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Classification of equivariant real vector bundles over a circle

By

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Abstract

This is a continuation of the previous work [CKMS01] by the authors on classification of equivariant complex vector bundles over a circle. In this paper equivariant real vector bundles over a circle with a compact Lie group action are classified, by characterizing their fiber representations, and by using the result of the complex case. Their triviality is also treated. The basic phenomenon is similar to the complex case but more complicated here.

1. Introduction

In [CKMS01] we classified equivariant *complex* vector bundles over a circle, and in this paper we classify equivariant *real* ones. The argument developed in this paper is similar to that in [CKMS01] but is rather more complicated. The complexity arises from two aspects: one is topology and the other is representation theory. For instance, any (nonequivariant) complex line bundle over a circle is trivial while there are two non-isomorphic real line bundles, that is, the Hopf line bundle and the trivial one. This is an evidence of the topological complexity in the real case. It is also recognized in general that real representation theory is more complicated than complex representation theory.

Our classification result can be applied to various problems in transformation group theory. For example, when G is a compact *abelian* Lie group, the first author showed in his thesis [Cho99, Theorem 4.3.8] using a similar classification that every closed smooth orientable G-manifold of dimension 3 is algebraically realized, that is, it is equivariantly diffeomorphic to a nonsingular real algebraic G-variety.

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In the following, some notation to state our results will be introduced. Let G be a compact Lie group and let $\rho\colon G \to O(2)$ be an orthogonal representation. The unit circle of the corresponding G-module is denoted by $S(\rho)$. It is well known that any circle with G-action is equivalent to $S(\rho)$ for some ρ. We set $H = \rho^{-1}(1)$, so that H acts trivially on $S(\rho)$ and the *fiber H-module* of a real G-vector bundle over $S(\rho)$ is determined uniquely up to isomorphism.

Let $\mathrm{Irr}(H)$ be the set of characters of irreducible real H-modules. It has a G-action defined as follows: For $\chi \in \mathrm{Irr}(H)$ and $g \in G$, ${}^g\chi \in \mathrm{Irr}(H)$ is defined by ${}^g\chi(h) = \chi(g^{-1}hg)$ for $h \in H$. Since a character is a class function, the isotropy subgroup G_χ of G at $\chi \in \mathrm{Irr}(H)$ contains H. We choose and fix a representative from each G-orbit in $\mathrm{Irr}(H)$ and denote the set of those representatives by $\mathrm{Irr}(H)/G$. Denote by $\mathrm{Vect}_G(S(\rho))$ the set of isomorphism classes of real G-vector bundles over $S(\rho)$ and by $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$ the subset of $\mathrm{Vect}_{G_\chi}(S(\rho))$ with a multiple of χ as the character of fiber H-modules. They are semi-groups under Whitney sum. The decomposition of a G-vector bundle into the χ-isotypical components induces an isomorphism

$$\mathrm{Vect}_G(S(\rho)) \cong \bigoplus_{\chi \in \mathrm{Irr}(H)/G} \mathrm{Vect}_{G_\chi}(S(\rho), \chi),$$

see [CKMS01, Section 2]. This reduces the study of $\mathrm{Vect}_G(S(\rho))$ to that of $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$, and since χ is G_χ-invariant and G_χ is again a compact Lie group, we are led to study $\mathrm{Vect}_G(S(\rho), \chi)$ where χ is G-invariant, namely ${}^g\chi = \chi$ for all $g \in G$.

Let $E \in \mathrm{Vect}_G(S(\rho), \chi)$. The fiber H-module of E has a multiple of χ as the character by definition. In fact, the fiber E_z of E over a point $z \in S(\rho)$ is a real module of the isotropy subgroup G_z at z, and unless $\rho(G) \subset SO(2)$, G_z properly contains H for some z. It turns out that these fiber G_z-modules almost distinguish elements in $\mathrm{Vect}_G(S(\rho), \chi)$. To be more specific, we shall introduce some more notation. Unless $\rho(G) \subset SO(2)$, $\rho(G)$ is $O(2)$ or a dihedral group D_n of order $2n$ for some positive integer n. We identify $S(\rho)$ with the unit circle of the complex line $\mathbb{C}$, so that the dihedral group D_n is generated by the rotation through an angle $2\pi/n$ and the reflection about the x-axis. Then the isotropy subgroups G_z at $z = 1$ (and $z = e^{\pi i/n}$ when $\rho(G) = D_n$) contain H as an index two subgroup unless $\rho(G) \subset SO(2)$. For a group K containing H we denote by $\mathrm{Rep}(K, \chi)$ the set of isomorphism classes of real K-modules whose characters restricted to H are multiples of χ. The set $\mathrm{Rep}(K, \chi)$ is a semi-group under direct sum. Restriction of elements in $\mathrm{Vect}_G(S(\rho), \chi)$ to fibers at 1 (and $\mu = e^{\pi i/n}$ when $\rho(G) = D_n$) yields a semi-group homomorphism

$$\Gamma\colon\ \mathrm{Vect}_G(S(\rho), \chi) \to \begin{cases} \mathrm{Rep}(H, \chi), & \text{if} \quad \rho(G) \subset SO(2), \\ \mathrm{Rep}(G_1, \chi), & \text{if} \quad \rho(G) = O(2), \\ \mathrm{Rep}(G_1, G_\mu, \chi), & \text{if} \quad \rho(G) = D_n, \end{cases}$$

where $\mathrm{Rep}(G_1, G_\mu, \chi)$ denotes the subsemi-group of $\mathrm{Rep}(G_1, \chi) \times \mathrm{Rep}(G_\mu, \chi)$ consisting of pairs of the same dimension. The map Γ can also be defined in the complex case and is proved to be an isomorphism in [CKMS01, Proposition 6.2]. However, Γ is not always an isomorphism in the real case, for instance the two non-isomorphic line bundles over a circle (when G is trivial) mentioned before have obviously the same Γ image. Nevertheless it turns out that Γ is an isomorphism in most cases. Remember that χ is called of *real*, *complex*, or *quaternion* type if the H-endomorphism algebra of the irreducible real H-module with χ as the character is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ respectively, and that χ is called K-*extendible* if it extends to a character of a group K containing H. There are two cases in which the classification works somewhat exceptionally.

Case A. $\rho(G) \subsetneq SO(2)$ and χ is of real type.

Case B. $\rho(G) = D_n$, χ is of real type and neither G_1- nor G_μ-extendible.

Theorem 1.1. *Except for Cases* A *and* B, *the semi-group homomorphism* Γ *is an isomorphism. In Case* A *or* B, Γ *is a two to one map*; *more precisely, there is a free involution on* $\mathrm{Vect}_G(S(\rho),\chi)$ *given by tensoring with a nontrivial G-line bundle* (*with trivial fiber H-module*) *and* Γ *induces an isomorphism on the orbit space.*

The study of $\mathrm{Vect}_G(S(\rho),\chi)$ is reduced to real representation theory by Theorem 1.1, especially to the study of $\mathrm{Rep}(K,\chi)$ where K is a compact Lie group containing H as an index two subgroup and χ is K-invariant. The complexity of real representation theory emerges here. Namely, the number of K-extensions of χ can be zero, one, or two, while it is always two in the complex case. Combining this observation with Theorem 1.1, one sees that the semi-group structures on $\mathrm{Vect}_G(S(\rho),\chi)$ are of five types depending on $\rho(G)$ and χ (see Theorem 5.1) while they are of three types in the complex case (see [CKMS01, Theorem B]).

