections at y with respect to S.

Applying the Riemann-Roch formula for generically finite morphisms (cf. Theorem 4.2.1), we can find a generalized metric g_n of $\det g_*(G_n)$ such that g_n is equal to the Quillen metric of $\overline{G}_n$ at each $z \in S$, $\widehat{c}_1(\det g_*(G_n), g_n) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$, and

$$\lim_{n \to \infty} \frac{\widehat{c}_1(\det g_*(G_n), g_n)}{n^{r^2+1}} = 0.$$

Let us consider the exact sequence:

$$0 \to f_*(F_n) \to g_*(G_n) \to R^1 f_*(F_n \otimes A^{\otimes -2})$$

induced by $0 \to F_n \otimes A^{\otimes -2} \to F_n \to G_n \to 0$. Let Q_n be the image of

$$g_*(G_n) \to R^1 f_*(F_n \otimes A^{\otimes -2}).$$

The natural L^2-metric of $g_*(G_n)$ around z induces the quotient metric $\tilde{q}_n$ of Q_n around z for each $z \in S$. Thus, we can find a C^∞ metric q_n of $\det Q_n$ such that q_n is equal to $\det \tilde{q}_n$ at each $z \in S$.

Since

$$\det Rf_*(F_n) = \det g_*(G_n) \otimes (\det Q_n)^{\otimes -1} \otimes \left(\det R^1 f_*(F_n)\right)^{\otimes -1},$$

we have the generalized metric t_n of $\det R^1 f_*(F_n)$ such that

$$(\det Rf_*(F_n), h_n) = (\det g_*(G_n), g_n) \otimes (\det Q_n, q_n)^{\otimes -1} \otimes (\det R^1 f_*(F_n), t_n)^{\otimes -1}.$$

Step 4. We set $a_n = \max_{z \in S}\{\log t_n(s_n, s_n)(z)\}$, where s_n is the canonical section of $\det R^1 f_*(F_n)$. In this step, we will show that $\widehat{c}_1(\det Q_n, q_n)$ is semi-ample at y with respect to S and $a_n \le O(n^{r^2}\log(n))$. The semi-ampleness of $\widehat{c}_1(\det Q_n, q_n)$ at y is derived from Proposition 3.7.1 and the fact that $g_*(\mathcal{E}nd(\overline{E})\big|_B) \otimes \overline{H}$ and $g_*(\overline{A}\big|_B) \otimes \overline{H}$ are generated by small sections at y with respect to S. The estimation of a_n involves asymptotic behavior of analytic torsion (cf. Corollary 6.5) and a comparison of sup-norm with L^2-norm (cf. Lemma 8.3.1).

Step 5. Thus, using the last equation in Step 3, we can get a decomposition

$$-\frac{\widehat{c}_1(\det Rf_*(F_n), h_n)}{n^{r^2+1}} = \alpha_n + \beta_n$$

such that α_n is semi-ample at y with respect to S and $\lim_{n\to\infty} \beta_n = 0$.

8.3. Preliminaries

First of all, we will prepare three lemmas for the proof of the relative Bogomolov's inequality.

Lemma 8.3.1. *Let M be a d-dimensional compact Kähler manifold, $\overline{E} = (E, h)$ a flat Hermitian vector bundle of rank r on M, and $\overline{V} = (V, k)$ a Hermitian line bundle. Then, there is a constant c such that, for any $n > 0$ and any $s \in H^0(M, \mathrm{Sym}^n(E) \otimes V)$,*

$$\|s\|_{\sup} \leq cn^{d+r-1}\|s\|_{L^2}.$$

Proof. Let $f : P = \mathrm{Proj}\left(\bigoplus_{i\geq 0} \mathrm{Sym}^i(E)\right) \to M$ be the projective bundle of E, and $L = \mathcal{O}_P(1)$ the tautological line bundle of E on P. Let h_L be the quotient metric of L induced by the surjective homomorphism $f^*(E) \to L$ and the Hermitian metric $f^*(h)$ of $f^*(E)$. Let Ω_M be a Kähler form of M. Since $\overline{E}$ is flat, $c_1(L, h_L)$ is positive semi-definite of rank $r-1$. Thus, $f^*(\Omega_M)+c_1(L, h_L)$ gives rise to a fundamental 2-form Ω_P on P. Moreover, by virtue of the flatness of $\overline{E}$, we have $c_1(L, h_L)^r = 0$. Thus,

$$\Omega_P^{d+r-1} = \binom{d+r-1}{d} f^*(\Omega_M^d) \wedge c_1(L, h_L)^{r-1}.$$

By [11, Lemma 30], there is a constant c such that

$$\|s'\|_{\sup} \leq cn^{d+r-1}\|s'\|_{L^2}$$

for any $n > 0$ and any $s' \in H^0(P, L^{\otimes n} \otimes f^*(V))$, where $\|s'\|_{L^2} = \displaystyle\int_P |s'|^2\Omega_P^{d+r-1}$. We denote a homomorphism

$$f^*(\mathrm{Sym}^n(E)) \otimes f^*(V) \to L^{\otimes n} \otimes f^*(V)$$

by α_n. As in the proof of [11, (44)], we can see that, for any $s \in H^0(M, \mathrm{Sym}^n(E) \otimes V)$,

$$|s|^2 = \binom{n+r-1}{r-1} \int_{P\to M} |\alpha_n(s)|^2 c_1(L, h_L)^{r-1}.$$

Thus,

$$|s|^2 \leq \binom{n+r-1}{r-1} \int_{P\to M} \|\alpha(s)\|_{\sup}^2 c_1(L, h_L)^{r-1} = \binom{n+r-1}{r-1} \|\alpha(s)\|_{\sup}^2.$$

Therefore, we get

$$\|s\|_{\sup}^2 \leq \binom{n+r-1}{r-1} \|\alpha_n(s)\|_{\sup}^2$$

for all $s \in H^0(M, \mathrm{Sym}^n(E) \otimes V)$.

On the other hand,

$$\begin{aligned}
\|\alpha_n(s)\|_{L^2}^2 &= \int_P |\alpha_n(s)|^2 \Omega_P^r \\
&= \binom{d+r-1}{d} \int_M \int_{P\to M} |\alpha_n(s)|^2 f^*(\Omega_M^d) \wedge c_1(L, h_L)^{r-1} \\
&= \binom{d+r-1}{d} \int_M \Omega_M^d \int_{P\to M} |\alpha_n(s)|^2 c_1(L, h_L)^{r-1} \\
&= \binom{d+r-1}{d} \binom{n+r-1}{r-1}^{-1} \int_M |s|^2 \Omega_M^d \\
&= \binom{d+r-1}{d} \binom{n+r-1}{r-1}^{-1} \|s\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|s\|_{\sup}^2 &\le \binom{n+r-1}{r-1} \|\alpha_n(s)\|_{\sup}^2 \\
&\le \binom{n+r-1}{r-1} c^2 n^{2(d+r-1)} \|\alpha_n(s)\|_{L^2}^2 \\
&= \binom{d+r-1}{d} c^2 n^{2(d+r-1)} \|s\|_{L^2}^2.
\end{aligned}$$

Thus, we get our lemma. □

Here we recall Einstein-Hermitian metrics of vector bundles. Let M be a d-dimensional compact Kähler manifold with a Kähler form Ω_M, and E a vector bundle on M. We say E is *stable* (resp. *semistable*) *with respect to* Ω_M if, for any subsheaf F of E with $0 \subsetneq F \subsetneq E$,

$$\frac{1}{\mathrm{rk}\, F} \int_M c_1(F) \wedge \Omega_M^{d-1} < \frac{1}{\mathrm{rk}\, E} \int_M c_1(E) \wedge \Omega_M^{d-1}.$$

$$\left(\text{resp.} \quad \frac{1}{\mathrm{rk}\, F} \int_M c_1(F) \wedge \Omega_M^{d-1} \le \frac{1}{\mathrm{rk}\, E} \int_M c_1(E) \wedge \Omega_M^{d-1}.\right)$$

Moreover, E is said to be *poly-stable with respect to* Ω_M if E is semistable with respect to Ω_M and E has a decomposition $E = E_1 \oplus \cdots \oplus E_s$ of vector bundles such that each E_i is stable with respect to Ω_M. Let h be a Hermitian metric of E. We say h is *Einstein-Hermitian with respect to* Ω_M if there is a constant ρ such that $K(E, h) \wedge \Omega_M^{d-1} = \rho \Omega_M^d \otimes \mathrm{id}_E$, where $K(E, h)$ is the curvature form given by (E, h) and id_E is the identity map in $\mathcal{H}om(E, E)$. The Kobayashi-Hitchin correspondence tells us that E has an Einstein-Hermitian metric with respect to Ω_M if and only if E is poly-stable with respect to Ω_M.

Lemma 8.3.2. *Let M be a compact Kähler manifold with a Kähler form Ω_M, and E a poly-stable vector bundle with respect to Ω_M on M. If h and h' are Einstein-Hermitian metrics of E with respect to Ω_M, then so is $h + h'$.*

Proof. Let $E = E_1 \oplus \cdots \oplus E_s$ be a decomposition into stable vector bundles. If we set $h_i = h|_{E_i}$ and $h'_i = h'|_{E_i}$ for each i, then h_i and h'_i are Einstein-Hermitian metrics of E_i and we have the following orthogonal decompositions:

$$(E, h) = \bigoplus_{i=1}^{s} (E_i, h_i) \quad \text{and} \quad (E, h') = \bigoplus_{i=1}^{s} (E_i, h'_i)$$

(cf. [16, Chater IV, Section 3]). Thus, we may assume that E is stable. In this case, by virtue of the uniqueness of Einstein-Hermitian metric, there is a positive constant c with $h' = ch$. Thus, $h + h' = (1 + c)h$. Hence $h + h'$ is Einstein-Hermitian. □

Lemma 8.3.3. *Let C be a compact Riemann surface. Considering C as a projective variety over $\mathbb{C}$, let $\overline{C} = C \otimes_{\mathbb{C}} \mathbb{C}$ be the tensor product via the complex conjugation. Let E be a vector bundle on C, and $\overline{E} = E \otimes_{\mathbb{C}} \mathbb{C}$ on $\overline{C}$. Then, E is poly-stable on C if and only if $\overline{E}$ is poly-stable on $\overline{C}$.*

Proof. This is an easy consequence of the fact that if F is a vector bundle on C, then $\deg(F) = \deg(\overline{F})$. □

8.4. Complete proof of the relative Bogomolov's inequality

Let us start the complete proof of the relative Bogomolov's inequality.

Considering $S \cup F_\infty(S)$ instead of S, we may assume that $F_\infty(S) = S$ by virtue of Lemma 8.3.3. For each $z \in S$, let Ω_z be the Kähler form induced by the metric of $\overline{\omega}_{X/Y}$ along $f_{\mathbb{C}}^{-1}(z)$. Since $E_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)}$ is poly-stable for all $z \in S$, there is a C^∞ Hermitian metric h' of $E_{\mathbb{C}}$ such that $h'|_{f_{\mathbb{C}}^{-1}(z)}$ is Einstein-Hermitian with respect to Ω_z for all $z \in S$. It is easy to see that $\left.\overline{F_\infty^*(h')}\right|_{f_{\mathbb{C}}^{-1}(z)}$ is Einstein-Hermitian with respect to Ω_z for all $z \in S$. Thus, if h' is not invariant under F_∞, then, considering $h' + \overline{F_\infty^*(h')}$, we may assume that h' is invariant under F_∞. For, by Lemma 8.3.2, $h' + \overline{F_\infty^*(h')}$ is Einstein-Hermitian with respect to Ω_z on $f_{\mathbb{C}}^{-1}(z)$ for each $z \in S$.

Here we claim:

Claim 8.4.1. There is a $\gamma \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $a(\gamma) \in \widehat{\mathrm{CH}}^1_{L^1}(Y; S)$ and $\gamma(z) \geq 0$ for each $z \in S$, and

$$\widehat{\mathrm{dis}}_{X/Y}(E, h) = \widehat{\mathrm{dis}}_{X/Y}(E, h') + a(\gamma).$$

Proof. We set $\phi = \sqrt[r]{\det(h')/\det(h)}$. Then, it is easy to see that $\widehat{\operatorname{dis}}_{X/Y}(E, \phi h) = \widehat{\operatorname{dis}}_{X/Y}(E, h)$. Thus, we may assume that $\det(h) = \det(h')$. Then, we have

$$\widehat{\operatorname{dis}}_{X/Y}(E, h) - \widehat{\operatorname{dis}}_{X/Y}(E, h') = a\left(-f_*(2r\widetilde{\operatorname{ch}}_2(E, h, h'))\right).$$

Hence if we set $\gamma = -f_*(2r\widetilde{\operatorname{ch}}_2(E, h, h'))$, then $a(\gamma) \in \widehat{\operatorname{CH}}^1_{L^1}(Y; S)$. On the other hand, by [2, (ii) of Corollary 1.30], $-f_*(\widetilde{\operatorname{ch}}_2(E, h, h'))(z)$ is nothing more than Donaldson's Lagrangian (for details, see [18, Section 6]). Thus, we get $\gamma(z) \geq 0$ for each $z \in S$. $\square$

By the above claim, we may assume that $h|_{f_{\mathbb{C}}^{-1}(z)}$ is Einstein-Hermitian for each $z \in S$. Let $\overline{A} = (A, h_A)$ be a Hermitian line bundle on X such that A is very ample, and $A \otimes \omega_{X/Y}^{\otimes -1}$ is ample. If we take a general member M' of $|A_{\mathbb{Q}}^{\otimes 2}|$, then, by Bertini's theorem (cf. [15, Theorem 6.10]), M' is smooth over $\mathbb{Q}$, and $M' \to Y_{\mathbb{Q}}$ is étale over y. Note that if Z is an algebraic set of $\mathbb{P}^N_{\mathbb{C}}$, U is a non-empty Zariski open set of $\mathbb{P}^N_{\mathbb{Q}}$, and $U(\mathbb{Q}) \subseteq Z(\mathbb{C})$, then $Z = \mathbb{P}^N_{\mathbb{C}}$. Hence, we may assume that $M'(\mathbb{C}) \to Y(\mathbb{C})$ is étale over z for all $z \in S$. Let $M' = M'_1 + \cdots + M'_{l_1} + M'_{l_1+1} + \cdots + M'_{l_2}$ be the decomposition of M' into irreducible components (actually, the decomposition into connected components because M' is smooth over $\mathbb{Q}$) such that $f_{\mathbb{Q}}(M'_i) = Y_{\mathbb{Q}}$ for $1 \leq i \leq l_1$ and $f_{\mathbb{Q}}(M'_j) \subsetneq Y_{\mathbb{Q}}$ for $l_1 + 1 \leq j \leq l_2$. Let M_i $(i = 1, \ldots, l_1)$ be the closure of M'_i in X. We set $M = M_1 + \cdots + M_{l_1}$ and $B = M_1 \coprod \cdots \coprod M_{l_1}$ (disjoint union). Then, there is a line bundle L on X with $M \in |A^{\otimes 2} \otimes L|$. Note that $L|_{X_y} \simeq \mathcal{O}_{X_y}$ and $L_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)} \simeq \mathcal{O}_{f_{\mathbb{C}}^{-1}(z)}$ for all $z \in S$ because $y \notin \bigcup_{j=l_1+1}^{l_2} f_{\mathbb{Q}}(M'_j)$ and $z \notin \bigcup_{j=l_1+1}^{l_2} f_{\mathbb{C}}(M'_j(\mathbb{C}))$. We denote the morphism $B \to M \to X$ by ι, and the morphism $B \xrightarrow{\iota} X \xrightarrow{f} Y$ by g. We remark that the morphism $B \to M$ is an isomorphism over $\mathbb{Q}$. Further, we set

$$\overline{F} = \mathcal{E}nd(E, h) = (E \otimes E^\vee, h \otimes h^\vee).$$

Then, $h \otimes h^\vee$ is a flat metric along $f_{\mathbb{C}}^{-1}(z)$ for each $z \in S$ because $h \otimes h^\vee$ is Einstein-Hermitian and $\deg(E \otimes E^\vee) = 0$ along $f_{\mathbb{C}}^{-1}(z)$. We choose a Hermitian line bundle $\overline{H} = (H, h_H)$ on Y such that $g_*(\iota^*(A)) \otimes H$ and $g_*(\iota^*(F)) \otimes H$ are generated by small sections at y with respect to S. Moreover, we set

$$\begin{aligned}\overline{F}_n &= \operatorname{Sym}^n\left(\overline{F} \otimes f^*(\overline{H})\right) \otimes \overline{A} \otimes f^*(\overline{H}) \\ &= \left(\operatorname{Sym}^n(F \otimes f^*(H)) \otimes A \otimes f^*(H), k_n\right).\end{aligned}$$

Claim 8.4.2. There are $Z_0, \ldots, Z_{r^2} \in \widehat{\operatorname{CH}}^1_{L^1}(Y; S)_{\mathbb{Q}}$ and $\beta \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$

such that $a(\beta) \in \widehat{\mathrm{CH}}^1_{L^1}(Y; S)$, and

$$f_*\left(\widehat{\mathrm{ch}}_2(\overline{F}_n) - \frac{1}{2}\widehat{c}_1(\overline{F}_n)\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) = \frac{n^{r^2+1}}{(r^2+1)!}f_*(\widehat{\mathrm{ch}}_2(\overline{F})) + \sum_{i=0}^{r^2} Z_i n^i + a(b_n\beta),$$

where $b_n = \displaystyle\sum_{\substack{\alpha_1+\cdots+\alpha_{r^2}=n,\\ \alpha_1\geq 0,\ldots,\alpha_{r^2}\geq 0}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_{r^2}!}\right)$.