The paper is organized as follows. In Section 2 we shall determine the semi-group structure of the target space of the semi-group homomorphism Γ. For this we need to study some representation theory, especially on extensions of representations. The semi-group structure of the target space of Γ is given in Corollary 2.2 and Lemma 2.3.

In Section 3 we prove Theorem 1.1 except for Cases A and B. Indeed, Proposition 3.2 shows that Γ is always surjective, and Proposition 3.3 shows that Γ is injective except for Cases A and B.

Cases A and B are treated in Section 4. Case A can easily be proved by reducing it to the nonequivariant case. For Case B we consider possible complex G vector bundle structures on an element of $\mathrm{Vect}_G(S(\rho),\chi)$. Then we use the classification results of complex G vector bundles over a circle in [CKMS01], and some counting argument to finish the proof for Case B.

The semi-group structure of $\mathrm{Vect}_G(S(\rho),\chi)$ is given in Theorem 5.1 of Section 5 by figuring out the generators with their relations. Theorem 6.2 in Section 6 shows which of the generators of $\mathrm{Vect}_G(S(\rho),\chi)$ are trivial.

2. Ingredients from representation theory

We shall determine the semi-group structure of the target space of Γ. For that, the following lemma from representation theory plays a key role. Throughout this section, H is an index two normal subgroup of a group K and U is a real irreducible H-module with K-invariant character. By the type of U we mean the type of the character of U. As is well known, any K-extension of U appears in $\operatorname{ind}_H^K U$ as a direct summand at least once, which follows from the Frobenius reciprocity, and $\operatorname{res}_H \operatorname{ind}_H^K U \cong 2U$ because H is of index two and the character of U is K-invariant.

Lemma 2.1. (1) *Suppose U is K-extendible.*

(a) *If U is of real type, there are two mutually non-isomorphic K-extensions of real type and they are related through tensor product with the nontrivial real K-module of dimension one with trivial H-action.*

(b) *If U is of complex type, there are either two mutually non-isomorphic K-extensions of complex type or unique K-extension of real type, not both.*

(c) *If U is of quaternionic type, there are either two mutually non-isomorphic K-extensions of quaternionic type or unique K-extension of complex type, not both.*

(2) *Suppose U is not K-extendible. Then U is not of quaternionic type, and $\operatorname{ind}_H^K U$ is irreducible of complex or quaternionic type according as U is of real or complex type, respectively. Moreover, any K-extension of $2U$ is isomorphic to $\operatorname{ind}_H^K U$.*

Proof. (1) Let W be a K-extension of U. Since U is irreducible, so is W. Set $d(W) = \dim_{\mathbb{R}} \operatorname{Hom}_K(W, W)$ and $d(U) = \dim_{\mathbb{R}} \operatorname{Hom}_H(U, U)$. Note that $d(U) = 1, 2$, or 4 according as U is of real, complex, or quaternionic type, respectively. Since $\operatorname{res}_H W \cong U$, it follows from the Frobenius reciprocity that

(*)
$$m(W) \cdot d(W) = \dim_{\mathbb{R}} \operatorname{Hom}_K(W, \operatorname{ind}_H^K U) = \dim_{\mathbb{R}} \operatorname{Hom}_H(\operatorname{res}_H W, U) = d(U),$$

where $m(W)$ denotes the multiplicity of W in $\operatorname{ind}_H^K U$. On the other hand, since

$$\dim_{\mathbb{R}} \operatorname{ind}_H^K U = 2 \dim_{\mathbb{R}} U = 2 \dim_{\mathbb{R}} W,$$

the multiplicity $m(W)$ must be either 1 or 2.

If $m(W) = 1$ (i.e., $d(W) = d(U)$), then $\operatorname{ind}_H^K U \cong W \oplus W'$ for some K-module W' which is not isomorphic to W. Since $\operatorname{res}_H \operatorname{ind}_H^K U \cong 2U$, $\operatorname{res}_H W \cong U$ is isomorphic to $\operatorname{res}_H W'$, so that W' is also a K-extension of U. Since W' appears in $\operatorname{ind}_H^K U$ once, $m(W') = 1$. Therefore the identity (*) applied to W' implies $d(W') = d(U)$. This together with the equality $d(W) = d(U)$ implies that the two mutually non-isomorphic K-extensions W and W' of U are of the same type as U.

If $m(W)=2$, then $\operatorname{ind}_H^K U \cong 2W$. Since any K-extension of U is contained in $\operatorname{ind}_H^K U$ as a direct summand, W is the unique K-extension of U. The type of W can be read from the equality $d(W)=d(U)/2$. This equality in particular implies that $d(U)$ must be even, in other words, U is not of real type when $m(W)=2$.

If U of real type has two K-extensions W and W', then $\operatorname{Hom}_H(W,W')$ is a nontrivial real K-module of dimension one with trivial H-action, and $W \otimes \operatorname{Hom}_H(W,W')$ is isomorphic to W'. This completes the proof of (1).

(2) Suppose U has no K-extension. If $\operatorname{ind}_H^K U$ is reducible, then each direct summand of $\operatorname{ind}_H^K U$ is a K-extension of U which contradicts the assumption that U has no K-extension. Therefore, $\operatorname{ind}_H^K U$ is irreducible. Noting that $\operatorname{res}_H \operatorname{ind}_H^K U \cong 2U$, it follows from the Frobenius reciprocity that

$$d(\operatorname{ind}_H^K U) = \dim_{\mathbb{R}} \operatorname{Hom}_K(\operatorname{ind}_H^K U, \operatorname{ind}_H^K U) = \dim_{\mathbb{R}} \operatorname{Hom}_H(2U, U) = 2d(U),$$

which implies the statement on the type of $\operatorname{ind}_H^K U$. In particular, U can not be of quaternionic type because $d(\operatorname{ind}_H^K U) = 1, 2$, or 4 and so $d(U) = 1$ or 2. The last statement in (2) follows again from the Frobenius reciprocity. □

Corollary 2.2. *Let χ be the K-invariant character of U, and let e be the number of K-extensions of U. Then $e = 0$, 1, or 2, and*

(1) *if $e=0$, then $\operatorname{Rep}(K,\chi)$ is generated by $\operatorname{ind}_H^K U$,*

(2) *if $e = 1$, then $\operatorname{Rep}(K,\chi)$ is generated by a K-module of the same dimension as U,*

(3) *if $e=2$, then $\operatorname{Rep}(K,\chi)$ is generated by two K-modules of the same dimension as U such that one is isomorphic to the tensor product of the other with the nontrivial real K-module of dimension one with trivial H-action.* □

Let K_1 and K_2 be two groups containing H as an index two subgroup, and let χ be a real irreducible character of H which is K_1- and K_2-invariant. We next consider the subset $\operatorname{Rep}(K_1,K_2,\chi)$ of $\operatorname{Rep}(K_1,\chi) \times \operatorname{Rep}(K_2,\chi)$ consisting of pairs of the same dimension. Denote by e_1 and e_2 the number of K_1- and K_2-extensions, respectively.

Lemma 2.3. *The semi-group $\operatorname{Rep}(K_1,K_2,\chi)$ is generated by*

$$\begin{cases} \textit{one element } R_\chi, & \textit{if} \quad (e_1,e_2) = (0,0), (1,0), (0,1), \ \textit{or } (1,1), \\ \textit{two elements } R_\chi^{\pm}, & \textit{if} \quad (e_1,e_2) = (2,1) \ \textit{or } (1,2), \\ \textit{three elements } \widetilde{R}_\chi^0, \widetilde{R}_\chi^{\pm}, & \textit{if} \quad (e_1,e_2) = (2,0) \ \textit{or } (0,2), \\ \textit{four elements } R_\chi^{\pm\pm}, & \textit{if} \quad (e_1,e_2) = (2,2), \end{cases}$$

with relations $2\widetilde{R}_\chi^0 = \widetilde{R}_\chi^+ + \widetilde{R}_\chi^-$ and $R_\chi^{++} + R_\chi^{--} = R_\chi^{+-} + R_\chi^{-+}$.

Proof. For $i=1$ and 2, denote by $\widetilde{R}_i$, R_i and $R_i^{\pm}$ the set of generators of $\operatorname{Rep}(K_i,\chi)$ according to $e_i = 0$, 1 and 2, respectively. Note that the dimension of $\widetilde{R}_i$ is twice that of R_i and $R_i^{\pm}$. Then it is easy to find the generators and relations of $\operatorname{Rep}(K_1,K_2,\chi)$ according to (e_1,e_2) as in Table 1. □

Table 1. Generators and relations of $\mathrm{Rep}(K_1, K_2, \chi)$ according to (e_1, e_2)

(e_1, e_2)	Generators	Relations
$(0,0)$	$R_\chi = (\widetilde{R}_1, \widetilde{R}_2)$	none
$(1,0)$	$R_\chi = (R_1 \oplus R_1, \widetilde{R}_2)$	
$(0,1)$	$R_\chi = (\widetilde{R}_1, R_2 \oplus R_2)$	
$(1,1)$	$R_\chi = (R_1, R_2)$	
$(2,1)$	$R_\chi^\pm = (R_1^\pm, R_2)$	none
$(1,2)$	$R_\chi^\pm = (R_1, R_2^\pm)$	
$(2,0)$	$\widetilde{R}_\chi^0 = (R_1^+ \oplus R_1^-, \widetilde{R}_2)$, $\widetilde{R}_\chi^\pm = (R_1^\pm \oplus R_1^\pm, \widetilde{R}_2)$	$2\widetilde{R}_\chi^0 = \widetilde{R}_\chi^+ + \widetilde{R}_\chi^-$
$(0,2)$	$\widetilde{R}_\chi^0 = (\widetilde{R}_1, R_2^+ \oplus R_2^-)$, $\widetilde{R}_\chi^\pm = (\widetilde{R}_1, R_2^\pm \oplus R_2^\pm)$	
$(2,2)$	$R_\chi^{\pm\pm} = (R_1^\pm, R_2^\pm)$	$R_\chi^{++} + R_\chi^{--} = R_\chi^{+-} + R_\chi^{-+}$

Remark. For $i = 1$ and 2, denote by $\mathbb{R}_i^+$ and $\mathbb{R}_i^-$, respectively, the trivial and the nontrivial real K_i-module of dimension one with trivial H-action. Then the set of pairs $(\mathbb{R}_1^\pm, \mathbb{R}_2^\pm)$ forms a group isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ under tensor product on each factor, and it acts by the same operation on the generators in Lemma 2.3. The action is transitive except for the third case where $\mathrm{Rep}(K_1, K_2, \chi)$ is generated by three elements. In that case, $\widetilde{R}_\chi^\pm$ constitute one orbit and $\widetilde{R}_\chi^0$ is fixed by the action of the pairs $(\mathbb{R}_1^\pm, \mathbb{R}_2^\pm)$.

We recall some facts on the extension of an H-module, which will be used in Section 6.

Lemma 2.4. *Let H be a normal subgroup of G and let U be a real irreducible H-module with G-invariant character.*

(1) *Suppose G/H is finite cyclic of odd order. Then U has a G-extension, and the G-extension is unique if U is of real type.*

(2) *Suppose G/H is a dihedral group of order $2n$ for odd n. Then*

(a) *U has a G-extension if and only if it has a K-extension for some subgroup K of G which contain H as an index two subgroup,*

(b) *$2U$ always has a G-extension.*

Proof. See [CMS01] for the former statements in (1) and (2-a). To see the uniqueness in (1), we note that if U_1 and U_2 are G-extensions of U, then $\mathrm{Hom}_H(U_1, U_2) \cong \mathbb{R}$. Since H acts trivially on $\mathrm{Hom}_H(U_1, U_2)$ and G/H is of odd order, $\mathrm{Hom}_H(U_1, U_2)$ must be a trivial G-module. Therefore $\mathrm{Hom}_G(U_1, U_2)$ is also isomorphic to $\mathbb{R}$, which means that U_1 and U_2 are isomorphic as G-modules. This proves the uniqueness of the G-extension of U.

It remains to prove (2-b). Let P be a normal subgroup of G which contains H and P/H is a normal cyclic subgroup of G/H of order n. By (1) above, U has a P-extension, say W. Then $\mathrm{ind}_P^G W$ is a G-extension of $2U$. $\square$

3. Fiber modules

In this section we prove Theorem 1.1 except for Cases A and B. The following two propositions can be proved by the same argument as in the complex case, see Theorem A and its subsequent remark in [CKMS01].

Proposition 3.1. *A real H-module is the fiber H-module of a real G-vector bundle over $S(\rho)$ if and only if its character is G-invariant and G_1-extendible (and G_μ-extendible when $\rho(G) = D_n$).*

Proposition 3.2. *Given G_z-extensions V_z of a real H-module with G-invariant character for $z = 1$ (and μ when $\rho(G) = D_n$), there exists a real G-vector bundle E over $S(\rho)$ such that the fiber E_z of E over z is isomorphic to V_z as G_z-modules.*

Proposition 3.1 gives a characterization of the fiber H-module of a real G-vector bundle over $S(\rho)$, and Proposition 3.2 shows that the semi-group homomorphism Γ in the introduction is surjective. On the other hand, the following proposition shows that Γ is injective except for Cases A and B, which proves Theorem 1.1 except for Cases A and B.

Proposition 3.3. *Let χ be a real irreducible character of H which is G-invariant. Except for Cases* A *and* B*, two real G-vector bundles E and E' in $\mathrm{Vect}_G(S(\rho), \chi)$ are isomorphic if and only if the fiber G_z-modules E_z and E'_z at $z \in S(\rho)$ are isomorphic for $z = 1$ (and for $z = \mu$ when $\rho(G) = D_n$).*

Proof. The proof of [CKMS01, Theorem 6.1] holds in the real category with slight modification. For reader's convenience we shall give the argument when $\rho(G)$ is finite. The case when $\rho(G)$ is infinite is easy since the action of G on $S(\rho)$ is transitive, see [CKMS01, Proposition 2.3] for details.

The necessity is obvious so we prove the sufficiency. We first note that if there exists an equivariant isomorphism $\Psi\colon E \to E'$, then it must satisfy the equivariance condition $\Psi_{\rho(g)z} = g\Psi_z g^{-1}$ for any $g \in G$ where $\Psi_z = \Psi|_{E_z}$.