Proof. Since $\mathrm{Sym}^n(\overline{F}\otimes f^*(\overline{H}))\otimes\overline{A}\otimes f^*(\overline{H})$ is isometric to $\mathrm{Sym}^n(\overline{F})\otimes f^*(\overline{H})^{\otimes(n+1)}\otimes\overline{A}$,

$$\widehat{\mathrm{ch}}_2(\overline{F}_n) = \widehat{\mathrm{ch}}_2(\mathrm{Sym}^n(\overline{F})) + \widehat{c}_1(\mathrm{Sym}^n(\overline{F}))\cdot\widehat{c}_1(f^*(\overline{H})^{\otimes(n+1)}\otimes\overline{A}) + \binom{n+r^2-1}{r^2-1}\widehat{\mathrm{ch}}_2(f^*(\overline{H})^{\otimes(n+1)}\otimes\overline{A}).$$

Here since $\det(\overline{F}) = \overline{\mathcal{O}}_X$, by Proposition 7.1.1,

$$\widehat{c}_1(\mathrm{Sym}^n(\overline{F})) = a(b_n) \quad\text{and}\quad \widehat{\mathrm{ch}}_2(\mathrm{Sym}^n(\overline{F})) = \binom{n+r^2}{r^2+1}\widehat{\mathrm{ch}}_2(\overline{F}).$$

Thus, by Proposition 2.4.1,

$$\begin{aligned} f_*&\left(\widehat{c}_1(\mathrm{Sym}^n(\overline{F}))\cdot\widehat{c}_1(f^*(\overline{H})^{\otimes(n+1)}\otimes\overline{A})\right)\\ &= f_*\left(b_n a\left((n+1)f^*(c_1(\overline{H})) + c_1(\overline{A})\right)\right)\\ &= a\left(b_n f_*(c_1(\overline{A}))\right). \end{aligned}$$

On the other hand, using the projection formula (cf. Proposition 2.4.1),

$$\begin{aligned} f_*&\left(\widehat{\mathrm{ch}}_2(f^*(\overline{H})^{\otimes(n+1)}\otimes\overline{A})\right)\\ &= \frac{1}{2}f_*\left[\left((n+1)\widehat{c}_1(f^*(\overline{H})) + \widehat{c}_1(\overline{A})\right)^2\right]\\ &= \frac{1}{2}f_*\left[(n+1)^2\widehat{c}_1(f^*(\overline{H}))^2 + 2(n+1)\widehat{c}_1(f^*(\overline{H}))\cdot\widehat{c}_1(\overline{A}) + \widehat{c}_1(\overline{A})^2\right]\\ &= (n+1)\deg_f(A)\widehat{c}_1(\overline{H}) + \frac{1}{2}f_*\left(\widehat{c}_1(\overline{A})^2\right), \end{aligned}$$

where $\deg_f(A)$ is the degree of A on the generic fiber of f. Therefore, we have

$$\begin{aligned} f_*&\widehat{\mathrm{ch}}_2(\overline{F}_n)\\ &= \binom{n+r^2}{r^2+1}f_*\widehat{\mathrm{ch}}_2(\overline{F})\\ &\quad + \binom{n+r^2-1}{r^2-1}\left((n+1)\deg_f(A)\widehat{c}_1(\overline{H}) + \frac{1}{2}f_*\left(\widehat{c}_1(\overline{A})^2\right)\right)\\ &\quad + a\left(b_n f_*(c_1(\overline{A}))\right). \end{aligned}$$

Thus, there are $Z'_0, \dots, Z'_{r^2} \in \widehat{\mathrm{CH}}^1(Y;S)_{\mathbb{Q}}$ such that

$$f_*\widehat{\mathrm{ch}}_2(\overline{F}_n) = \frac{n^{r^2+1}}{(r^2+1)!}f_*\widehat{\mathrm{ch}}_2(\overline{F}) + \sum_{i=0}^{r^2} Z'_i n^i + a\left(b_n f_*(c_1(\overline{A}))\right). \tag{8.4.3}$$

Further, since $\widehat{c}_1(\overline{F}_n)\cdot\widehat{c}_1(\overline{\omega}_{X/Y})$ is equal to

$$\left(\widehat{c}_1(\mathrm{Sym}^n(\overline{F})) + \binom{n+r^2-1}{r^2-1}((n+1)\widehat{c}_1(f^*(\overline{H})) + \widehat{c}_1(\overline{A}))\right)\cdot\widehat{c}_1(\overline{\omega}_{X/Y}),$$

we have

$$\begin{aligned} &f_*\left(\widehat{c}_1(\overline{F}_n)\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) \\ &\qquad = a\left(b_n f_*(c_1(\overline{\omega}_{X/Y}))\right) \\ &\qquad\quad + \binom{n+r^2-1}{r^2-1}\left((n+1)(2g-2)\widehat{c}_1(\overline{H}) + f_*\left(\widehat{c}_1(\overline{A})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right)\right). \end{aligned}$$

Thus, there are $Z''_0, \dots, Z''_{r^2} \in \widehat{\mathrm{CH}}^1(Y;S)_{\mathbb{Q}}$ such that

$$f_*\left(\widehat{c}_1(\overline{F}_n)\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) = \sum_{i=0}^{r^2} Z''_i n^i + a\left(b_n f_*(c_1(\overline{\omega}_{X/Y}))\right). \tag{8.4.4}$$

Thus, combining (8.4.3) and (8.4.4), we get our claim. □

Let $h_{X/Y}$ be a C^∞ Hermitian metric of $\det Rf_*\mathcal{O}_X$ over $Y(\mathbb{C})$ such that $h_{X/Y}$ is invariant under F_∞. Then, since the Quillen metric $h_Q^{\overline{\mathcal{O}_X}}$ of $\det Rf_*\mathcal{O}_X$ is a generalized metric, there is a real valued $\phi \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $h_Q^{\overline{\mathcal{O}_X}} = e^{\phi}h_{X/Y}$ and $F_\infty^*(\phi) = \phi$ (a.e.). Adding a suitable real valued C^∞ function ϕ' with $F_\infty^*(\phi') = \phi'$ to ϕ (replace $h_{X/Y}$ by $e^{-\phi'}h_{X/Y}$ accordingly), we may assume that $\phi(z) = 0$ for all $z \in S$. Here, we set $h_n = \exp\left(-\binom{n+r^2-1}{r^2-1}\phi\right)h_Q^{\overline{F}_n}$. Then, h_n is a generalized metric of $\det Rf_*F_n$ with $F_\infty^*(h_n) = \overline{h}_n$ (a.e.). Moreover,

$$\begin{aligned} &\widehat{c}_1\left(\det Rf_*F_n, h_n\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*\mathcal{O}_X, h_{X/Y}\right) \\ &\qquad\qquad = \widehat{c}_1\left(\det Rf_*F_n, h_Q^{\overline{F}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*\mathcal{O}_X, h_Q^{\overline{\mathcal{O}_X}}\right). \end{aligned}$$

Here, since

$$\widehat{c}_1\left(\det Rf_*F_n, h_Q^{\overline{F}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*\mathcal{O}_X, h_Q^{\overline{\mathcal{O}_X}}\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$$

by Theorem 5.2.1 and $\widehat{c}_1(\det Rf_*\mathcal{O}_X, h_{X/Y}) \in \widehat{\mathrm{CH}}^1(Y;S)$, we have

$$\widehat{c}_1\left(\det Rf_*F_n, h_n\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}.$$

Further, by the arithmetic Riemann-Roch theorem for stable curves (cf. Theorem 5.2.1),

$$\widehat{c}_1\left(\det Rf_*(F_n), h_n\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_{X/Y}\right) \\ = f_*\left(\widehat{\mathrm{ch}}_2(\overline{F}_n) - \frac{1}{2}\widehat{c}_1(\overline{F}_n)\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right).$$

Therefore, by Claim 8.4.2, there are $W_0,\ldots,W_{r^2} \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$ and $\beta \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $a(\beta) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)$, and

$$\widehat{c}_1\left(\det Rf_*(F_n), h_n\right) = \frac{n^{r^2+1}}{(r^2+1)!}f_*(\widehat{\mathrm{ch}}_2(\overline{F})) + \sum_{i=0}^{r^2} W_i n^i + a(b_n\beta). \tag{8.4.5}$$

Claim 8.4.6. $\dfrac{1}{(r^2+1)!}\widehat{\mathrm{dis}}_{X/Y}(\overline{E}) = -\lim_{n\to\infty}\dfrac{\widehat{c}_1\left(\det Rf_*(F_n), h_n\right)}{n^{r^2+1}}$ in $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$.

Proof. By virtue of Proposition 7.2.1, $f_*(\widehat{\mathrm{ch}}_2(\overline{F})) = -\widehat{\mathrm{dis}}_{X/Y}(\overline{E})$. Thus, by (8.4.5), it is sufficient to show that $0 \le b_n \le O(n^{r^2})$.

It is well known that

$$\frac{\log(\theta_1)+\cdots+\log(\theta_N)}{N} \le \log\left(\frac{\theta_1+\cdots+\theta_N}{N}\right)$$

for positive numbers $\theta_1,\ldots,\theta_N$. Thus, noting $\displaystyle\sum_{\substack{\alpha_1+\cdots+\alpha_{r^2}=n,\\ \alpha_1\ge 0,\ldots,\alpha_{r^2}\ge 0}} \frac{n!}{\alpha_1!\cdots\alpha_{r^2}!} = (r^2)^n$, we have

$$0 \le \sum_{\substack{\alpha_1+\cdots+\alpha_{r^2}=n,\\ \alpha_1\ge 0,\ldots,\alpha_{r^2}\ge 0}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_{r^2}!}\right) \le \binom{n+r^2-1}{r^2-1}\log\left(\frac{(r^2)^n}{\binom{n+r^2-1}{r^2-1}}\right) \le O(n^{r^2}).$$

$\square$

We set $\overline{G}_n = \iota^*(\overline{F}_n)$. Then, by Theorem 4.2.1,

$$\widehat{c}_1\left(\det Rg_*(G_n), h_Q^{\overline{G}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rg_*(\mathcal{O}_B), h_Q^{\overline{\mathcal{O}}_B}\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)$$

and

$$\widehat{c}_1\left(\det Rg_*(G_n), h_Q^{\overline{G}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rg_*(\mathcal{O}_B), h_Q^{\overline{\mathcal{O}}_B}\right) = g_*\left(\widehat{c}_1(\overline{G}_n)\right).$$

As before, we can take a C^∞ Hermitian metric $h_{B/Y}$ of $\det Rg_*(\mathcal{O}_B)$ over $Y(\mathbb{C})$ and a real valued $\varphi \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $h_Q^{\overline{\mathcal{O}}_B} = e^{\varphi} h_{B/Y}$, $F_\infty^*(h_{B/Y}) = \overline{h}_{B/Y}$, $F_\infty^*(\varphi) = \varphi$ (a.e.), and $\varphi(z) = 0$ for all $z \in S$. We set

$$g_n = \exp\left(-\binom{n+r^2-1}{r^2-1}\varphi\right) h_Q^{\overline{G}_n}.$$

Then,

$$\widehat{c}_1\left(\det Rg_*(G_n), g_n\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rg_*(\mathcal{O}_B), h_{B/Y}\right) = g_*\left(\widehat{c}_1(\overline{G}_n)\right)$$

and $\widehat{c}_1\left(\det Rg_*(G_n), g_n\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)$. Moreover, in the same as in Claim 8.4.2, we can see that

$$g_*\left(\widehat{c}_1(\overline{G}_n)\right) = a(\deg(g)b_n) + \binom{n+r^2-1}{r^2-1}\left((n+1)g_*\widehat{c}_1(g^*(\overline{H})) + g_*\widehat{c}_1(\iota^*(\overline{A}))\right).$$

Thus, there are $W'_0, \ldots, W'_{r^2} \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$ such that

$$\widehat{c}_1\left(\det Rg_*(G_n), g_n\right) = \sum_{i=0}^{r^2} W'_i n^i + a(b_n \deg(g)).$$

Therefore, we have

$$\lim_{n\to\infty} \frac{\widehat{c}_1\left(\det Rg_*(G_n), g_n\right)}{n^{r^2+1}} = 0 \tag{8.4.7}$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$.

Let us consider an exact sequence:

$$0 \to F_n \otimes A^{\otimes -2} \otimes L^{\otimes -1} \to F_n \to F_n|_M \to 0.$$

Since F is semi-stable and of degree 0 along X_y and $L|_{X_y} = \mathcal{O}_{X_y}$, we have

$$f_*(F_n \otimes A^{\otimes -2} \otimes L^{\otimes -1}) = 0$$

on Y. Thus, the above exact sequence gives rise to

$$0 \to f_*(F_n) \to (f|_M)_*(F_n|_M) \to R^1 f_*(F_n \otimes A^{\otimes -2} \otimes L^{\otimes -1}) \to R^1 f_*(F_n).$$

Let Q_n be the cokernel of

$$f_*(F_n) \to (f|_M)_*(F_n|_M) \to g_*(G_n).$$

Let U be the maximal Zariski open set of Y such that f is smooth over U and g is étale over U. Moreover, let U_n be the maximal Zariski open set of Y such that

$$\begin{cases} \text{(a) } U_n \subset U, \\ \text{(b) } (f|_M)_*(F_n|_M) \text{ coincides with } g_*(G_n) \text{ over } U_n, \\ \text{(c) } R^1 f_*(F_n) = 0 \text{ over } U_n, \text{ and} \\ \text{(d) } f_*(F_n),\ g_*(G_n) \text{ and } Q_n \text{ are locally free over } U_n. \end{cases}$$

Then, $y \in (U_n)_{\mathbb{Q}}$ and $S \subseteq U_n(\mathbb{C})$. For, since $A \otimes \omega_{X/Y}^{-1}$ is ample on X_y and E is semi-stable on X_y, we can see that $R^1 f_*(F_n) = 0$ around y, which implies that $f_*(F_n)$ is locally free around y. Further, since $f_*(F_n)$ and $(f|_M)_*(F_n|_M)$ are free at y, $R^1 f_*(F_n) = 0$ around y, and $(f|_M)_*(F_n|_M)$ coincides with $g_*(G_n)$ around y, we can easily check that Q_n is free at y. Thus, $y \in (U_n)_{\mathbb{Q}}$. In the same way, we can see that $S \subseteq U_n(\mathbb{C})$.

Next let us consider a metric of $\det Q_n$. $g_*(G_n)$ has the Hermitian metric $(f|_M)_*(k_n|_M)$ over $U_n(\mathbb{C})$, where k_n is the Hermitian metric of $\overline{F}_n$. Let $\tilde{q}_n$ be the quotient metric of Q_n over $U_n(\mathbb{C})$ induced by $(f|_M)_*(k_n|_M)$. Let q_n be a C^∞ Hermitian metrics of $\det Q_n$ over $Y(\mathbb{C})$ such that $F_\infty^*(q_n) = q_n$ and $q_n(z) = \det \tilde{q}_n(z)$ for all $z \in S$. (If q_n is not invariant under F_∞, then consider $(1/2)\left(q_n + \overline{F_\infty^*(q_n)}\right)$.)

Here since $\det Rf_*(F_n) \simeq \det f_*(F_n) \otimes \left(\det R^1 f_*(F_n)\right)^{-1}$ and $\det f_*(F_n) \simeq \det g_*(G_n) \otimes (\det Q_n)^{-1}$, we have

$$\det Rf_*(F_n) \simeq \det g_*(G_n) \otimes (\det Q_n)^{-1} \otimes \left(\det R^1 f_*(F_n)\right)^{-1}.$$

Further, we have generalized metrics h_n, g_n and q_n of $\det Rf_*(F_n)$, $\det g_*(G_n)$ and $\det Q_n$. Thus, there is a generalized metric t_n of $\det R^1 f_*(F_n)$ such that the above is an isometry.

As in the proof of Proposition 3.7.1, let us construct a section of $\det R^1 f_*(F_n)$. First, we fix a locally free sheaf P_n on Y and a surjective homomorphism $P_n \to R^1 f_*(F_n)$. Let P'_n be the kernel of $P_n \to R^1 f_*(F_n)$. Then, P'_n is a torsion free sheaf and has the same rank as P_n because $R^1 f_*(F_n)$ is a torsion sheaf. Noting that $\left(\bigwedge^{\operatorname{rk} P'_n} P'_n\right)^*$ is an invertible sheaf on Y, we can identify $\det R^1 f_*(F_n)$ with

$$\bigwedge^{\operatorname{rk} P_n} P_n \otimes \left(\bigwedge^{\operatorname{rk} P'_n} P'_n\right)^*.$$

Moreover, the homomorphism $\bigwedge^{\operatorname{rk} P'_n} P'_n \to \bigwedge^{\operatorname{rk} P_n} P_n$ induced by $P'_n \hookrightarrow P_n$ gives rise to a non-zero section s_n of $\det R^1 f_*(F_n)$. Note that $s_n(y) \neq 0$ and $s_n(z) \neq 0$ for all $z \in S$ because $R^1 f_*(F_n) = 0$ at y and z.

Here we set

$$a_n = \max_{z \in S}\{\log t_n(s_n, s_n)(z)\}.$$

By our construction, we have

$$\widehat{c}_1(\det R^1 f_*(F_n), e^{-a_n} t_n) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S).$$

and an isometry

(8.4.8)
$(\det Rf_*(F_n), h_n) \simeq$
$(\det g_*(G_n), g_n) \otimes (\det Q_n, q_n)^{-1} \otimes (\det R^1 f_*(F_n), e^{-a_n} t_n)^{-1} \otimes (\mathcal{O}_Y, e^{-a_n} h_{can}).$

Here we claim:

Claim 8.4.9. $(\det Q_n, q_n)$ is generated by small sections at y with respect to S.

Proof. First of all,

$$g_* \left(\iota^*(F) \otimes g^*(H)\right) = g_*(\iota^*(F)) \otimes H \quad \text{and} \quad g_* \left(\iota^*(A) \otimes g^*(H)\right) = g_*(\iota^*(A)) \otimes H$$

are generated by small section at y with respect to S. Thus, by (2) and (3) of Proposition 3.7.1,

$$g_*(G_n) = g_* \left(\mathrm{Sym}^n(\iota^*(F) \otimes g^*(H)) \otimes \iota^*(A) \otimes g^*(H)\right)$$

is generated by small sections at y with respect to S. Thus, by (1) of Proposition 3.7.1, $(Q_n, \tilde{q}_n)$ is generated by small sections at y with respect to S. Hence, by (4) of Proposition 3.7.1, $(\det Q_n, q_n)$ is generated by small sections at y with respect to S because $q_n(z) = \det \tilde{q}_n(z)$ for all $z \in S$. □

Next we claim:

Claim 8.4.10. $a_n \le O(n^{r^2} \log(n))$.

Proof. It is sufficient to show that $\log t_n(s_n, s_n)(z) \le O(n^{r^2} \log(n))$ for each $z \in S$. Let $\{e_1, \ldots, e_{l_n}\}$ be an orthonormal basis of $g_*(G_n) \otimes \kappa(z)$ with respect to $g_*(k_n|_B)(z)$ such that $\{e_1, \ldots, e_{m_n}\}$ forms a basis of $f_*(F_n) \otimes \kappa(z)$. Then, $e_1 \wedge \cdots \wedge e_{m_n}$, $e_1 \wedge \cdots \wedge e_{l_n}$ and $\bar{e}_{m_n+1} \wedge \cdots \wedge \bar{e}_{l_n}$ form bases of $\det(f_*(F_n)) \otimes \kappa(z)$, $\det(g_*(G_n)) \otimes \kappa(z)$, and $\det(Q_n) \otimes \kappa(z)$ respectively, and $(e_1 \wedge \cdots \wedge e_{m_n}) \otimes (\bar{e}_{m_n+1} \wedge \cdots \wedge \bar{e}_{l_n}) = e_1 \wedge \cdots \wedge e_{l_n}$, where $\bar{e}_{m_n+1}, \ldots, \bar{e}_{l_n}$ are images of $e_{m_n+1}, \ldots, e_{l_n}$ in $Q_n \otimes \kappa(z)$. Then,

$$\left|(e_1 \wedge \cdots \wedge e_{m_n}) \otimes s_n^{\otimes -1}\right|^2_{h_n}(z) = \frac{|e_1 \wedge \cdots \wedge e_{l_n}|^2_{g_n}(z)}{|\bar{e}_{m_n+1} \wedge \cdots \wedge \bar{e}_{l_n}|^2_{q_n}(z) |s_n|^2_{t_n}(z)} = |s_n|^{-2}_{t_n}(z),$$

where $|a|_\lambda = \sqrt{\lambda(a,a)}$ for $\lambda = h_n, g_n, q_n, t_n$. Moreover, let Ω_z be the Kähler form induced by the metric of $\overline{\omega}_{X/Y}$ along $f_{\mathbb{C}}^{-1}(z)$. Then, there is a Hermitian metric v_n of $H^0(f_{\mathbb{C}}^{-1}(z), F_n)$ defined by

$$v_n(s, s') = \int_{f_{\mathbb{C}}^{-1}(z)} k_n(s, s') \Omega_z.$$

Here $R^1 f_*(F_n) = 0$ at z. Thus, $(\det R^1 f_*(F_n))_z$ is canonically isomorphic to $\mathcal{O}_{Y(\mathbb{C}),z}$. Since $(P'_n)_z = (P_n)_z$, under the above isomorphism, s_n goes to the determinant of $(P_n)_z \xrightarrow{\mathrm{id}} (P_n)_z$, namely $1 \in \mathcal{O}_{Y(\mathbb{C}),z}$. Hence, by the definition of Quillen metric,

$$\left|(e_1 \wedge \cdots \wedge e_{m_n}) \otimes s_n^{\otimes -1}\right|_{h_n}^2 (z) = \det(v_n(e_i, e_j)) \exp\left(-T\left(\left.\overline{F}_n\right|_{f_{\mathbb{C}}^{-1}(z)}\right)\right).$$

Therefore,

$$\log |s_n|_{t_n}^2(z) = T\left(\left.\overline{F}_n\right|_{f_{\mathbb{C}}^{-1}(z)}\right) - \log\det(v_n(e_i, e_j)).$$

By Corollary 6.5,

$$T\left(\left.\overline{F}_n\right|_{f_{\mathbb{C}}^{-1}(z)}\right) \leq O(n^{r^2}\log(n)).$$

Thus, in order to get our claim, it is sufficient to show that

$$-\log\det(v_n(e_i, e_j)) \leq O(n^{r^2-1}\log(n)).$$

Let s be an arbitrary section of $H^0(f_{\mathbb{C}}^{-1}(z), F_n)$. Then, by Lemma 8.3.1,

$$\begin{aligned} &g_*\left(\left.k_n\right|_B\right)(s, s) \\ &\quad = \sum_{x \in g_{\mathbb{C}}^{-1}(z)} |s|_{k_n}^2(x) \leq \deg(g) \sup_{x \in f_{\mathbb{C}}^{-1}(z)} \{|s|_{k_n}^2(x)\} \leq \deg(g) c^2 n^{2r^2} \|s\|_{L^2}^2 \end{aligned}$$

for some constant c independent of n. Thus, by [18, Lemma 3.4] and our choice of e_i's,

$$1 = \det\left(g_*\left(\left.k_n\right|_B\right)(e_i, e_j)\right) \leq \left(\deg(g) c^2 n^{2r^2}\right)^{\dim_{\mathbb{C}} H^0(f_{\mathbb{C}}^{-1}(z), F_n)} \det\left(v_n(e_i, e_j)\right).$$

Using Riemann-Roch theorem, we can easily see that

$$\dim_{\mathbb{C}} H^0(f_{\mathbb{C}}^{-1}(z), F_n) \leq O(n^{r^2-1}).$$

Thus, we have

$$-\log\det(v_n(e_i, e_j)) \leq O(n^{r^2-1}\log(n)).$$

Hence, we obtain our claim. □

Let us go back to the proof of our theorem. By the isometry (8.4.8), we get

$$\begin{aligned} &-\widehat{c}_1(\det Rf_*(F_n), h_n) \\ &\quad = -\widehat{c}_1(\det g_*(G_n), g_n) + \widehat{c}_1(\det Q_n, q_n) + \widehat{c}_1(\det R^1 f_*(F_n), e^{-a_n} t_n) - a(a_n) \\ &\quad = \left[\widehat{c}_1(\det Q_n, q_n) + \widehat{c}_1(\det R^1 f_*(F_n), e^{-a_n} t_n) + a\left(\max\{-a_n, 0\}\right)\right] \\ &\qquad\quad + \left[-\widehat{c}_1(\det g_*(G_n), g_n) + a(\min\{-a_n, 0\})\right]. \end{aligned}$$

Here we set

$$\begin{cases} \alpha_n = \dfrac{(r^2+1)!}{n^{r^2+1}}[\widehat{c}_1(\det Q_n, q_n) \\ \qquad\qquad +\widehat{c}_1(\det R^1 f_*(F_n), e^{-a_n}t_n) + a\,(\max\{-a_n, 0\})], \\ \\ \beta_n = \dfrac{(r^2+1)!}{n^{r^2+1}}\left[-\widehat{c}_1(\det g_*(G_n), g_n) + a(\min\{-a_n, 0\})\right]. \end{cases}$$

Then,

$$\frac{-(r^2+1)!\widehat{c}_1(\det Rf_*(F_n), h_n)}{n^{r^2+1}} = \alpha_n + \beta_n.$$

By (8.4.7) and Claim 8.4.10, $\lim_{n\to\infty} \beta_n = 0$ in $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$. Therefore, by Claim 8.4.6,

$$\widehat{\mathrm{dis}}_{X/Y}(\overline{E}) = \lim_{n\to\infty} \frac{-(r^2+1)!\widehat{c}_1\left(\det Rf_*(F_n), h_n\right)}{n^{r^2+1}} = \lim_{n\to\infty}(\alpha_n + \beta_n) = \lim_{n\to\infty} \alpha_n$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$. On the other hand, it is obvious that

$$\widehat{c}_1(\det R^1 f_*(F_n), e^{-a_n}t_n) \quad \text{and} \quad a\,(\max\{-a_n, 0\})$$

is semi-ample at y with respect to S. By Claim 8.4.9, $\widehat{c}_1(\det Q_n, q_n)$ is semi-ample at y with respect to S. Thus, α_n is semi-ample at y with respect to S. Hence we get our theorem.

9. Preliminaries for Cornalba-Harris-Bost's inequality

This section is a preparatory one for the next section, where we will prove the relative Cornalba-Harris-Bost's inequality (cf. Theorem 10.1.4). Moreover, in the next section, we will see how the relative Bogomolov's inequality (Theorem 8.1) and the relative Cornalba-Harris-Bost's inequality (Theorem 10.1.4) are related (cf. Proposition 10.2.2).

9.1. Normalized Green forms

Let Y be a smooth quasi-projective variety over $\mathbb{C}$, $\overline{E} = (E, h)$ a Hermitian vector bundle of rank r on Y. Let $\pi : \mathbb{P}(E) \to Y$ be the canonical morphism, where $\mathbb{P}(E) = \mathrm{Proj}(\bigoplus_{i\geq 0} \mathrm{Sym}^i(E^\vee))$. We equip the canonical quotient bundle $\mathcal{O}_E(1)$ on $\mathbb{P}(E)$ with the quotient metric via $\pi^*(E^\vee) \to \mathcal{O}_E(1)$. We will denote this Hermitian line bundle by $\overline{\mathcal{O}_E(1)}$. Furthermore, let $\Omega = c_1(\overline{\mathcal{O}_E(1)})$ be the first Chern form.

The purpose of this subsection is that, for every cycle $X \subset \mathbb{P}(E)$ whose all irreducible components map surjectively to Y, we give a Green form g_X such that on a general fiber, it is an Ω-normalized Green current in the sense of [5, 2.3.2].

Let X be a cycle of codimension p on $\mathbb{P}(E)$ such that every irreducible component of X maps surjectively to Y. An L^1-form g_X on $\mathbb{P}(E)$ satisfying the following conditions is called an *Ω-normalized Green form*, (or simply a *normalized Green form* when no confusion is likely).

(i) There are d-closed L^1-forms γ_i of type $(p-i, p-i)$ on Y $(i = 0, \dots, p)$ with

$$dd^c([g_X]) + \delta_X = \sum_{i=0}^{p} \left[\pi^*(\gamma_i) \wedge \Omega^i\right].$$

(ii) $\pi_*(g_X \wedge \Omega^{r-p}) = 0$.
Note that γ_p is the degree of X along a general fiber of π.