Suppose $\rho(G) \subsetneq SO(2)$. Then $G_1 = H$, $\rho(G)$ is finite cyclic, say, of order n, and since Case A is excluded, χ is not of real type. Choose an element $a \in G$ such that $\rho(a)$ is the rotation through the angle $2\pi/n$. By the assumption we have an H-linear isomorphism $\Psi_1\colon E_1 \to E'_1$. Set $\Psi_{\rho(a)1} = a\Psi_1 a^{-1}$, which is also an H-linear isomorphism. Then we connect Ψ_1 and $\Psi_{\rho(a)1}$ continuously in the set of H-linear isomorphisms of the fiber H-module along the arc of $S(\rho)$ joining 1 and $\rho(a)1 = e^{2\pi i/n}$. This is possible because the set of H-linear isomorphisms of the fiber H-module is homeomorphic to a general linear group over $\mathbb{C}$ or $\mathbb{H}$ depending on the type of χ (remember that χ is not of real type), and it is arcwise connected. Thus we have a bundle isomorphism between E and E' restricted to the arc of $S(\rho)$ joining 1 and $\rho(a)1$. We extend this isomorphism to an entire isomorphism over $S(\rho)$ using the equivariance condition $\Psi_{\rho(a)z} = a\Psi_z a^{-1}$.

When $\rho(G) = D_n$, we choose a G_1-linear isomorphism Ψ_1 and a G_μ-linear isomorphism Ψ_μ. Similarly to the above, we connect Ψ_1 and Ψ_μ as H-linear isomorphisms along the arc of $S(\rho)$ joining 1 and $\mu = e^{\pi i/n}$, and then extend it to an isomorphism over $S(\rho)$ using the equivariance condition. But it is not always possible to connect Ψ_1 and Ψ_μ when χ is of real type because the set of H-linear isomorphisms of the fiber H-module, which is homeomorphic to $\mathrm{GL}(m, \mathbb{R})$, is not arcwise connected. In this case, however, another assumption is that χ is G_z-extendible for $z = 1$ or μ since Case B is excluded. By Lemma 2.1 (1-a) χ has two G_z-extensions, say $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$. Thus the character of $E_z \cong E'_z$ as a G_z-module is of the form $m_1\widetilde{\chi}_1 + m_2\widetilde{\chi}_2$ for some nonnegative integers m_1 and m_2 with $m = m_1 + m_2$, so that the set of G_z-linear isomorphisms between E_z and E'_z is homeomorphic to $\mathrm{GL}(m_1, \mathbb{R}) \times \mathrm{GL}(m_2, \mathbb{R})$. Since the inclusion map from $\mathrm{GL}(m_1, \mathbb{R}) \times \mathrm{GL}(m_2, \mathbb{R})$ to $\mathrm{GL}(m, \mathbb{R})$ induces a surjection on the π_0 level, it is possible to choose a G_z-linear isomorphism Ψ_z so that Ψ_1 and Ψ_μ can be connected in the set of H-linear isomorphisms of the fiber H-module. □

4. Topological complexity: Cases A and B

Propositions 3.2 and 3.3 show that the map Γ is an isomorphism except for Cases A and B. In this section we investigate the structure on $\mathrm{Vect}_G(S(\rho), \chi)$ for Cases A and B, and complete the proof of Theorem 1.1.

Case A. The case where $\rho(G) \subsetneq SO(2)$ and χ is of real type. In this case one can reduce the study of $\mathrm{Vect}_G(S(\rho), \chi)$ to the nonequivariant case.

Lemma 4.1. *In Case* A*, the semi-group* $\mathrm{Vect}_G(S(\rho), \chi)$ *is generated by two elements* $N_\chi^\pm$ *with relation* $2N_\chi^+ = 2N_\chi^-$*. Moreover,* $N_\chi^\pm$ *have* χ *as the character of the fiber* H*-modules, and they are related in such a way that* N_χ^- *is obtained from* N_χ^+ *by tensoring with a real* G*-line bundle over* $S(\rho)$ *with trivial fiber* H*-module.*

Proof. There is an element L in $\mathrm{Vect}_G(S(\rho), \chi)$ with χ as the character of the fiber H-module by Proposition 3.1, and we have the semi-group isomorphisms

$$\mathrm{Vect}_G(S(\rho), \chi) \cong \mathrm{Vect}_{G/H}(S(\rho)) \cong \mathrm{Vect}(S^1),$$

where the former isomorphism is given by sending E to $\mathrm{Hom}_H(L, E)$ and the latter is given by taking orbit spaces by the G/H-action. In fact, since χ is of real type, the map sending $F \in \mathrm{Vect}_{G/H}(S(\rho))$ to $L \otimes F$ is the inverse of the former isomorphism, where F is viewed as a G-vector bundle through the quotient map from G to G/H (see Lemma 2.2 in [CKMS01] for details), and the latter is an isomorphism because the action of G/H on $S(\rho)$ is free. As is well known, $\mathrm{Vect}(S^1)$ is generated by the trivial line bundle ϵ and the Hopf line bundle η with relation $2\epsilon = 2\eta$. Therefore, if we denote by $N_\chi^\pm$ the two generators of $\mathrm{Vect}_G(S(\rho), \chi)$ corresponding to ϵ and η in $\mathrm{Vect}(S^1)$ through the above isomorphism, then the lemma follows except the last statement. To see the last statement, we note that $\mathrm{Hom}_H(N_\chi^+, N_\chi^-)$ is a real G-line bundle over $S(\rho)$ with trivial fiber H-module and that

$$N_\chi^+ \otimes \mathrm{Hom}_H(N_\chi^+, N_\chi^-) \cong N_\chi^-,$$

proving the last statement. □

Case B. The case where $\rho(G) = D_n$, χ is of real type, and neither G_1- nor G_μ-extendible. In this case we investigate complex structures on the bundles in $\mathrm{Vect}_G(S(\rho), \chi)$.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and set

$$\mathcal{J}(\mathbb{F}^k) \equiv \{J \in \mathrm{GL}(k, \mathbb{F}) \mid J^2 = -I\},$$

which is the set of complex structures on $\mathbb{F}^k$. Needless to say, $\mathcal{J}(\mathbb{R}^k)$ is empty unless k is even. Viewing $\mathbb{C}$ as $\mathbb{R}^2$ in a natural way induces an injective homomorphism from $\mathrm{GL}(k, \mathbb{C})$ to $\mathrm{GL}(2k, \mathbb{R})$, so that it induces an injection from $\mathcal{J}(\mathbb{C}^k)$ to $\mathcal{J}(\mathbb{R}^{2k})$ and we view $\mathcal{J}(\mathbb{C}^k)$ as a subset of $\mathcal{J}(\mathbb{R}^{2k})$ through this map.

Lemma 4.2.
(1) *$\mathcal{J}(\mathbb{C}^k)$ has $k+1$ connected components.*
(2) *$\mathcal{J}(\mathbb{R}^{2k})$ has two connected components.*
(3) *If k is odd, then each connected component of $\mathcal{J}(\mathbb{R}^{2k})$ contains $(k+1)/2$ connected components of $\mathcal{J}(\mathbb{C}^k)$, while if k is even, then one connected component of $\mathcal{J}(\mathbb{R}^{2k})$ contains $k/2$ and the other contains $k/2+1$ connected components of $\mathcal{J}(\mathbb{C}^k)$.*

Proof. (1) We note that $\mathrm{GL}(k, \mathbb{C})$ acts on $\mathcal{J}(\mathbb{C}^k)$ by conjugation. Since the minimal polynomial of any element in $\mathcal{J}(\mathbb{C}^k)$ has distinct roots it is diagonalizable. So two elements in $\mathcal{J}(\mathbb{C}^k)$ are in the same orbit if and only if they have the same eigenvalues which are $\pm i$ because $J^2 = -I$. This implies that $\mathcal{J}(\mathbb{C}^k)$ has exactly $k+1$ connected components because there are $k+1$ possibilities of the k eigenvalues.