Let $X = \sum_i a_i X_i$ be the irreducible decomposition of X as cycles. Let $\tilde{X}_i \to X_i$ be a desingularization of X_i, and $\tilde{f}_i : \tilde{X}_i \to Y$ the induced morphism. The main result of this subsection is the following.

Proposition 9.1.1. *With notation as above, there exists an Ω-normalized Green form g_X on $\mathbb{P}(E)$ satisfying the following property. If $y \in Y$ and $\tilde{f}_i$ is smooth over y for every i, then there is an open set U containing y such that $\gamma_0, \dots, \gamma_p$ are C^∞ on U and that $g_X|_{\pi^{-1}(U)}$ is a Green form of logarithmic type for X_U, where $\gamma_0, \dots, \gamma_p$ are L^1-forms in the definition of Ω-normalized Green form.*

To prove the above proposition, let us begin with the following two lemmas.

Lemma 9.1.2. *There exist a Green form g of logarithmic type along X, and d-closed C^∞ forms β_i of type $(p-i, p-i)$ on Y $(i = 0, \dots, p)$ such that*

$$dd^c([g]) + \delta_X = \sum_{i=0}^{p} \left[\pi^*(\beta_i) \wedge \Omega^i\right].$$

Proof. We divide the proof into three steps.

Step 1.: The case where Y is projective.
Let g_1 be a Green form of logarithmic type along X such that

$$dd^c([g_1]) + \delta_X = [\omega]$$

where ω is a smooth form on $\mathbb{P}(E)$. Then, we can find a smooth form η on $\mathbb{P}(E)$ of the form

$$\eta = \sum_{i=0}^{p} \pi^*(\beta_i) \wedge \Omega^i$$

which represents the same cohomology class as ω, where β_i is a d-closed C^∞-form of type $(p-i, p-i)$ on Y. Since $\omega - \eta$ is d-exact (p,p)-form, by the

dd^c-lemma, there is a smooth $(p-1,p-1)$-form ϕ with $\omega - \eta = dd^c(\phi)$. Thus, if we set $g = g_1 - \phi$, then g is of logarithmic type along X and

$$dd^c([g]) + \delta_X = dd^c([g_1]) - dd^c(\phi) + \delta_X = [\eta].$$

Step 2.: Let h' be another Hermitian metric of E, and Ω' the Chern form of $\mathcal{O}_E(1)$ arising from h'. In this step, we will prove that if the lemma holds for h', then so does it for h.

By our assumption, there exist a Green form g' of logarithmic type along X, and d-closed C^∞ forms β'_i $(i = 0, \dots, p)$ of type $(p-i, p-i)$ on Y such that

$$dd^c([g']) + \delta_X = \sum_{i=0}^{p} \left[\pi^*(\beta'_i) \wedge {\Omega'}^i\right].$$

On the other hand, there is a real C^∞-function a on $\mathbb{P}(E)$ with $\Omega' - \Omega = dd^c(a)$. Here note that if v is a ∂ and $\overline{\partial}$-closed form on $\mathbb{P}(E)$, then $dd^c(v \wedge a) = v \wedge dd^c(a)$. Thus, it is easy to see that there is a C^∞ form θ on $\mathbb{P}(E)$ such that

$$\sum_{i=1}^{p} \pi^*(\beta'_i) \wedge {\Omega'}^i = dd^c(\theta) + \sum_{i=1}^{p} \pi^*(\beta'_i) \wedge \Omega^i.$$

Therefore, if we set $g = g' - \theta$ and $\beta_i = \beta'_i$, then we have our assertion for h.

Step 3.: General case.

Using Hironaka's resolution [14], there is a smooth projective variety Y' over $\mathbb{C}$ such that Y is an open set of Y'. Moreover, using [13, Exercise 5.15 in Chapter II], there is a coherent sheaf E' on Y' with $E'|_Y = E$. Further, taking a birational modification along $Y' \setminus Y$ if necessary, we may assume that E' is locally free. Let h' be a Hermitian metric of E' over Y'. Since $\mathbb{P}(E)$ is an Zariski open set of $\mathbb{P}(E')$, let X' be the closure of X in $\mathbb{P}(E')$. Then, by Step 1, our assertion holds for (E', h') and X'. Thus, so does it for $(E, h'|_Y)$ and X. Therefore, by Step 2, we can conclude our lemma. □

Lemma 9.1.3. *Let g be a Green form of logarithmic type along X and ω a C^∞-form with $dd^c([g]) + \delta_X = [\omega]$. If we set $\varsigma = \pi_*(g \wedge \Omega^{r-p})$, then $\varsigma \in L^1_{loc}(Y)$ and $dd^c([\varsigma]) \in L^1_{loc}(\Omega^{1,1}_Y)$. Moreover, if $y \in Y$ and $\tilde{f}_i$ is smooth over y for every i, then ς is C^∞ around y.*

Proof. By Proposition 1.2.5, ς is an L^1-function on Y and

$$\begin{aligned} dd^c([\varsigma]) &= dd^c(\pi_*([g \wedge \Omega^{r-p}])) = \pi_* dd^c([g \wedge \Omega^{r-p}]) \\ &= \pi_* dd^c([g]) \wedge \Omega^{r-p} = \pi_*([\omega] \wedge \Omega^{r-p}) - \pi_*(\delta_X \wedge \Omega^{r-p}) \\ &= \pi_*[\omega \wedge \Omega^{r-p}] - \sum_i a_i \pi_*(\delta_{X_i} \wedge \Omega^{r-p}) \\ &= \pi_*[\omega \wedge \Omega^{r-p}] - \sum_i a_i (\tilde{f}_i)_* [\tilde{f}_i^*(\Omega^{r-p})]. \end{aligned}$$

Thus, $dd^c([\varsigma]) \in L^1_{loc}(\Omega^{1,1}_Y)$. Moreover, if $y \in Y$ and $\tilde{f}_i$ is smooth over y for every i, then, by the above formula, $dd^c([\varsigma])$ is C^∞ around y. Thus, by virtue of [9, (i) of Theorem 1.2.2], ς is C^∞ around y. □

Let us start the proof of Proposition 9.1.1. Let g be a Green form constructed in Lemma 9.1.2. Then, there are d-closed β_i's with $\beta_i \in A^{p-i,p-i}(Y)$ and

$$dd^c([g]) + \delta_X = \sum_{i=0}^{p} \left[\pi^*(\beta_i) \wedge \Omega^i\right].$$

If we set $\varsigma = \pi_*(g \wedge \Omega^{r-p})$, then by Lemma 9.1.3, ς is locally an L^1-form. We put

$$g_X = g - \pi^*(\varsigma)\Omega^{p-1},$$

which is clearly locally an L^1-form on $\mathbb{P}(E)$. We will show that g_X satisfies the conditions (i) and (ii). Using $\int_{\mathbb{P}(E)\to Y} \Omega^{r-1} = 1$, (ii) can be readily checked. Moreover,

$$\begin{aligned} dd^c([g_X]) + \delta_X &= \sum_{i=0}^{p} \left[\pi^*(\beta_i) \wedge \Omega^i\right] - dd^c[\pi^*(\varsigma)\Omega^{p-1}] \\ &= \beta_p\Omega^p + \pi^*([\beta_{p-1}] - dd^c([\varsigma])) \wedge \Omega^{p-1} + \sum_{i=0}^{p-2} \left[\pi^*(\beta_i) \wedge \Omega^i\right]. \end{aligned}$$

The remaining assertion is easily derived from Lemma 9.1.3. □

Remark 9.1.4. Let y be a point of Y such that $\tilde{f}_i$ is smooth over y for every i. Then, by Proposition 9.1.1, on the fiber $\pi^{-1}(y)$, $g_X|_{\pi^{-1}(y)}$ is a Green form of logarithmic type along X_y. Moreover,

$$dd^c([g_X|_{\pi^{-1}(y)}]) + \delta_{X_y} = \deg(X_y)[\Omega^p|_{\pi^{-1}(y)}]$$

and

$$\int_{\pi^{-1}(y)} \left(g_X|_{\pi^{-1}(y)}\right)\left(\Omega^{r-p}|_{\pi^{-1}(y)}\right) = 0.$$

Thus, $g_X|_{\pi^{-1}(y)}$ is a Ω-normalized Green form on $\pi^{-1}(y)$, and it is also a Ω-normalized Green current in the sense of [5, 2.3.2].

9.2. Associated Hermitian vector bundles

Let $\mathrm{GL}_r = \operatorname{Spec}\mathbb{Z}[X_{11}, X_{12}, \cdots, X_{rr}]_{\det(X_{ij})}$ be the general linear group of rank r and $\mathrm{SL}_r = \operatorname{Spec}\mathbb{Z}[X_{11}, X_{12}, \cdots, X_{rr}]/(\det(X_{ij}) - 1)$ be the special linear group of rank r.

Let $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ be a morphism of group schemes. First, we note that

$$\rho(\mathbb{C})(\overline{A}) = \overline{\rho(\mathbb{C})(A)},$$

where $\rho(\mathbb{C}) : \mathrm{GL}_r(\mathbb{C}) \to \mathrm{GL}_R(\mathbb{C})$ is the induced morphism and $A \in \mathrm{GL}_r(\mathbb{C})$. Indeed, the above equality is nothing but the associativity of the map

$$\operatorname{Spec}\mathbb{C} \xrightarrow{-} \operatorname{Spec}\mathbb{C} \xrightarrow{A} \mathrm{GL}_r \xrightarrow{\rho} \mathrm{GL}_R\,.$$

Next, we consider the following condition for ρ;

$$\rho({}^tA) = {}^t\rho(A) \qquad \text{for any } A \in \mathrm{GL}_r. \tag{9.2.1}$$

In the group scheme language, this condition means ρ commutes with the transposed morphism.

Let $\mathrm{U}_r(\mathbb{C}) = \{A \in \mathrm{GL}_r(\mathbb{C}) \mid {}^tA \cdot \overline{A} = I_r\}$ be the unitary group of rank r. If a group morphism $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ commutes with the transposed morphism, then

$$\begin{aligned} I_R = \rho(\mathbb{C})(I_r) &= \rho(\mathbb{C})({}^tA \cdot \overline{A}) \\ &= \rho(\mathbb{C})({}^tA) \cdot \rho(\mathbb{C})(\overline{A}) = {}^t\rho(\mathbb{C})(A) \cdot \overline{\rho(\mathbb{C})(A)}, \end{aligned}$$

namely, $\rho(\mathbb{C})$ maps $\mathrm{U}_r(\mathbb{C})$ into $\mathrm{U}_R(\mathbb{C})$.

Let k be an integer. A morphism $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ of group schemes is said to be *of degree* k if

$$\rho(tI_r) = t^k I_R \qquad \text{for any } t.$$

In the group scheme language, this means that the diagram

$$\begin{tikzcd} \mathrm{GL}_1 \arrow[r, "\lambda_r"] \arrow[d, "\alpha"'] & \mathrm{GL}_r \arrow[d, "\rho"] \\ \mathrm{GL}_1 \arrow[r, "\lambda_R"] & \mathrm{GL}_R \end{tikzcd}$$

commutes, where λ_r and λ_R are given by $t \mapsto \operatorname{diag}(t, t, \cdots, t)$ and α is given by $t \mapsto t^k$.

Let Y be an arithmetic variety, $\overline{E} = (E, h)$ a Hermitian vector bundle of rank r on Y and $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ be a morphism of group schemes satisfying commutativity with the transposed morphism. In the following, we will show that we can naturally construct a Hermitian vector bundle $\overline{E}^\rho = (E^\rho, h^\rho)$, which we will call the *associated Hermitian vector bundle* with respect to $\overline{E}$ and ρ.

First, we construct E^ρ. Let $\{Y_\alpha\}$ be an affine open covering such that $\phi_\alpha : E|_{Y_\alpha} \xrightarrow{\sim} \mathcal{O}_{Y_\alpha}^{\oplus r}$ gives a local trivialization. On $Y_\alpha \cap Y_\beta$, we set the transition function $g_{\alpha\beta} = \phi_\alpha \cdot \phi_\beta^{-1}$, which can be seen as an element of $\mathrm{GL}_r(\Gamma(\mathcal{O}_{Y_\alpha \cap Y_\beta}))$.

Then we define the associated vector bundle E^ρ as the vector bundle of rank R on Y with the transition functions $\rho(\Gamma(\mathcal{O}_{Y_\alpha \cap Y_\beta}))(g_{\alpha\beta})$;

$$E^\rho = \coprod_\alpha \mathcal{O}_{Y_\alpha}^{\oplus R} / \sim .$$

Next, we define metric on E^ρ. Let h^α be the Hermitian metric on $\mathcal{O}_{Y_\alpha}^{\oplus r}$ over Y_α such that $\phi_\alpha : E|_{Y_\alpha} \overset{\sim}{\longrightarrow} \mathcal{O}_{Y_\alpha}^{\oplus r}$ becomes isometry over $Y_\alpha(\mathbb{C})$. Let

$$\begin{aligned} e_1^\alpha &= {}^t(1,0,\cdots,0), \cdots, e_r^\alpha = {}^t(0,\cdots,0,1) \in \Gamma(\mathcal{O}_{Y_\alpha}^{\oplus r}), \\ f_1^\alpha &= {}^t(1,0,\cdots,0), \cdots, f_R^\alpha = {}^t(0,\cdots,0,1) \in \Gamma(\mathcal{O}_{Y_\alpha}^{\oplus R}) \end{aligned}$$

be the standard local frames of $\mathcal{O}_{Y_\alpha}^{\oplus r}$ and $\mathcal{O}_{Y_\alpha}^{\oplus R}$. We set

$$H_\alpha = (h^\alpha(e_i^\alpha, e_j^\alpha))_{1 \le i,j \le r}.$$

Then H_α is a C^∞-map over $Y_\alpha(\mathbb{C})$ and, for each point y in $Y_\alpha(\mathbb{C})$, $H_\alpha(y)$ is a positive definite Hermitian matrix. Let $\rho(C^\infty(Y_\alpha(\mathbb{C}))) : \mathrm{GL}_r(C^\infty(Y_\alpha(\mathbb{C}))) \to \mathrm{GL}_R(C^\infty(Y_\alpha(\mathbb{C})))$ be the induced map.

Claim 9.2.2. $\rho(C^\infty(Y_\alpha(\mathbb{C})))(H_\alpha)$ is a C^∞-map over $Y_\alpha(\mathbb{C})$ and, for each point y in $Y_\alpha(\mathbb{C})$, $\rho(C^\infty(Y_\alpha(\mathbb{C})))(H_\alpha)(y)$ is a positive definite Hermitian matrix.

Proof. The first assertion is obvious. For the second one, we note that there is a matrix $A \in \mathrm{GL}_r(\mathbb{C})$ such that ${}^tA \cdot \overline{A} = H_\alpha(y)$. Then it is easy to see that $\rho(C^\infty(Y_\alpha(\mathbb{C})))(H_\alpha)(y)$ is a positive definite Hermitian matrix by using (9.2.1). $\square$

Now we define a metric h^{ρ_α} on $\mathcal{O}_{Y_\alpha}^{\oplus R}$ over Y_α by

$$h^{\rho_\alpha}(f_k^\alpha, f_l^\alpha) = \rho(C^\infty(Y_\alpha(\mathbb{C})))(H_\alpha)_{kl}$$

for $1 \le k, l \le R$.

Claim 9.2.3. $\{h^{\rho_\alpha}\}_\alpha$ glue together to form a Hermitian metric on E^ρ.