(2) $\mathrm{GL}(2k, \mathbb{R})$ acts transitively on $\mathcal{J}(\mathbb{R}^{2k})$, and the isotropy subgroup at any element of $\mathcal{J}(\mathbb{R}^{2k})$ is isomorphic to $\mathrm{GL}(k, \mathbb{C})$; so $\mathcal{J}(\mathbb{R}^{2k})$ is homeomorphic to a homogeneous space $\mathrm{GL}(2k, \mathbb{R})/\mathrm{GL}(k, \mathbb{C})$ which has two connected components (see [MS98, Proposition 2.48] for more details).

(3) As observed in (1) above, $k+1$ elements

$$\mathrm{diag}(i, i, \ldots, i), \mathrm{diag}(-i, i, \ldots, i), \ldots, \mathrm{diag}(-i, -i, \ldots, -i)$$

respectively lie in the $k+1$ different connected components of $\mathcal{J}(\mathbb{C}^k)$. Through the inclusion map from $\mathcal{J}(\mathbb{C}^k)$ to $\mathcal{J}(\mathbb{R}^{2k})$, they respectively are mapped to

$$\operatorname{diag}(J_0, J_0, \ldots, J_0), \operatorname{diag}(-J_0, J_0, \ldots, J_0), \ldots, \operatorname{diag}(-J_0, -J_0, \ldots, -J_0),$$

where J_0 is the 2×2 matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $-J_0$ and J_0 are conjugate by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ whose determinant is negative, the $k+1$ elements above in $\mathcal{J}(\mathbb{C}^k)$ are in a same connected component of $\mathcal{J}(\mathbb{R}^{2k})$ if and only if the number of J_0's as entries are congruent modulo 2. This implies (3). □

For a real G-module V, we denote the set of G-invariant complex structures on V by

$$\mathcal{J}(V)^G \equiv \{J \in \mathrm{GL}(V)^G \mid J^2 = -I\},$$

where $\mathrm{GL}(V)^G$ denotes the G-linear automorphisms of V. A pair (V, J) is a complex G-module whose realification is V. We note that $\mathrm{GL}(V)^G$ acts on $\mathcal{J}(V)^G$ by conjugation and that two complex G-modules (V, J) and (V, J') are isomorphic if and only if J and J' are in the same orbit of the $\mathrm{GL}(V)^G$ action.

We consider the following setting for later use.

Lemma 4.3. *Let K be a group and let H be a normal subgroup of K. Suppose*

(a) *W is an irreducible real K-module of complex type,*

(b) *U is an irreducible real H-module of real type,*

(c) $\operatorname{res}_H W \cong 2U$.

Then, for any positive integer k, $\mathcal{J}(kW)^K$ can naturally be viewed as a subspace of $\mathcal{J}(2kU)^H$, and we have

(1) *$\mathcal{J}(kW)^K$ has $k+1$ connected components,*

(2) *$\mathcal{J}(2kU)^H$ has two connected components,*

(3) *if k is odd, then each connected component of $\mathcal{J}(2kU)^H$ contains $(k+1)/2$ connected components of $\mathcal{J}(kW)^K$, while if k is even, then one connected component of $\mathcal{J}(2kU)^H$ contains $k/2$ and the other contains $k/2+1$ connected components of $\mathcal{J}(kW)^K$.*

Proof. It follows from the assumptions (a) and (b) that $\mathrm{GL}(kW)^K \cong \mathrm{GL}(k, \mathbb{C})$ and $\mathrm{GL}(2kU)^H \cong \mathrm{GL}(2k, \mathbb{R})$. Therefore the lemma follows from Lemma 4.2. □

We return to the original setting of Case B. Denote by U a real irreducible H-module with χ as its character. Since χ is neither G_1- nor G_μ-extendible and of real type, $\operatorname{ind}_H^{G_z} U$ is the unique G_z-extension of $2U$, which is of complex type, for $z = 1$ and μ by Lemma 2.1 (2). Therefore we are in a setting to which Lemma 4.3 can be applied. Moreover this shows that an element E in $\mathrm{Vect}_G(S(\rho), \chi)$ must have the fibers at $z = 1$ and μ isomorphic to $k(\operatorname{ind}_H^{G_z} U)$ for some integer k. In particular, its fiber H-module is $2kU$.

A G-invariant complex structure on E is a G-vector bundle automorphism J of E such that $J^2 = -I$. A pair (E, J) is a complex G-vector bundle whose realification is E. We say that two G-invariant complex structures J and J' on E are equivalent if (E, J) and (E, J') are isomorphic as complex G-vector bundles, that is, if J and J' are conjugate by a real G-vector bundle automorphism of E.

Lemma 4.4. *The number of inequivalent G-invariant complex structures on E is $(k+1)^2/2$ if k is odd, and $k(k/2+1)$ or $k(k/2+1)+1$ if k is even.*

Proof. Let $\mathcal{J}(E_z)$ be the set of (not necessarily invariant) complex structures on the fiber E_z. The collection $\mathcal{J}(E)$ of $\mathcal{J}(E_z)$ over $z \in S(\rho)$ forms a G-fiber bundle over $S(\rho)$, the G-action on $\mathcal{J}(E)$ being induced from that on E. Then a G-invariant complex structure on E can be viewed as a continuous G-equivariant cross section of the G-fiber bundle. The image of the cross section lies in $\mathcal{J}(E)^H$ because H acts trivially on $S(\rho)$.

In order to construct a continuous G-equivariant cross section of $\mathcal{J}(E) \to S(\rho)$, we choose a pair of points from $\mathcal{J}(E_1)^{G_1}$ and $\mathcal{J}(E_\mu)^{G_\mu}$ (i.e., one point from each), which can be connected by a continuous cross section of $\mathcal{J}(E)^H$ restricted to the arc R in $S(\rho)$ joining 1 and $\mu = e^{\pi i/n}$. Not all pairs of those points are connected by such a cross section as observed later. But, once we find such a cross section, we can extend it to an entire G-equivariant cross section using the equivariance as is done in the proof of Proposition 3.3. On the other hand, it is known in [CKMS01, Theorem 6.1] that isomorphism classes of complex G-vector bundles over $S(\rho)$ are distinguished by the complex fiber G_1- and G_μ-modules. Therefore, the number $CS(E)^G$ of inequivalent G-invariant complex structures on E is equal to the number of pairs of connected components in $\mathcal{J}(E_1)^{G_1}$ and $\mathcal{J}(E_\mu)^{G_\mu}$ which are connected through $\mathcal{J}(E)^H|_R$.