Proof. Let $s_\alpha = {}^t(s_1^\alpha, \cdots, s_R^\alpha) \in \Gamma(\mathcal{O}_{Y_\alpha}^{\oplus R}|_{Y_\alpha \cap Y_\beta})$ and $s_\beta = {}^t(s_1^\beta, \cdots, s_R^\beta) \in \Gamma(\mathcal{O}_{Y_\beta}^{\oplus R}|_{Y_\alpha \cap Y_\beta})$. Then they give the same section of $E^\rho|_{Y_\alpha \cap Y_\beta}$ if ${}^t(s_1^\alpha, \cdots, s_R^\alpha) = \rho(g_{\alpha\beta})^t(s_1^\beta, \cdots, s_R^\beta)$. In this case, we write $s_\alpha \sim s_\beta$. Now we take $s_\alpha \sim s_\beta$ and $t_\alpha \sim t_\beta$. Then by a straightforward calculation using (9.2.1) and $H_\beta = {}^tg_{\alpha\beta} H_\alpha \overline{g_{\alpha\beta}}$, we get $h^{\rho_\alpha}(s_\alpha, t_\alpha) = h^{\rho_\beta}(s_\beta, t_\beta)$ on $Y_\alpha \cap Y_\beta$. $\square$

Remark 9.2.4. Let $\mathrm{id}_r : \mathrm{GL}_r \to \mathrm{GL}_r$ be the identity morphism, $\rho_1 = (\mathrm{id}_r)^{\otimes k}$, $\rho_2 = \mathrm{Sym}^k(\mathrm{id}_r)$, and $\rho_3 = \bigwedge^k(\mathrm{id}_r)$. Further, let $\rho_4 : \mathrm{GL}_r \to \mathrm{GL}_r$ be the group homomorphism given by $A \mapsto {}^tA^{-1}$. Then ρ_1, ρ_2, ρ_3 and ρ_4 are of degree k, k, k and -1, respectively. Let (E, h) be a Hermitian vector bundle of rank r. Then the associated vector bundles are $(E^{\otimes k}, h^{\otimes k})$, $(\mathrm{Sym}^k(E), h^{\rho_2})$, $(\bigwedge^k(E), h^{\rho_3})$ and $(E^\vee, h^\vee)$. Note, for example, that h^{ρ_2} is not the quotient metric h_{quot} given by $E^{\otimes k} \to \mathrm{Sym}^k(E)$; Indeed, for a locally orthogonal basis $e_1, \cdots, e_r$ of $\overline{E}$ and $\alpha_1, \cdots, \alpha_r \in \mathbb{Z}$, $h^{\rho_2}(e_1^{\alpha_1} \cdots e_r^{\alpha_r}, e_1^{\alpha_1} \cdots e_r^{\alpha_r}) = 1$, while $h_{quot}(e_1^{\alpha_1} \cdots e_r^{\alpha_r}, e_1^{\alpha_1} \cdots e_r^{\alpha_r}) = \alpha_1! \cdots \alpha_r!/r!$.

9.3. Chow forms and their metrics

Let Y be a regular arithmetic variety, and $\overline{E} = (E, h)$ a Hermitian vector bundle of rank r on Y.

Let $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ be a group scheme morphism of degree k commuting with the transposed morphism and $\overline{E}^{\rho} = (E^{\rho}, h^{\rho})$ the associated Hermitian bundle of rank R. We give the quotient metric on $\mathcal{O}_{E^{\rho}}(1)$ via $\pi^*(E^{\rho\vee}) \to \mathcal{O}_{E^{\rho}}(1)$. We denote this Hermitian line bundle by $\overline{\mathcal{O}_{E^{\rho}}(1)}$. Further, let $\Omega_{\rho} = c_1(\overline{\mathcal{O}_{E^{\rho}}(1)})$ be the first Chern form.

Let X be an effective cycle in $\mathbb{P}(E^{\rho})$ such that X is flat over Y with the relative dimension d and degree δ on the generic fiber. Let g_X be a Ω_{ρ}-normalized Green form for X and we set $\widehat{X} = (X, g_X)$. Then $\widehat{X} \in \widehat{Z}^{R-1-d}_{L^1}(\mathbb{P}(E^{\rho}))$. Thus $\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X}$ belongs to $\widehat{\mathrm{CH}}^{R}_{L^1}(\mathbb{P}(E^{\rho}))_{\mathbb{Q}}$. Hence,

$$\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X}\right) \in \widehat{\mathrm{CH}}^{1}_{L^1}(Y)_{\mathbb{Q}}.$$

Let us consider elementary properties of $\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X}\right)$.

Proposition 9.3.1. *Let $X = \sum_{k=1}^{l} a_k X_k$ be the irreducible decomposition of X as cycles. Let $\phi_k : \tilde{X}_k \to X_k$ be a generic resolution of singularities of X_k for each k, i.e., ϕ_k is a proper birational morphism such that $(\tilde{X}_k)_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$. Let $i_k : X_k \hookrightarrow \mathbb{P}(E^{\rho})$ be the inclusion map and $j_k : \tilde{X}_k \to \mathbb{P}(E^{\rho})$ the composition map $i_k \cdot \phi_k$. Also we let $f_k : X_k \to Y$ be the composition map $\pi \cdot i_k$ and $\tilde{f}_k : \tilde{X}_k \to Y$ the composition map $\pi \cdot j_k$. Let Y_0 be the maximal open set of Y such that $\tilde{f}_k$ is smooth over there for every k. Then, we have the following.*

(1) $$\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X}\right) = \sum_{k=1}^{l} a_k \tilde{f}_{k*}(\widehat{c}_1(j_k^*\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1}).$$

In particular, $\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X}\right)$ is independent of the choice of normalized Green forms g_X for X, and $\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X}\right) \in \widehat{\mathrm{CH}}^{1}_{L^1}(Y; Y_0(\mathbb{C}))$.

(2) *Let y be a closed point of $(Y_0)_{\mathbb{Q}}$, and Γ' the closure of $\{y\}$ in Y. Here we choose g_X as in Proposition 9.1.1. Then, there is a representative (Z, g_Z) of $\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X}$ such that $\pi^{-1}(\Gamma')$ and Z intersect properly, and $g_Z|_{\pi^{-1}(z)}$ is locally integrable for each $z \in O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)$.*

Proof. We may assume that X is reduced and irreducible, so that we will omit index k in the following.

(1) Let g_X be a Ω_{ρ}-normalized Green form for X. Then, by virtue of Proposition 2.4.2,

$$\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X} = j_*\left(\widehat{c}_1(j^*\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1}\right) + a(\Omega_{\rho}^{d+1} \wedge [g_X]).$$

Therefore, since $\pi_*(g_X \wedge \Omega_\rho^{d+1}) = 0$ by the definition of g_X, we get

$$\pi_* \left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot \widehat{X}\right) = \pi_* j_* \left(\widehat{c}_1(j^*\overline{\mathcal{O}_{E^\rho}(1)})^{d+1}\right) = \tilde{f}_* \left(\widehat{c}_1(j^*\overline{\mathcal{O}_{E^\rho}(1)})^{d+1}\right).$$

(2) First of all, we need the following lemma.

Lemma 9.3.2. *Let T be a quasi-projective integral scheme over $\mathbb{Z}$, $L_1, \ldots, L_n$ line bundles on T, and Γ a cycle on T. Then, there is a cycle Z on T such that Z is rationally equivalent to $c_1(L_1)\cdots c_1(L_n)$, and that Z and Γ intersect properly.*

Proof. We prove this lemma by induction on n. First, let us consider the case $n = 1$. Let $\Gamma = \sum_{i=1}^r a_i\Gamma_i$ be the irreducible decomposition as cycles. Let γ_i be a closed point of $\Gamma_i \setminus \bigcup_{j\neq i}\Gamma_j$, and m_i the maximal ideal at γ_i. Let H be an ample line bundle on X. Choose a sufficiently large integer N such that

$$H^1(T, H^{\otimes N} \otimes m_1 \otimes \cdots \otimes m_r) = H^1(T, H^{\otimes N} \otimes L_1 \otimes m_1 \otimes \cdots \otimes m_r) = 0.$$

Then, the natural homomorphisms

$$H^0(T, H^{\otimes N}) \to \bigoplus_{i=1}^r H^{\otimes N} \otimes \kappa(\gamma_i)$$
$$\text{and} \quad H^0(T, H^{\otimes N} \otimes L_1) \to \bigoplus_{i=1}^r H^{\otimes N} \otimes L_1 \otimes \kappa(\gamma_i)$$

are surjective. Thus, there are sections $s_1 \in H^0(T, H^{\otimes N})$ and $s_2 \in H^0(T, H^{\otimes N} \otimes L_1)$ such that $s_1(\gamma_i) \neq 0$ and $s_2(\gamma_i) \neq 0$ for all i. Then, $\operatorname{div}(s_2) - \operatorname{div}(s_1) \sim c_1(L_1)$, and $\operatorname{div}(s_2) - \operatorname{div}(s_1)$ and Γ intersect properly.

Next we assume $n > 1$. Then, by hypothesis of induction, there is a cycle Z' such that $Z' \sim c_1(L_1)\cdots c_1(L_{n-1})$, and Z' and Γ intersect properly. Let $Z' = \sum_j b_j T_j$ be the decomposition as cycles. We set $\Gamma_j = (T_j \cap \operatorname{Supp}(\Gamma))_{red}$. Then, using the case $n = 1$, there is a cycle Z_j such that $Z_j \sim c_1(L_n|_{T_j})$, and Z_j and Γ_j intersect properly. Thus, if we set $Z = \sum_j b_j Z_j$, then $Z \sim c_1(L_1)\cdots c_1(L_n)$, and Z and Γ intersect properly. □

Let us go back to the proof of (2) of Proposition 9.3.1. By virtue of Lemma 9.3.2, there is a cycle Z on X such that $Z \sim c_1\left(i^*\mathcal{O}_{E^\rho}(1)\right)^{d+1}$, and that Z and $f^{-1}(\Gamma')$ intersect properly. Then, $Z \sim c_1(\mathcal{O}_{E^\rho}(1))^{d+1} \cdot X$, and Z and $\pi^{-1}(\Gamma')$ intersect properly. Let ϕ_X be a Green form of logarithmic type for X. Then, since

$$\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot (X, \phi_X) \in \widehat{\mathrm{CH}}^R(\mathbb{P}(E^\rho)),$$

there is a Green form ϕ_Z of logarithmic type for Z such that (Z, ϕ_Z) is a representative of $\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot (X, \phi_X)$. Thus, if we set

$$g_Z = \phi_Z + c_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \wedge (g_X - \phi_X),$$

then (Z, g_Z) is a representative of $\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot \widehat{X}$. Since Z and $\pi^{-1}(\Gamma')$ intersect properly and g_X has the property in Proposition 9.1.1, we can easily see that g_Z is locally integrable along $\pi^{-1}(z)$ for each $z \in O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)$. $\square$

Here we recall some elementary results of Chow forms. Details can be found in [4]. Consider the incidence subscheme Γ in the product

$$\mathbb{P}(E^\rho) \times_Y \mathbb{P}(E^{\rho\vee})^{d+1} = \mathbb{P}(E^\rho) \times_Y \mathbb{P}(E^{\rho\vee}) \times_Y \cdots \times_Y \mathbb{P}(E^{\rho\vee}).$$

Let $\imath : \Gamma \to \mathbb{P}(E^\rho)$ and $\jmath : \Gamma \to \mathbb{P}(E^{\rho\vee})^{d+1}$ be projection maps. The Chow divisor $\mathrm{Ch}(X)$ of X is defined by

$$\mathrm{Ch}(X) = \jmath_* \imath^*(X).$$

The following facts are well-known:

1. $\mathrm{Ch}(X)$ is an effective cycle of codimension 1 in $\mathbb{P}(E^{\rho\vee})^{d+1}$;
2. $\mathrm{Ch}(X)$ is flat over Y;
3. For any $y \in Y$, $\mathrm{Ch}(X)_y$ is a divisor of type $(\delta, \delta, \cdots, \delta)$ in $\mathbb{P}(E^{\rho\vee})^{d+1}_y$.

Let $p : \mathbb{P}(E^{\rho\vee})^{d+1} \to Y$ be the canonical morphism, and $p_i : \mathbb{P}(E^{\rho\vee})^{d+1} \to \mathbb{P}(E^{\rho\vee})$ the projection to the i-th factor. Then, by the above properties, there is a line bundle L on Y and a section Φ_X of

$$H^0\left(\mathbb{P}(E^{\rho\vee})^{d+1}, p^*(L) \otimes \bigotimes_{i=1}^{d+1} p_i^* \mathcal{O}_{E^{\rho\vee}}(\delta)\right)$$

such that $\mathrm{div}(\Phi_X) = \mathrm{Ch}(X)$. Since

$$p_*\left(p^*(L) \otimes \bigotimes_{i=1}^{d+1} p_i^* \mathcal{O}_{E^{\rho\vee}}(\delta)\right) = L \otimes (\mathrm{Sym}^\delta(E^\rho))^{\otimes d+1},$$

we may view Φ_X as an element of $H^0(Y, L \otimes (\mathrm{Sym}^\delta(E^\rho))^{\otimes d+1})$. We say Φ_X is a *Chow form* of X. Clearly Φ_X is uniquely determined up to $H^0(Y, \mathcal{O}_Y^\times)$.

As in [4, Proposition 1.2 and its remark], we have

$$c_1(L) = \pi_*\left(c_1(\mathcal{O}_{E^\rho}(1))^{d+1} \cdot X\right).$$

We give a generalized metric h_L on L so that $\overline{L} = (L, h_L)$ satisfies the equality

$$\widehat{c}_1(\overline{L}) = \pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot \widehat{X}\right) \tag{9.3.3}$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y)$.

Note that we can also give a metric L by the equation

$$\mathcal{O}_{\mathbb{P}(E^{\rho\vee})^{d+1}}(\mathrm{Ch}(X)) = p^*(L) \otimes \bigotimes_{i=1}^{d+1} p_i^* \mathcal{O}_{E^\vee}(\delta)$$

and by suitably metrizing other terms, as is implicitly done in [27, 1.5]. We do not however pursue this here.

9.4. Restriction of Chow forms on fibers

In this section we will consider the restriction of Chow forms on fibers.