Suppose k is even. Denote by C_z^1 and C_z^2, for $z = 1$ and μ, the connected components of $\mathcal{J}(E_z)^H|_R$ containing $k/2$ and $k/2+1$ connected components of $\mathcal{J}(E_z)^{G_z}$, respectively. If C_1^1 and C_μ^1 are connected through $\mathcal{J}(E)^H|_R$, then so are C_1^2 and C_μ^2. Counting the number of choices of pairs of connected components in $\mathcal{J}(E_1)^{G_1}$ and $\mathcal{J}(E_\mu)^{G_\mu}$ which are connected through $\mathcal{J}(E)^H|_R$, one has

$$CS(E)^G = \left(\frac{k}{2}\right)^2 + \left(\frac{k}{2}+1\right)^2 = k(k/2+1)+1.$$

On the other hand, if C_1^1 and C_μ^2 are connected through $\mathcal{J}(E)^H|_R$, then so are C_1^2 and C_μ^1 and one has

$$CS(E)^G = \frac{k}{2}\left(\frac{k}{2}+1\right) + \frac{k}{2}\left(\frac{k}{2}+1\right) = k(k/2+1).$$

For k odd, a similar argument proves that

$$CS(E)^G = \left(\frac{k+1}{2}\right)^2 + \left(\frac{k+1}{2}\right)^2 = (k+1)^2/2.$$

□

Lemma 4.5. *In Case* B*, the semi-group* $\mathrm{Vect}_G(S(\rho), \chi)$ *is generated by two elements* $M_\chi^\pm$ *with relation* $2M_\chi^+ = 2M_\chi^-$*. Moreover,* $M_\chi^\pm$ *have* 2χ *as the character of the fiber* H*-modules, and they are related in such a way that* M_χ^- *is obtained from* M_χ^+ *by tensoring with a real* G*-line bundle over* $S(\rho)$ *with trivial fiber* H*-module.*

Proof. Since χ is of real type, the character of $U \otimes \mathbb{C}$ is also χ; so we may view χ as a complex irreducible character of H. We have proved in [CKMS01, Theorem B] that the semi-group $\mathrm{Vect}_G^{\mathbb{C}}(S(\rho), \chi)$ of isomorphism classes of complex G-vector bundles over $S(\rho)$ with multiples of χ as the character of fiber H-modules is generated by four elements $L_\chi^{\pm\pm}$ with relation $L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$, where $L_\chi^{\pm\pm}$ are complex G-vector bundles over $S(\rho)$ with $U \otimes \mathbb{C}$ as the fiber H-module such that the fiber G_1-modules (resp. G_μ-modules) of L_χ^{st} and $L_\chi^{s't'}$, where s, s', t and t' denote $+$ or $-$, agree if and only if $s = s'$ (resp. $t = t'$). In fact, the two non-isomorphic fiber G_1-modules (resp. G_μ-modules) of $L_\chi^{\pm\pm}$ are complex conjugate to each other, so the complex conjugate (or dual) bundles of L_χ^{++} and L_χ^{+-} are respectively L_χ^{--} and L_χ^{-+}.

Let $\Phi\colon \mathrm{Vect}_G^{\mathbb{C}}(S(\rho), \chi) \to \mathrm{Vect}_G(S(\rho), \chi)$ be the realification map. It is surjective by Lemma 4.4. Since any complex G-vector bundle is isomorphic to its complex conjugate bundle as real G-vector bundles, $\Phi(L_\chi^{++}) = \Phi(L_\chi^{--})$ and $\Phi(L_\chi^{+-}) = \Phi(L_\chi^{-+})$. Therefore the relation $L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$ on $\mathrm{Vect}_G^{\mathbb{C}}(S(\rho), \chi)$ reduces to $2\Phi(L_\chi^{++}) = 2\Phi(L_\chi^{+-})$ on $\mathrm{Vect}_G(S(\rho), \chi)$. It follows that for each fixed fiber dimension there are at most two elements in $\mathrm{Vect}_G(S(\rho), \chi)$. We claim that there is no other relation. It suffices to show that there are exactly two elements in $\mathrm{Vect}_G(S(\rho), \chi)$ for a fixed fiber dimension. If there is only one element for a fixed fiber dimension, say $2k \dim U$, then the unique bundle must have $(k+1)^2$ inequivalent G-invariant complex structures because the number of elements in $\mathrm{Vect}_G^{\mathbb{C}}(S(\rho), \chi)$ of (real) fiber dimension $2k \dim U$ is exactly $(k+1)^2$ [CKMS01, Corollary 5.2]. This contradicts Lemma 4.4.

It remains to show that the two generators $\Phi(L_\chi^{++})$ and $\Phi(L_\chi^{+-})$ are related by tensoring with a real G-line bundle with trivial fiber H-module. The fiber G_1-modules of L_χ^{+-} and L_χ^{++} at 1 are isomorphic but the fiber G_μ-modules of them at μ are not, more precisely, they are related through the tensor product with the nontrivial real 1-dimensional G_μ-module defined by $G_\mu \to G_\mu/H \cong \{\pm 1\}$, see Lemma 2.1 (1). Therefore $\Phi(L_\chi^{+-})$ is obtained from $\Phi(L_\chi^{++})$ by tensoring with a real G-line bundle with trivial fiber H-module, whose fiber at 1 is the trivial G_1-module and the fiber at μ is the nontrivial G_μ-module. The existence of such line bundle is guaranteed by Proposition 3.2. ∎

Proof of Theorem 1.1. The map Γ is surjective by Proposition 3.2 and injective except for Cases A and B by Proposition 3.3. In both Cases A and B the target of Γ is a semi-group generated by one element by Lemma 2.3 while the domain of Γ is generated by two elements with the relation shown in Lemmas 4.1 and 4.5. This implies that Γ is two to one.

Finally, we note that tensoring elements in $\mathrm{Vect}_G(S(\rho),\chi)$ with the real G-line bundles with trivial H-module does not change the fiber G_1-modules (resp. fiber G_1- and G_μ-modules) in Case A (resp. Case B). This implies the last statement in the theorem. $\square$

5. The semi-group structure on $\mathrm{Vect}_G(S(\rho),\chi)$

In this section we determine the semi-group structure on $\mathrm{Vect}_G(S(\rho),\chi)$. Let e_1 and e_μ denote the numbers of G_1- and G_μ-extensions of χ, respectively. When $\rho(G)$ agrees with $O(2)$ or is contained in $SO(2)$, we define e_μ to be 1 for convenience. In both real and complex category, the semi-group structure on the target of Γ is determined by the numbers e_1 and e_μ. The numbers e_1 and e_μ depend only on the types of $\rho(G)$ in the complex category, but this is not true in the real category. This is another complexity in our study arising from real representation theory.

The possible values of e_1 and e_μ according to $\rho(G)$ and the type of χ are given by Tables 2 and 3.

Table 2. The possible values of (e_1, e_μ) according to $\rho(G)$

(e_1,e_μ)	$(0,0)$	$(1,0)$	$(0,1)$	$(2,0)$	$(0,2)$	$(1,1)$	$(2,1)$	$(1,2)$	$(2,2)$
$\rho(G)\subset SO(2)$	×	×	×	×	×	○	×	×	×
$\rho(G)=O(2)$	×	×	○	×	×	○	○	×	×
$\rho(G)=D_n$	○	○	○	○	○	○	○	○	○

Table 3. The possible values of (e_1, e_μ) according to the type of χ when $\rho(G) = D_n$

(e_1,e_μ)	$(0,0)$	$(1,0)$	$(0,1)$	$(2,0)$	$(0,2)$	$(1,1)$	$(2,1)$	$(1,2)$	$(2,2)$
real	○	×	×	○	○	×	×	×	○
complex	○	○	○	○	○	○	○	○	○
quaternionic	×	×	×	×	×	○	○	○	○

We state here the semi-group structure on $\mathrm{Vect}_G(S(\rho),\chi)$ according to the values of e_1 and e_μ.