Let $Y, \overline{E}, \rho, X$ be as in Section 9.3. Let y be a closed point of $Y_{\mathbb{Q}}$. Let Γ' be the closure of $\{y\}$ in Y, and Γ the normalization of Γ'. Let $f : \Gamma \to Y$ be the natural morphism. We set $E_\Gamma = f^*(E)$ and $\overline{E_\Gamma} = (E_\Gamma, f^*(h))$. Also we put $(E^\rho)_\Gamma = f^*(E^\rho)$ and $\overline{(E^\rho)_\Gamma} = ((E^\rho)_\Gamma, f^*(h^\rho))$. Then $(\overline{E_\Gamma})^\rho$ is equal to $\overline{(E^\rho)_\Gamma}$, so that we denote $(E^\rho)_\Gamma$ by E^ρ_Γ, and $\overline{(E^\rho)_\Gamma}$ by $\overline{E^\rho_\Gamma}$. Considering the following fiber product

$$\begin{array}{ccc} \mathbb{P}(E^\rho_\Gamma) & \xrightarrow{f'} & \mathbb{P}(E^\rho) \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ \Gamma & \xrightarrow{f} & Y \end{array}$$

we set $X_\Gamma = f'^*(X)$. Then X_Γ is flat over Γ with the relative dimension d and the degree δ on the generic fiber. For this quadruplet $(\Gamma, \overline{E_\Gamma}, \rho, X_\Gamma)$ in place of the quadruplet $(Y, \overline{E}, \rho, X)$, we can define in the same way the Hermitian line bundle $\overline{\mathcal{O}_{E^\rho_\Gamma}(1)}$ on $\mathbb{P}(E^\rho_\Gamma)$, an arithmetic L^1-divisor $\widehat{X_\Gamma} = (X_\Gamma, g_{X_\Gamma})$ on $\mathbb{P}(E^\rho_\Gamma)$ and the arithmetic L^1-divisor $\pi'_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho_\Gamma}(1)})^{d+1} \cdot \widehat{X_\Gamma}\right)$ on Γ. Further, we have $\overline{L_\Gamma} = (L_\Gamma, h_\Gamma)$ satisfying

$$\widehat{c}_1(\overline{L_\Gamma}) = \pi'_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho_\Gamma}(1)})^{d+1} \cdot \widehat{X_\Gamma}\right)$$

in $\widehat{\mathrm{CH}}^1_{L^1}(\Gamma)$. We also have $\mathrm{Ch}(X_\Gamma)$. Moreover, letting $p'_i : \mathbb{P}((E^\rho_\Gamma)^\vee)^{d+1} \to \mathbb{P}((E^\rho_\Gamma)^\vee)$ be the i-th projection, there is a section Φ_{X_Γ} of

$$\begin{aligned} H^0&\left(\mathbb{P}((E^\rho_\Gamma)^\vee)^{d+1}, p'^*(L_\Gamma) \otimes \bigotimes_{i=1}^{d+1} {p'_i}^* \mathcal{O}_{(E^\rho_\Gamma)^\vee}(\delta)\right) \\ &= H^0\left(\Gamma, L_\Gamma \otimes (\mathrm{Sym}^\delta((E^\rho_\Gamma)^\vee))^{\otimes d+1}\right), \end{aligned}$$

such that $\mathrm{div}(\Phi_{X_\Gamma}) = \mathrm{Ch}(X_\Gamma)$.

Let us consider the following fiber product,

$$\begin{array}{ccc} \mathbb{P}((E^\rho_\Gamma)^\vee)^{d+1} & \xrightarrow{g'} & \mathbb{P}((E^\rho)^\vee)^{d+1} \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ \Gamma & \xrightarrow{f} & Y \end{array}$$

Then, we have the following proposition.

Proposition 9.4.1. (i) $g'^* \operatorname{Ch}(X) = \operatorname{Ch}(X_\Gamma)$. *Moreover, we can choose* Φ_{X_Γ} *to be* $f^*\Phi_X$.

(ii) *Let* $X_1, \cdots, X_l$ *be the irreducible components of* X_{red}. *Assume that, for every* $1 \le i \le l$, *there is a generic resolution of singularities* $\phi_i : \tilde{X}_i \to X_i$ *such that the induced map* $\tilde{X}_i \to Y$ *is smooth over* y *for every* i. *Then the equality*

$$\begin{aligned} f^*\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1}\cdot\widehat{X}\right) &= \pi'_* f'^*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1}\cdot\widehat{X}\right) \\ &= \pi'_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho_\Gamma}(1)})^{d+1}\cdot\widehat{X_\Gamma}\right). \end{aligned}$$

holds in $\widehat{\operatorname{CH}}^1_{L^1}(\Gamma)$. *In other words,* $f^*(L, h_L) = (L_\Gamma, h_{L_\Gamma})$.

Proof. (i) If f is flat, then this follows from the base change theorem. In the case f is not flat, we refer to the remark [5, 4.3.2(i)], or we can easily see this using Appendix A.

(ii) We take g_X as in Proposition 9.1.1. Let $\alpha = \widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1}\cdot\widehat{X} \in \widehat{\operatorname{CH}}^1_{L^1}(Y)$. By Proposition 9.3.1, we can take a representative (Z, g_Z) of α such that Z and $\pi^{-1}(\Gamma')$ intersect properly, and g_Z is locally integrable along $\pi^{-1}(w)$ for all $w \in O_{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)$. Now we have

$$\begin{aligned} f^*\pi_*(\alpha) &= f^*(\pi_* Z, [\pi_* g_Z]) \\ &= \left(f^*\pi_* Z, \sum_{w\in O_{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)}\left(\int_{\pi^{-1}(w)} g_Z\right)\cdot w\right). \end{aligned}$$

On the other hand, we have

$$\pi'_* f'^*(\alpha) = \left(\pi'_* f'^* Z, \sum_{w\in O_{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)}\left(\int_{\pi^{-1}(w)} g_Z\right)\cdot w\right).$$

Moreover, by Appendix A, $f^*\pi_* Z$ is equal to $\pi'_* f'^* Z$ as a cycle. Thus we have proven the first equality.

Now we will prove the second equality. Let ϕ_X be a Green form of logarithmic type for X. Since

$$f'^* : \bigoplus_{i\ge 0}\widehat{\operatorname{CH}}^i(\mathbb{P}(E^\rho)) \to \bigoplus_{i\ge 0}\widehat{\operatorname{CH}}^i(\mathbb{P}(E^\rho_\Gamma))$$

is a homomorphism of rings (cf. [9, 5) of Theorem in 4.4.3]). Thus,

$$f'^*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1}\cdot(X,\phi_X)\right) = f'^*\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1}\cdot f'^*(X,\phi_X).$$

Therefore, since we take g_X as in Proposition 9.1.1, we can see

$$\begin{aligned}
f'^* \left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot \widehat{X}\right) &= f'^* \left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot ((X, \phi_X) + (0, g_X - \phi_X))\right) \\
&= f'^* \left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot (X, \phi_X)\right) \\
&\quad + a \left(f'^*(c_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \wedge (g_X - \phi_X))\right) \\
&= f'^* \widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot f'^*(X, \phi_X) \\
&\quad + a \left(f'^*(c_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \wedge (g_X - \phi_X))\right) \\
&= f'^* \widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot f'^*(X, g_X)
\end{aligned}$$

Moreover, as pointed out in Remark 9.1.4, $f'^* g_X$ is a normalized Green form for $f'^* X$. Thus we have got the second equality. □

9.5. Chow stability and field extensions

Let $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ be a group scheme morphism of degree k commuting with the transposed morphism. Let S be a ring (commutative, with the multiplicative identity). For a positive integer δ and d, we consider $\mathrm{Sym}^\delta(S^R)^{\otimes d+1}$. Then through the induced group homomorphism $\rho(S) : \mathrm{GL}_r(S) \to \mathrm{GL}_R(S)$, $\mathrm{GL}_r(S)$ (or $\mathrm{SL}_r(S)$) acts on $\mathrm{Sym}^\delta(S^R)^{\otimes d+1}$.

Proposition 9.5.1. *Let K be an infinite field and L an extension field of K. Let P be a homogeneous polynomial of degree e on $\mathrm{Sym}^\delta(L^R)^{\otimes d+1}$, i.e., $P \in \mathrm{Sym}^e(\mathrm{Sym}^\delta(L^R)^{\otimes d+1\vee})$. Then we have the following.*

(i) *P is $\mathrm{SL}_r(K)$-invariant if and only if P is $\mathrm{SL}_r(L)$-invariant.*

(ii) *If P is $\mathrm{SL}_r(K)$-invariant, then*

$$P(v^\sigma)^r = (\det \sigma)^{ek(d+1)\delta} P(v)^r$$

in L for all $v \in \mathrm{Sym}^\delta(L^R)^{\otimes d+1}$ and $\sigma \in \mathrm{GL}_r(L)$.

Proof. (i) We only need to prove the 'only if' part. Let $S_L(P) = \{\sigma \in \mathrm{SL}_r(L) \mid P^\sigma = P\}$ be the stabilizer of P in $\mathrm{SL}_r(L)$. $S_L(P)$ is a closed algebraic set of $\mathrm{SL}_r(L)$ and contains $\mathrm{SL}_r(K)$. Since $\mathrm{SL}_r(K)$ is Zariski dense in $\mathrm{SL}_r(L)$, $S_L(P)$ must coincide with $\mathrm{SL}_r(L)$.

(ii) Let M be an extension field of L such that it has an r-th root ξ of $\det \sigma$. If σ' is defined by $\sigma = \xi\sigma'$, then $\sigma' \in \mathrm{SL}_r(M)$. Since P is $\mathrm{SL}_r(M)$-invariant by (i), we find

$$\begin{aligned}
P(v^\sigma)^r = P\left((\rho(\mathbb{C})(\xi\sigma')) \cdot v\right)^r &= P\left((\xi^k \rho(\mathbb{C})(\sigma')) \cdot v\right)^r \\
&= \xi^{rek(d+1)\delta} P(v)^r = (\det \sigma)^{ek(d+1)\delta} P(v)^r.
\end{aligned}$$

in M. Hence the equality holds in L because both sides belong to L. □

Remark 9.5.2. More strongly, we can show that, for any integral domain S of characteristic zero, if $P \in \mathrm{Sym}^e(\mathrm{Sym}^\delta(S^R)^{\otimes d+1\vee})$ is $\mathrm{SL}_r(\mathbb{Z})$-invariant, then P is $\mathrm{SL}_r(S)$-invariant.

Now let Y, $\overline{E}$, ρ and X be as in Section 9.3. Recall that for a closed point y of $Y_\mathbb{Q}$, $\mathrm{Ch}(X)_y$ is a divisor of type $(\delta, \delta, \cdots, \delta)$ in $\mathbb{P}(E^{\rho\vee})_y^{d+1}$. Hence the Chow form restricted on y, i.e., $\Phi_X|_y = \Phi_{X_y}$ is an element of $\mathrm{Sym}^\delta(K^R)^{d+1}$. We say that X_y is *Chow semi-stable* if $\Phi_{X_y} \in \mathrm{Sym}^\delta(K^R)^{d+1}$ is semi-stable under the action of $\mathrm{SL}_r(K)$, where K is the residue field of y.

Lemma 9.5.3. *There are a positive integer e and $\mathrm{SL}_r(\mathbb{Q})$-invariant homogeneous polynomials $P_1, \cdots, P_l \in \mathrm{Sym}^e(\mathrm{Sym}^\delta(\mathbb{Z}^R)^{d+1\vee})$, which depend only on ρ, d and δ, with the following property. For any closed points $y \in Y_\mathbb{Q}$, if X_y is Chow semistable, then there is a P_i such that $P_i(\Phi_{X_y}) \neq 0$.*

Proof. $\mathrm{SL}_r(\mathbb{Q})$ acts linearly on $\mathrm{Sym}^\delta(\mathbb{Q}^R)^{d+1}$. Since $\mathrm{SL}_r(\mathbb{Q})$ is a reductive group, we can take $\mathrm{SL}_r(\mathbb{Q})$-invariant homogeneous polynomials $Q_1, \cdots, Q_l$ such that they form generators of the algebra of $\mathrm{SL}_r(\mathbb{Q})$ invariant polynomials on $\mathrm{Sym}^\delta(\mathbb{Q}^R)^{d+1}$. By clearing the denominators, we may assume that $Q_1, \cdots, Q_l$ is defined over $\mathbb{Z}$. Let e_i be the degree of Q_i for $i = 1, \cdots, l$. We take a positive integer e such that $e_i | e$ for $i = 1, \cdots, l$. We set $P_i = Q_i^{e/e_i}$.

Let us check that P_i's have the desired property. Since X_y is Chow semistable, there is a $\mathrm{SL}_r(K)$-invariant homogeneous polynomial F on $\mathrm{Sym}^\delta(K^R)^{d+1}$ with $F(\Phi_{X_y}) \neq 0$, where K is the residue field of y. Let us choose $\alpha_1, \ldots, \alpha_n \in K$ and homogeneous polynomials $F_1, \ldots, F_n$ over $\mathbb{Q}$ such that $F = \alpha_1 F_1 + \cdots + \alpha_n F_n$ and that $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Here, for $\sigma \in \mathrm{SL}_r(\mathbb{Q})$,

$$F^\sigma = \alpha_1 F_1^\sigma + \cdots + \alpha_n F_n^\sigma$$

and F_i^σ's are homogeneous polynomials over $\mathbb{Q}$. Thus, we can see that F_i's are $\mathrm{SL}_r(\mathbb{Q})$-invariant. Moreover, since

$$F(\Phi_{X_y}) = \alpha_1 F_1(\Phi_{X_y}) + \cdots + \alpha_n F_n(\Phi_{X_y}),$$

there is F_i with $F_i(\Phi_{X_y}) \neq 0$. On the other hand, F_i is an element of $\mathbb{Q}[Q_1, \ldots, Q_l]$. Thus, we can find Q_j with $Q_j(\Phi_{X_y}) \neq 0$, namely $P_j(\Phi_{X_y}) \neq 0$. $\square$

10. Semi-stability and positiveness in a relative case

10.1. Cornalba-Harris-Bost's inequality in a relative case

Let Y be an arithmetic variety and $\overline{E} = (E, h)$ a Hermitian vector bundle of rank r on Y. Let $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ be a group scheme morphism of degree k commuting with the transposed morphism.

Before we prove the relative Cornalba-Harris-Bost's inequality, we need three lemmas.

Lemma 10.1.1. *Let L be a line bundle on Y. Let P be an element of $\operatorname{Sym}^e(\operatorname{Sym}^\delta(\mathbb{Z}^R)^{d+1\vee})\backslash\{0\}$ and suppose that P is $\mathrm{SL}_r(\mathbb{Q})$-invariant. Then there is a polynomial map of sheaves*

$$\varphi_P : L \otimes \operatorname{Sym}^\delta(E^\rho)^{\otimes d+1} \to L^{\otimes er} \otimes (\det E)^{\otimes ek(d+1)\delta}$$

given by P^r, namely, φ_P is locally defined by the evaluation in terms of P^r.

Proof. Let U be a Zariski open set, and $\phi : E|_U \xrightarrow{\sim} \mathcal{O}_U^{\oplus n}$ and $\psi : L|_U \xrightarrow{\sim} \mathcal{O}_U$ local trivializations of E and L respectively. Then, by the construction of E^ρ, we have

$$\phi_{\rho,\delta,d} : \left(\operatorname{Sym}^\delta\left(E^\rho\right)^{\otimes d+1}\right)\Big|_U \xrightarrow{\sim} \operatorname{Sym}^\delta\left(\mathcal{O}_U^{\oplus R}\right)^{\otimes d+1}.$$

Thus we get

$$\psi \otimes \phi_{\rho,\delta,d} : \left(L \otimes \operatorname{Sym}^\delta\left(E^\rho\right)^{\otimes d+1}\right)\Big|_U \xrightarrow{\sim} \operatorname{Sym}^\delta\left(\mathcal{O}_U^{\oplus R}\right)^{\otimes d+1}.$$

Here, we define

$$\varphi_P|_U : \left(L \otimes \operatorname{Sym}^\delta(E^\rho)^{\otimes d+1}\right)\Big|_U \to \left(L^{\otimes er} \otimes (\det E)^{\otimes ek(d+1)\delta}\right)\Big|_U$$

such that the following diagram is commutative.

$$\begin{array}{ccc}
\left(L \otimes \operatorname{Sym}^\delta\left(E^\rho\right)^{\otimes d+1}\right)\Big|_U & \xrightarrow{\psi\otimes\phi_{\rho,\delta,d}} & \operatorname{Sym}^\delta\left(\mathcal{O}_U^{\oplus R}\right)^{\otimes d+1} \\
{\scriptstyle \varphi_P|_U}\downarrow & & \downarrow{\scriptstyle P^r} \\
\left(L^{\otimes er} \otimes (\det E)^{\otimes ek(d+1)\delta}\right)\big|_U & \xrightarrow{\psi^{er}\otimes\det(\phi)^{ek(d+1)\delta}} & \mathcal{O}_U,
\end{array}$$

where P^r is the map given by the evaluation in terms of the polynomial P^r. In order to see that the local $\varphi_P|_U$ glues together on Y, it is sufficient to show that $\varphi_P|_U$ does not depend on the choice of local trivializations ϕ and ψ. Let $\phi' : E|_U \xrightarrow{\sim} \mathcal{O}_U^{\oplus n}$ and $\psi' : L|_U \xrightarrow{\sim} \mathcal{O}_U$ be another local trivializations. In the same way, we have the following commutative diagram.

$$\begin{array}{ccc}
\left(L \otimes \operatorname{Sym}^\delta\left(E^\rho\right)^{\otimes d+1}\right)\Big|_U & \xrightarrow{\psi'\otimes\phi'_{\rho,\delta,d}} & \operatorname{Sym}^\delta\left(\mathcal{O}_U^{\oplus R}\right)^{\otimes d+1} \\
{\scriptstyle \varphi'_P|_U}\downarrow & & \downarrow{\scriptstyle P^r} \\
\left(L^{\otimes er} \otimes (\det E)^{\otimes ek(d+1)\delta}\right)\big|_U & \xrightarrow{\psi'^{er}\otimes\det(\phi')^{ek(d+1)\delta}} & \mathcal{O}_U
\end{array}$$

We set the transition functions $g = \phi \cdot (\phi')^{-1}$ and $h = \psi \cdot (\psi')^{-1}$. Then by a straightforward calculation using (ii) of Proposition 9.5.1, we get, on U,

$$P^r \cdot (\psi \otimes \phi_{\rho,\delta,d}) = h^{re} \det(g)^{ek(d+1)\delta} P^r \cdot (\psi' \otimes \phi'_{\rho,\delta,d}),$$

which implies

$$\begin{aligned}&\left(\psi^{er} \otimes \det(\phi)^{ek(d+1)\delta}\right) \cdot \left(\varphi_P|_U\right) \\ &\qquad = h^{re} \det(g)^{ek(d+1)\delta} \left(\psi'^{er} \otimes \det(\phi')^{ek(d+1)\delta}\right) \cdot \left(\varphi'_P|_U\right).\end{aligned}$$

Here note that

$$h^{re} \det(g)^{ek(d+1)\delta} = \left(\psi^{er} \otimes \det(\phi)^{ek(d+1)\delta}\right) \cdot \left(\psi'^{er} \otimes \det(\phi')^{ek(d+1)\delta}\right)^{-1}.$$

Thus, we obtain $\varphi_P|_U = \varphi'_P|_U$. □

Suppose now L is given a generalized metric h_L. Since both sides of

$$\varphi_P : L \otimes \mathrm{Sym}^\delta(E^\rho)^{\otimes d+1} \to L^{\otimes er} \otimes (\det E)^{\otimes ek(d+1)\delta}$$

in the lemma above are then equipped with metrics, we can consider the norm of φ_P. Before evaluating the norm of φ_P, we define the norm of P as follows; We first define the metric $\| \cdot \|_{can}$ on $\mathrm{Sym}^\delta(\mathbb{C}^n)^{\otimes d+1}$ induced from the usual Hermitian metric on $\mathbb{C}$; We then define $\|\!|P|\!\|$ by

$$\|\!|P|\!\| = \sup_{v \in \mathrm{Sym}^\delta(\mathbb{C}^n)^{\otimes d+1} \setminus \{0\}} \frac{|P(v)|}{\|v\|_{can}^e},$$

where P is regarded as an element of $\mathrm{Sym}^e((\mathrm{Sym}^\delta(\mathbb{C}^m))^{\otimes d+1})^\vee)$.

Lemma 10.1.2. *For any section $s \in H^0(Y, L \otimes (\mathrm{Sym}^\delta(E^\rho))^{\otimes d+1})$ and any complex point $y \in Y(\mathbb{C})$ around which $h_L \otimes (\mathrm{Sym}^\delta(h^\rho))^{\otimes d+1}$ is C^∞, we have*

$$\|\varphi_P(s)\|(y) \le \|\!|P|\!\|^r \, \|s\|^{er}(y).$$

Proof. By choosing bases, $\overline{E}(y)$ and in $\overline{L}(y)$ are isometric to $\mathbb{C}^n$ and $\mathbb{C}$ with the canonical metrics, respectively. Then, with respect to these bases, $\overline{E}^\rho$ is by its construction isomorphic to $\mathbb{C}^R$ with the canonical metric. Recalling that φ_P is given by the evaluation by P^r once we fix local trivializations of E and L, the desired inequality follows from the definition of $\|\!|P|\!\|$. □

Now let X be an effective cycle in $\mathbb{P}(E^\rho)$ such that X is flat over Y with the relative dimension d and the degree δ on the generic fiber. In Section 9.3 we constructed a Chow form Φ_X of X, which is an element of $H^0(Y, L \otimes (\mathrm{Sym}^\delta(E^\rho))^{\otimes d+1})$. Recall that L is given a generalized metric by (9.3.3). For each irreducible component X_i of X_{red}, let $\tilde{X}_i \to X_i$ be a generic resolution of singularities of X_i. Moreover, let Y_0 be the maximal open set of Y such that the induced morphism $\tilde{X}_i \to Y$ is smooth over Y_0 for every i.

Further, we fix terminologies. Let T be a quasi-projective scheme over $\mathbb{Z}$, t a closed point of $T_{\mathbb{Q}}$, and K the residue field of t. By abuse use of notation, let $t : \mathrm{Spec}(K) \to T$ be the induced morphism by t. We say t is *extensible in* T if $t : \mathrm{Spec}(K) \to T$ extends to $\tilde{t} : \mathrm{Spec}(O_K) \to T$, where O_K is the ring of integers in K. Note that if T is projective over $\mathbb{Z}$, then every closed point of $T_{\mathbb{Q}}$ is extensible in T.

Let V be a set, ϕ a non-negative function on V, and S a finite subset of V. We define the geometric mean $\mathrm{g.m.}(\phi; S)$ of ϕ over S to be

$$\mathrm{g.m.}(\phi; S) = \left(\prod_{s \in S} \phi(s) \right)^{1/\#(S)}.$$

We will evaluate the norm of Φ_X.

Lemma 10.1.3. *There is a constant $c_1(R, d, \delta)$ depending only on R, d and δ with the following property. For any closed points y of $(Y_0)_{\mathbb{Q}}$ with y extensible in Y,*

$$\mathrm{g.m.} \left(\|\Phi_X\|_{\overline{L} \otimes (\mathrm{Sym}^{\delta}(\overline{E}^{\rho}))^{\otimes d+1}};\ O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y) \right) \leq c_1(R, d, \delta).$$

Proof. Let K be the residue field of y. Let Γ be the normalization of the closure of $\{y\}$ in Y. Then, since y is extensible in Y, $\Gamma = \mathrm{Spec}(O_K)$. Thus, by virtue of Proposition 9.4.1, we may assume $Y = \mathrm{Spec}(O_K)$. In this case, the estimate of the Chow form was already given in [4, Proposition 1.3] and [5, 4.3]. Indeed if we let k_L be the metric on L such that

$$\|\Phi_X\|_{(L, k_L) \otimes (\mathrm{Sym}^{\delta}(\overline{E}^{\rho}))^{\otimes d+1}}(w) = 1$$

for every $w \in O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)$, then $\widehat{\deg}(L, h_L) = h_{\overline{\mathcal{O}_E(1)}}(X)$ and $\widehat{\deg}(L, k_L) = h_{Herm}(\mathrm{Ch}(X))$, in the notation of [5]. $\square$

Now we will state a relative case of Cornalba-Harris-Bost's inequality.

Theorem 10.1.4. *Let Y be a regular arithmetic variety, $\overline{E} = (E, h)$ a Hermitian vector bundle of rank r on Y, $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ a group scheme morphism of degree k commuting with the transposed morphism. Let X be an effective cycle in $\mathbb{P}(E^{\rho})$ such that X is flat over Y with the relative dimension d and degree δ on the generic fiber. Let $X_1, \ldots, X_l$ be the irreducible components of X_{red}, and $\tilde{X}_i \to X_i$ a generic resolution of singularities of X_i. Let Y_0 be the maximal open set of Y such that the induced morphism $\tilde{X}_i \to Y$ is smooth over Y_0 for every i. Let (B, h_B) be a line bundle equipped with a generalized metric on Y given by the equality*:

$$\widehat{c}_1(B, h_B) = r\pi_* \left(\widehat{c}_1(\overline{\mathcal{O}_{E^{\rho}}(1)})^{d+1} \cdot \widehat{X} \right) + k\delta(d+1)\widehat{c}_1(\overline{E}).$$

Then, h_B is C^{∞} over Y_0. Moreover, there are a positive integer $e = e(\rho, d, \delta)$, a positive integer $l = l(\rho, d, \delta)$, a positive constant $C = C(\rho, d, \delta)$, and sections $s_1, \ldots, s_l \in H^0(Y, B^{\otimes e})$ with the following properties.

(i) *e, l, and C depend only on ρ, d, and δ.*

(ii) *For a closed point y of $Y_{\mathbb{Q}}$, if X_y is Chow semistable, then $s_i(y) \neq 0$ for some i.*

(iii) *For all i and all closed points y of $(Y_0)_{\mathbb{Q}}$ with y extensible in Y,*

$$\text{g.m.}\left(\left(h_B^{\otimes e}\right)(s_i, s_i);\ O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)\right) \leq C.$$

In particular, if we set

$$\beta = e\left(r\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot \widehat{X}\right) + k\delta(d+1)\widehat{c}_1(\overline{E})\right) + a(\log C),$$

then, for any closed point $y \in (Y_0)_{\mathbb{Q}}$ with X_y Chow semistable, there is a representative (D, g) of β such that D is effective, $y \notin D$, and that

$$\sum_{w \in O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(z)} g(w) \geq 0$$

for all $z \in (Y_0)_{\mathbb{Q}}$ with z extensible in Y.

Note that if ρ is the identity morphism, then, by the proof below, $C(\rho, d, \delta)$ is depending only on r, d, δ.

Proof. First of all, by Proposition 9.3.1,

$$r\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\rho}(1)})^{d+1} \cdot \widehat{X}\right) + k\delta(d+1)\widehat{c}_1(\overline{E}) \in \widehat{\mathrm{CH}}^1_{L^1}(Y; Y_0(\mathbb{C})).$$

Thus, h_B is C^∞ over $Y_0(\mathbb{C})$.

By Lemma 9.5.3, there are a positive integer e and $\mathrm{SL}_r(\mathbb{Q})$-invariant homogeneous polynomials $P_1, \cdots, P_l \in \mathrm{Sym}^e(\mathrm{Sym}^\delta(\mathbb{Z}^R)^{d+1\vee})$ depending only on ρ, d and δ such that if X_y is Chow semistable for a closed point y of $Y_{\mathbb{Q}}$, then $P_i(\Phi_{X_y}) \neq 0$ for some P_i. For later use, we put $c_2(\rho, d, \delta) = \max\{\|P_1\|, \cdots, \|P_l\|\}$, which is a constant depending only on ρ, d and δ.

Recall that the Chow form Φ_X is an element of $H^0(Y, L \otimes (\mathrm{Sym}^\delta(E^\rho))^{\otimes d+1})$ and by Lemma 10.1.1 P_i induces a polynomial map of sheaves

$$\varphi_{P_i} : L \otimes \mathrm{Sym}^\delta(E^\rho)^{\otimes d+1} \to L^{\otimes er} \otimes (\det E)^{ek(d+1)\delta}.$$

Hence we have

$$\varphi_{P_i}(\Phi_X) \in H^0\left(Y, L^{\otimes er} \otimes (\det E)^{ek(d+1)\delta}\right) = H^0(Y, B^{\otimes e})$$

by (9.3.3). Here we set $s_i = \varphi_{P_i}(\Phi_X)$. Then, the property (ii) is obvious by the construction of φ_{P_i} and (i) of Proposition 9.4.1.

Now we will evaluate $\|s_i\|$. Let y be a closed point of $(Y_0)_{\mathbb{Q}}$ with y extensible in Y. Combining Lemmas 10.1.2 and 10.1.3, we have

$$\begin{aligned} \text{g.m.}\left(\|s_i\|;\ O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)\right) &\leq \text{g.m.}\left(\|P_i\|^r\, \|\Phi_X\|^{er};\ O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(y)\right) \\ &\leq c_2(\rho, d, \delta)^r c_1(R, d, \delta)^{er}. \end{aligned}$$

Now we put

$$C(\rho, d, \delta) = c_1(R, d, \delta)^{2r} c_2(\rho, d, \delta)^{2er},$$

which is a positive constant depending only on ρ, d and δ. Thus, we get (iii). $\square$

Remark 10.1.5. Here let us consider the geometric analogue of Theorem 10.1.4. Let Y be an algebraic variety over an algebraically closed field k, E a vector bundle of rank r, $\rho : \mathrm{GL}_r \to \mathrm{GL}_R$ a group scheme morphism of degree l commuting with the transposed morphism. Let X be an effective cycle in $\mathbb{P}(E^\rho)$ such that X is flat over Y with the relative dimension d and degree δ on the generic fiber. Here we set

$$b_{X/Y}(E, \rho) = r\pi_* \left(c_1(\mathcal{O}_{E^\rho}(1))^{d+1} \cdot X \right) + l\delta(d+1)c_1(E),$$

which is a divisor on Y. In the same way as in the proof of Theorem 10.1.4 (actually, this case is much easier than the arithmetic case), we can show the following.

> There is a positive integer e depending only on ρ, d, and δ such that, if X_y is Chow semi-stable for some $y \in Y$, then
>
> $$H^0(Y, \mathcal{O}_Y(eb_{X/Y}(E, \rho))) \otimes \mathcal{O}_Y \to \mathcal{O}_Y(eb_{X/Y}(E, \rho))$$
>
> is surjective at y.

This gives a refinement of [4, Theorem 3.2].

10.2. Relationship of two theorems

In this subsection we will see some relationship between the relative Bogomolov's inequality (Theorem 8.1) and the relative Cornalba-Harris-Bost's inequality (Theorem 10.1.4). For this purpose, we will first show a more intrinsic version of Theorem 10.1.4.

Proposition 10.2.1. *Let $f : X \to Y$ be a flat morphism of regular projective arithmetic varieties with $\dim f = d$. Let L be a relatively very ample line bundle such that $E = f_*(L)$ is a vector bundle of rank r on Y. Let η be the generic point of X and $\delta = \deg(L_\eta^d)$. Moreover, let $i : X \to \mathbb{P}(E^\vee)$ be the embedding over Y. Assume that E is equipped with an Hermitian structure h so that L is also endowed with the Hermitian structure by $i^*\mathcal{O}_{E^\vee}(1) \simeq L$. Let Y_0 be the maximal open set of Y such that f is smooth over Y_0. Then, there is a positive integer $e(r, d, \delta)$ and a positive constant $C(r, d, \delta)$ depending only on r, d, δ with the following properties. If we set*

$$\beta = e(r, d, \delta) \left(rf_*(\widehat{c}_1(\overline{L})^{d+1}) - \delta(d+1)\widehat{c}_1(\overline{E}) \right) + a(\log C(r, d, \delta)),$$

then, for any closed point $y \in (Y_0)_{\mathbb{Q}}$ with X_y Chow semistable, there is a representative (D, g) of β such that D is effective, $y \notin D$, and

$$\sum_{w \in O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(z)} g(w) \geq 0$$

for all $z \in (Y_0)_{\mathbb{Q}}$.