Theorem 5.1. *Except for Cases* A *and* B*, the semi-group* $\mathrm{Vect}_G(S(\rho),\chi)$ *is generated by*

(1) *one element* L_χ, *if* $(e_1,e_\mu)=(0,0),(1,0),(0,1)$ *or* $(1,1)$,

(2) *two elements* $L_\chi^\pm$, *if* $(e_1,e_\mu)=(2,1)$ *or* $(1,2)$,

(3) *three elements* $\widetilde{L}_\chi^0, \widetilde{L}_\chi^\pm$ *with relation* $2\widetilde{L}_\chi^0=\widetilde{L}_\chi^+ + \widetilde{L}_\chi^-$, *if* $(e_1,e_\mu)=(2,0)$ *or* $(0,2)$,

(4) *four elements* $L_\chi^{\pm\pm}$ *with relation* $L_\chi^{++}+L_\chi^{--}=L_\chi^{+-}+L_\chi^{-+}$, *if* $(e_1,e_\mu)=(2,2)$.
In Case A *or* B, *it is generated by*

(5) *two elements* $\widetilde{\widetilde{L}}{}_\chi^{\pm}$ *with relation* $2\widetilde{\widetilde{L}}{}_\chi^{+}=2\widetilde{\widetilde{L}}{}_\chi^{-}$.
Moreover, except for $\widetilde{L}_\chi^0$ *in the case* (3), *the generators are related through tensor product with real* G*-line bundles over* $S(\rho)$ *with trivial fiber* H*-module in each case.*

Proof. The statements (1)–(4) follow from Corollary 2.2, Lemma 2.3 and Theorem 1.1, and the statement (5) follows from Lemmas 4.1 and 4.5.

We now prove the last statement in the theorem. After setting $K_1=G_1$ and $K_2=G_\mu$, it is obvious that the inverse images of the pairs $(\mathbb{R}_1^\pm,\mathbb{R}_2^\pm)$ in the remark after Lemma 2.3 by the semi-group homomorphism Γ in Theorem 1.1 are real G-line bundles with trivial fiber H-module. Moreover, Γ preserves the two tensor product operations, one on $\mathrm{Vect}_G(S(\rho),\chi)$ with real G-line bundles and the other on $\mathrm{Rep}(G_1,G_\mu,\chi)$ by the pairs $(\mathbb{R}_1^\pm,\mathbb{R}_2^\pm)$. Therefore, Proposition 3.3 implies that, except for Cases A and B, the generators of $\mathrm{Vect}_G(S(\rho),\chi)$ are related through tensor product with real G-line bundles with trivial fiber H-module. The same argument also holds for $\mathrm{Rep}(G_1,\chi)$ by Corollary 2.2. For Cases A and B, the statement follows from the last statement in Lemmas 4.1 and 4.5. $\square$

Corollary 5.2. *Let* N *be the number of isomorphism classes of real* G*-vector bundles over* $S(\rho)$ *with* $m\chi$ *as the character of the fiber* H*-modules. If* m *is odd and either* e_1 *or* e_μ *is zero then* N *is zero. Otherwise, except for Cases* A *and* B,

$$N=\begin{cases}1, & \text{if}\quad (e_1,e_\mu)=(0,0),(1,0),(0,1),\text{or }(1,1),\\ m+1, & \text{if}\quad (e_1,e_\mu)=(2,0),(0,2),(2,1),\text{or }(1,2),\\ (m+1)^2, & \text{if}\quad (e_1,e_\mu)=(2,2).\end{cases}$$

In Case A *or* B, *the number* N *is exactly two.*

Proof. The proof is elementary and left to the reader. $\square$

6. Triviality of real G-vector bundles over a circle

In this section we investigate triviality of the generators in Theorem 5.1 when $\rho(G)$ is finite. Triviality of a G-vector bundle is closely related to the existence of a G-extension of the fiber H-module in the following sense: For a given H-module V, there exists at least one trivial G-vector bundle with V as its fiber H-module if V extends to a G-module. In the following we denote by Z_n the finite cyclic subgroup of $SO(2)$ generated by the rotation through an angle $2\pi/n$. Then $\rho(G)=Z_n$ for some n if $\rho(G)\subsetneq SO(2)$. Denote by 1 the trivial real H-module of dimension one, in other words, H acts trivially on 1. In the notation of Lemma 4.1 and Theorem 5.1, real G-line bundles over

$S(\rho)$ with trivial fiber H-module are denoted by $N_1^{\pm}$ and $L_1^{\pm\pm}$ according as $\rho(G) = Z_n$ and D_n, respectively.

Lemma 6.1. (1) *Suppose $\rho(G) = Z_n$. If n is even, then $N_1^{\pm}$ are both trivial. If n is odd, then one of them, say N_1^{+}, is trivial and N_1^{-} is nontrivial.*

(2) *Suppose $\rho(G) = D_n$. If n is even, then $L_1^{\pm\pm}$ are all trivial. If n is odd, then two of them, say L_1^{++} and L_1^{--}, are trivial and the other two are nontrivial.*

Proof. (1) Since G/H acts freely on $S(\rho)$, every real G-line bundle over $S(\rho)$ with trivial fiber H-module is the pull-back of a real line bundle over S^1 by the quotient map $\pi\colon S(\rho) \to S(\rho)/G \cong S^1$. Suppose n is even. Then $\pi^*\colon H^1(S(\rho)/G, \mathbb{Z}/2) \to H^1(S(\rho), \mathbb{Z}/2)$ is trivial, so pullback line bundles by π have trivial first Whitney classes, which means that the underlying line bundles over $S(\rho)$ are trivial. According to [KM94, Proposition 1.1], an equivariant line bundle is trivial if and only if its underlying bundle is trivial. Thus, $N_1^{\pm}$ are both trivial when n is even.

If n is odd, then π^* above is an isomorphism. Therefore, exactly one of $N_1^{\pm}$ has trivial first Whitney class. This together with the result in [KM94] mentioned above shows that exactly one of $N_1^{\pm}$ is trivial equivariantly.

(2) Set $P = \rho^{-1}(Z_n)$. Since P/H acts freely on $S(\rho)$, $L_1^{\pm\pm}$ are pullback of real G/P-line bundles over $S(\rho)/P$ by the quotient map $\pi\colon S(\rho) \to S(\rho)/P$. Here G/P is of order two and acts on the circle $S(\rho)/P$ as reflection, so we may think of G/P as D_1. According to Theorem 5.1 (or Corollary 5.2) there are four real D_1-line bundles over $S(\rho)/P$. Since the map Γ is an isomorphism in this case, they are distinguished by their fiber D_1-modules over the points $\pm 1 \in S(\rho)/P$. More precisely, there are two possibilities for the fiber D_1-modules at 1 and -1 respectively since there are two real one-dimensional D_1-modules (the trivial one and the nontrivial one), and hence altogether there are four real D_1-line bundles over $S(\rho)/P$. Moreover, D_1-line bundles are trivial if and only if the fiber D_1-modules at ± 1 are isomorphic (see also [Kim94]).