Proof. We identify X with its image by i. Let $\pi : \mathbb{P}(E) \to Y$ be the projection. Then, by Proposition 9.3.1, we get

$$\pi_*\left(\widehat{c}_1(\overline{\mathcal{O}_{E^\vee}(1)})^{d+1}\cdot \widehat{X}\right) = f_*(\widehat{c}_1(\overline{L})^{d+1})$$

Thus, applying Theorem 10.1.4 for $(Y, E^\vee, \mathrm{id}, X)$, we get our assertion. □

The following proposition will be derived from Theorem 8.1.

Proposition 10.2.2. *Let* $f : X \to Y$ *be a projective morphism of regular arithmetic varieties such that every fiber of* $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ *is a reduced and connected curve with only ordinary double singularities. We assume that the genus* g *of the generic fiber of* f *is greater than or equal to* 1. *Let* L *be a line bundle on* X *such that* (1)*the degree* δ *of* L *on the generic fiber is greater than or equal to* $2g+1$, (2) $E = f_*(L)$ *is a vector bundle of rank* r *on* Y (*actually* $r = \delta + 1 - g$), *and that* (3) $f^*(E) \to L$ *is surjective. Assume that* E *is equipped with an Hermitian structure* h *so that* L *is also endowed with the quotient metric by* $f^*(E) \to L$. *Let* Y_0 *be the maximal open set of* Y *such that* f *is smooth over* Y_0. *Then, for any closed points* y *of* $(Y_0)_{\mathbb{Q}}$,

$$r f_*(\widehat{c}_1(\overline{L})^2) - 2\delta\widehat{c}_1(\overline{E})$$

is weakly positive at y *with respect to any finite subsets of* $Y_0(\mathbb{C})$.

Note that if the base space is $\mathrm{Spec}(O_K)$, then the second author showed in [21, Theorem 1.1] the above inequality (under weaker assumptions) using [18, Corollary 8.9]. Since we can prove Proposition 10.2.2 in the same way as [21, Theorem 1.1], we will only sketch the proof.

Proof. Let $S = \mathrm{Ker}(f^*(E) \to L)$ and h_S the submetric of S induced by h. Then, by [7], S_z is stable for all $z \in Y_0(\mathbb{C})$. Applying Theorem 8.1 for $\overline{S} = (S, h_S)$, we obtain that if y is a closed point of $(Y_0)_{\mathbb{Q}}$, then

$$f_*(2(r-1)\widehat{c}_2(\overline{S}) - (r-2)\widehat{c}_1(\overline{S})^2)$$

is weakly positive at y with respect to any finite subsets of $Y_0(\mathbb{C})$. If we set $\rho = \widehat{c}_2(f^*\overline{E}) - \widehat{c}_2(\overline{S}\oplus\overline{L})$, then there is $g \in L^1_{loc}(Y(\mathbb{C}))$ such that $f_*(\rho) = a(g)$, g is C^∞ over $Y_0(\mathbb{C})$, and $g > 0$ on $Y_0(\mathbb{C})$. Now by a straightforward calculation, we have

$$\begin{aligned} & f_*(2(r-1)\widehat{c}_2(\overline{S}) - (r-2)\widehat{c}_1(\overline{S})^2) + 2(r-1)f_*(\rho) \\ &\quad = f_*\left(2(r-1)\widehat{c}_2(f^*\overline{E}) - (r-2)\widehat{c}_1(f^*\overline{E})^2\right) + f_*\left(r\widehat{c}_1(\overline{L})^2 - 2\widehat{c}_1(f^*\overline{E})\cdot\widehat{c}_1(\overline{L})\right) \\ &\quad = r f_*(\widehat{c}_1(\overline{L})^2) - 2\delta\widehat{c}_1(\overline{E}). \end{aligned}$$

□

Let us compare Proposition 10.2.1 with Proposition 10.2.2. Both of them give some arithmetic positivity of the same divisor (although $d = 1$ in Proposition 10.2.2), under the assumption of some semi-stability (of Chow or of vector bundles). The former has advantage since it treats varieties of arbitrary relative dimension. On the other hand, the latter has advantage since it shows that the anonymous constant in the former is zero (see also [27]). Moreover, in the complex case, the counterpart of the relative Bogomolov's inequality of Theorem 8.1 has a wonderful application to the moduli of stable curves ([22]).

Appendix A. Commutativity of push-forward and pull-back

Let $f : X \to Y$ be a smooth proper morphism of regular noetherian schemes, and $u : Y' \to Y$ a morphism of regular noetherian schemes. Let $X' = X \times_Y Y'$ and

$$\begin{array}{ccc} X & \xleftarrow{u'} & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ Y & \xleftarrow{u} & Y' \end{array}$$

the induced diagram. Let Z be a cycle of X of codimension p and $|Z|$ the support of Z. We assume that $\operatorname{codim}_{X'}({u'}^{-1}(|Z|)) \geq p$. Then, it is easy to see that $\operatorname{codim}_{Y'}(u^{-1}(|f_*(Z)|)) \geq p-d$, where $d = \dim X - \dim Y$. Thus, we can define $f'_*({u'}^*(Z))$ and $u^*(f_*(Z))$ as elements of $Z^{p-d}(Y')$. It is well known, we believe, that $f'_*({u'}^*(Z)) = u^*(f_*(Z))$ in $Z^{p-d}(Y')$. We could not however find any suitable references for the above fact, so that in this section, we would like to give the proof of it.

Let X be a regular noetherian scheme, and T a closed subscheme of X. We denote by $K'_T(X)$ the Grothendieck group generated by coherent sheaves F with $\operatorname{Supp}(F) \subseteq T_{red}$ modulo the following relation: $[F] = [F'] + [F'']$ if there is an exact sequence $0 \to F' \to F \to F'' \to 0$.

Let p be a non-negative integer, and $X^{(p)}$ the set of all points x of X with $\operatorname{codim}_X \overline{\{x\}} = p$. We define $Z^p_T(X)$ to be

$$Z^p_T(X) = \bigoplus_{x \in X^{(p)} \cap T} \mathbb{Z} \cdot \overline{\{x\}}.$$

We assume that $\operatorname{codim}_X T \geq p$. Then, we can define the natural homomorphism

$$z^p : K'_T(X) \to Z^p_T(X)$$

to be

$$z^p([F]) = \sum_{x \in X^{(p)} \cap T} l_{\mathcal{O}_{X,x}}(F_x) \cdot \overline{\{x\}},$$

where $l_{\mathcal{O}_{X,x}}(F_x)$ is the length of F_x as $\mathcal{O}_{X,x}$-modules. Note that if $\operatorname{codim}_X T > p$, then $z^p = 0$.

Let $f : X \to Y$ be a proper morphism of regular noetherian schemes, and T a closed subscheme of X. Then, we define the homomorphism $f_* : K'_T(X) \to K'_{f(T)}(Y)$ to be

$$f_*([F]) = \sum_{i \geq 0} [R^i f_*(F)].$$

Here we set $d = \dim X - \dim Y$. Let p be a non-negative integer with $\operatorname{codim}_X T \geq p$ and $p \geq d$. Then, $\operatorname{codim}_Y f(T) \geq p - d$. First, let us consider the following proposition.

Proposition A.1. *With notation as above, the diagram*

$$\begin{array}{ccc} K'_T(X) & \xrightarrow{z^p} & Z^p_T(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K'_{f(T)}(Y) & \xrightarrow{z^{p-d}} & Z^{p-d}_{f(T)}(Y) \end{array}$$

is commutative.

Proof. Since $\operatorname{codim}(\operatorname{Supp}(R^i f_*(F))) > p - d$ for all $i > 0$, it is sufficient to show that $z^{p-d}([f_*(F)]) = f_*(z^p([F]))$. This is a local question with respect to Y, so that we may assume that $f(T)$ is irreducible and $\operatorname{codim}(f(T)) = p - d$. Let $T = T_1 \cup \cdots \cup T_n$ be an irreducible decomposition of T. Clearly we may assume that $f(T_i) = f(T)$ for all i. Let x_i (resp. y) be the generic point of T_i (resp. $f(T)$). Then, our assertion is equivalent to saying that

$$l_{\mathcal{O}_{Y,y}}(f_*(F)_y) = \sum_{i=1}^{n} l_{\mathcal{O}_{X,x_i}}(F_{x_i})[\kappa(x_i) : \kappa(y)].$$

Considering $X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}) \to \operatorname{Spec}(\mathcal{O}_{Y,y})$ instead of $X \to Y$, we may assume that $y, x_1, \ldots, x_n$ are closed points. Then, there are subsheaves $F_1, \ldots, F_n$ of F with $F = F_1 \oplus \cdots \oplus F_n$ and $\operatorname{Supp}(F_i) \subseteq \{x_i\}$ for all i. Thus, we may assume that $n = 1$. In this case, there is a filtration $0 = G_0 \subset G_1 \subset \cdots \subset G_l = F$ of F with $G_j/G_{j-1} = \kappa(x_1)$, so that we get our proposition. □

Let $g : Z \to X$ be a morphism of regular noetherian schemes, and T a closed subscheme of X. Then, we define the homomorphism $g^* : K'_T(X) \to K'_{f^{-1}(T)}(Z)$ to be

$$g^*([F]) = \sum_{i \geq 0} (-1)^i [L_i f^*(F)].$$

Let p be a non-negative integer with $\operatorname{codim}_X T \geq p$ and $\operatorname{codim}_Z(g^{-1}(T)) \geq p$. Here let us consider the following proposition.

Proposition A.2. *Let F and G be coherent sheaves on X with* $\mathrm{Supp}(F)$, $\mathrm{Supp}(G) \subseteq T_{red}$. *If $z^p([F]) = z^p([G])$, then $z^p(g^*([F])) = z^p(g^*([G]))$.*

Proof. This is a local question with respect to X, so that we may assume that X is affine. Let $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = F$ and $0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m = F$ be filtrations of F and G respectively such that $F_i/F_{i-1} \simeq \mathcal{O}_X/P_i$ and $G_j/G_{j-1} \simeq \mathcal{O}_X/Q_j$ for some prime ideal sheaves P_i and Q_j on X. Then,

$$\begin{cases} z^p(g^*([F])) = \sum_{i=1}^n z^p(g^*([\mathcal{O}_X/P_i])) \\ z^p(g^*([G])) = \sum_{j=1}^m z^p(g^*([\mathcal{O}_X/Q_j])) \end{cases}$$

Thus, it is sufficient to show that $z^p(g^*([\mathcal{O}_X/P])) = 0$ for all prime ideals P with

$$\mathrm{Supp}(\mathcal{O}_X/P) \subseteq T_{red}, \mathrm{codim}_X(\mathrm{Supp}(\mathcal{O}_X/P)) > p$$
$$\text{and } \mathrm{codim}_Z(g^{-1}(\mathrm{Supp}(\mathcal{O}_X/P))) = p.$$

This is a consequence of the following lemma. □

Lemma A.3. *Let (A, m) and (B, n) be regular local rings, $\phi : A \to B$ a homomorphism of local rings, and M an A-module of finite type. If* $\mathrm{Supp}(M \otimes_A B) = \{n\}$ *and*

$$\mathrm{codim}_{\mathrm{Spec}(B)}(\mathrm{Supp}(M \otimes_A B)) < \mathrm{codim}_{\mathrm{Spec}(A)}(\mathrm{Supp}(M)),$$

then

$$\sum_{i \geq 0} (-1)^i l_B(\mathrm{Tor}_i^A(M, B)) = 0.$$

Proof. We freely use notations in [25, Chapter I]. Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism induced by $\phi : A \to B$. We set $Y = \mathrm{Supp}(M)$ and $q = \mathrm{codim}_{\mathrm{Spec}(A)}(\mathrm{Supp}(M))$. Let $P. \to M$ be a free resolution of M. Then, $[P.] \in F^q K_0^Y(\mathrm{Spec}(A))$. Thus, by [25, (iii) of Theorem 3 in I.3],

$$[f^*(P.)] = [P. \otimes_A B] \in F^q K_0^{\{n\}}(\mathrm{Spec}(B))_{\mathbb{Q}}$$

because $f^{-1}(Y) = \mathrm{Supp}(M \otimes_A B) = \{n\}$. On the other hand, since

$$q > \mathrm{codim}_{\mathrm{Spec}(B)}(\mathrm{Supp}(M \otimes_A B)) = \dim B,$$

we have $F^q K_0^{\{n\}}(\mathrm{Spec}(B))_{\mathbb{Q}} = \{0\}$. Thus, $[P. \otimes_A B] = 0$ in $K_0^{\{n\}}(\mathrm{Spec}(B))$ because

$$K_0^{\{n\}}(\mathrm{Spec}(B)) \simeq \mathbb{Z}$$

has no torsion. This shows us our assertion. □

As a corollary of Proposition A.2, we have the following.

Corollary A.4. *With notation as in Proposition* A.2,

$$\begin{array}{ccc} K'_T(X) & \xrightarrow{z^p} & Z^p_T(X) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ K'_{f^{-1}(T)}(Z) & \xrightarrow{z^p} & Z^p_{f^{-1}(T)}(Z) \end{array}$$

is commutative. Note that $g^* : Z^p_T(X) \to Z^p_{f^{-1}(T)}(Z)$ *is defined by* $g^*(Z) = z^p(g^*([\mathcal{O}_Z]))$ *for each integral cycle* Z *in* $Z^p_T(X)$.

Let $f : X \to Y$ be a flat proper morphism of regular noetherian schemes, and $u : Y' \to Y$ a morphism of regular noetherian schemes. Let $X' = X \times_Y Y'$ and

$$\begin{array}{ccc} X & \xleftarrow{u'} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{u} & Y' \end{array}$$

the induced diagram. We assume that X' is regular. Note that if f is smooth, then X' is regular. We set $d = \dim X - \dim Y = \dim X' - \dim Y'$. Let T be a closed subscheme of X, and p a non-negative integer with $\operatorname{codim}_X T \geq p$, $\operatorname{codim}_{X'}(u'^{-1}(T)) \geq p$ and $p \geq d$. Note that $\operatorname{codim}_Y f(T) \geq p - d$ and $\operatorname{codim}_{Y'}(u^{-1}(f(T))) \geq p - d$ because $u^{-1}(f(T)) = f'(u'^{-1}(T))$. Then, we have the following proposition.

Proposition A.5. *The diagram*

$$\begin{array}{ccc} Z^p_T(X) & \xrightarrow{u'^*} & Z^p_{u'^{-1}(T)}(X') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ Z^{p-d}_{f(T)}(Y) & \xrightarrow{u^*} & Z^{p-d}_{u^{-1}(f(T))}(Y') \end{array}$$

is commutative.

Proof. Since f is flat, by [12, Proposition 3.1.0 in IV], for any coherent sheaves F on X,

$$L.u^*\left(R^{\cdot} f_*(F)\right) \xrightarrow{\sim} R^{\cdot} f'_*\left(L.u'^*(F)\right),$$

which shows that the diagram

$$\begin{array}{ccc} K'_T(X) & \xrightarrow{u'^*} & K'_{u'^{-1}(T)}(X') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ K'_{f(T)}(Y) & \xrightarrow{u^*} & K'_{u^{-1}(f(T))}(Y') \end{array}$$

is commutative. Thus, by virtue of Proposition A.1 and Corollary A.4, we can see our proposition. $\square$

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