If n is even, then all pullback line bundles by π are trivial as discussed in (1); so $L_1^{\pm\pm}$ are all trivial. If n is odd, then the pullback by π preserves the triviality of line bundles because $\pi^*\colon H^1(S(\rho)/P; \mathbb{Z}/2) \to H^1(S(\rho); \mathbb{Z}/2)$ is an isomorphism. Since there are exactly two trivial D_1-line bundles over $S(\rho)/P$, two of $L_1^{\pm\pm}$ are trivial and the other two are nontrivial. $\square$

Remark. Suppose $\rho(G) = D_n$. For $z = 1$ and μ, denote by $\mathbb{R}_z^{+}$ and $\mathbb{R}_z^{-}$, respectively, the trivial and the nontrivial real G_z-module of dimension one with trivial H-action, see also the remark after Lemma 2.3. Then we may assume without loss of generality that the images of L_1^{st} by Γ in Theorem 1.1 are $(\mathbb{R}_1^{s}, \mathbb{R}_\mu^{t})$, where s and t denote a sign $+$ or $-$.

Theorem 6.2. *Let $\rho(G) = Z_n$ or D_n, and let χ be a real irreducible character of H which is G-invariant. If n is even, then the generators in Theorem* 5.1 *except for $\widetilde{L}_\chi^0$ are all trivial or all nontrivial in each case. If n is odd, then*

(1) *L_χ is trivial,*
(2) *$L_\chi^\pm$ are both trivial,*
(3) *$\widetilde{L}_\chi^0$ and $\widetilde{L}_\chi^\pm$ are all trivial,*
(4) *two of $L_\chi^{\pm\pm}$ are trivial and the other two are nontrivial,*
(5) *one of $\widetilde{\widetilde{L}}_\chi^\pm$ is trivial and the other is nontrivial.*

Proof. Recall from the last statement in Theorem 5.1 that all generators are related through tensor product with the real G-line bundles $N_1^\pm$ and $L_1^{\pm\pm}$ according as $\rho(G) = Z_n$ and D_n, respectively. These line bundles are all trivial if n is even by Lemma 6.1. So the existence of one trivial generator implies triviality of the other generators, and this finishes the proof in case that n is even.

In the following we assume that n is odd. Denote by U a real irreducible H-module with χ as its character. Recall that the fiber H-module of a generator is U if both e_1 and e_μ are nonzero, and $2U$ otherwise. In case $\rho(G) = D_n$, we choose elements a and b in G such that $\rho(a)$ is the rotation through the angle $2\pi/n$ and $\rho(b)$ is the reflection about the x-axis. Then G_1 (resp. G_μ) is generated by H and b (resp. ab).

(1) It suffices to show that the fiber H-module of a generator extends to a G-module. In case that $e_1 = e_\mu = 1$, the fiber H-module of L_χ is U and it is G-extendible by Lemma 2.4. The other case is that either $e_1 = 0$ or $e_\mu = 0$ and in this case the fiber H-module of L_χ is $2U$ which is G-extendible by Lemma 2.4 (2-b).

(2) In this case $\rho(G) = D_n$ by Table 2 and the fiber H-modules of generators are U which is G-extendible by Lemma 2.4 (2). So there is at least one trivial generator, say L_χ^+. Since $(e_1, e_\mu) = (2,1)$ or $(1,2)$, the tensor product of L_χ^+ with L_1^{--} has different fiber G_z-module from that of L_χ^+ at the point z such that $e_z = 2$. Thus we get the other generator $L_\chi^- \cong L_\chi^+ \otimes L_1^{--}$. Since L_1^{--} is trivial by Lemma 6.1, so is L_χ^-.

(3) In this case $\rho(G) = D_n$ by Table 2 and the fiber H-modules of generators are $2U$ because either e_1 or e_μ is zero. Set $P = \rho^{-1}(Z_n)$. Then U has a P-extension, say V, by Lemma 2.4 (1). Note that the fiber modules of $\widetilde{L}_\chi^0$ at 1 and μ are isomorphic to $\operatorname{ind}_H^{G_1} U$ and $\operatorname{ind}_H^{G_\mu} U$, respectively. Thus $\widetilde{L}_\chi^0$ is isomorphic to the product bundle $S(\rho) \times \operatorname{ind}_P^G V$ by Proposition 3.3.

We next consider triviality of the generators $\widetilde{L}_\chi^\pm$. It suffices to show that at least one generator, say $\widetilde{L}_\chi^+$, is trivial. Then so is the other generator $\widetilde{L}_\chi^- \cong \widetilde{L}_\chi^+ \otimes L_1^{--}$. We assume that $(e_1, e_\mu) = (2,0)$. The other case $(e_1, e_\mu) = (0,2)$ can be proved similarly.

Claim. χ is of real type.

Proof of Claim. Since χ is not of quaternionic type by Table 3, it suffices to prove that χ is not of complex type. Suppose that χ is of complex type. Then there is a complex H-module V such that $U \otimes \mathbb{C} \cong V \oplus \overline{V}$ and $V \not\cong \overline{V}$

as complex H-modules. We note that the realifications of V and $\overline{V}$ are U, and since χ is G-invariant, ${}^g\chi_V = \chi_V$ or $\chi_{\overline{V}}$ for $g \in G$ where χ_V and $\chi_{\overline{V}}$ denote the characters of V and $\overline{V}$ respectively.

Since $e_1 = 2$ and $e_\mu = 0$ by assumption, U has two G_1-extensions of complex type by Lemma 2.1 but no G_μ-extension. It follows that V is G_1-extendible but not G_μ-extendible, so χ_V is G_1-invariant but not G_μ-invariant. Namely, ${}^b\chi_V = \chi_V$ and ${}^{ab}\chi_V = \chi_{\overline{V}}$, so that ${}^a\chi_V = \chi_{\overline{V}}$. Therefore ${}^{a^n}\chi_V = \chi_{\overline{V}}$ because n is odd. On the other hand, since a^n is an element of H, ${}^{a^n}\chi_V = \chi_V$. Therefore $\chi_V = \chi_{\overline{V}}$, but this contradicts that $V \not\cong \overline{V}$. Thus χ must be of real type. □

Since χ is of real type by the claim above, $U \otimes \mathbb{C}$ is irreducible and its character is G-invariant. It follows that there is a trivial complex G-vector bundle F over $S(\rho)$ with $U \otimes \mathbb{C}$ as the fiber H-module, see [CKMS01, Theorem C (3)]. Since $e_1 = 2$, there are two G_1-extensions of U, say $\widetilde{U}_1$ and $\widetilde{U}_2$. Their complexifications $\widetilde{U}_1 \otimes \mathbb{C}$ and $\widetilde{U}_2 \otimes \mathbb{C}$ are non-isomorphic because $\widetilde{U}_1 \not\cong \widetilde{U}_2$. Moreover these modules are both G_1-extensions of $U \otimes \mathbb{C}$. Thus the fiber G_1-module, say F_1, of F at 1 must be either $\widetilde{U}_1 \otimes \mathbb{C}$ or $\widetilde{U}_2 \otimes \mathbb{C}$. It follows that the realification of F_1 is either $2\widetilde{U}_1$ or $2\widetilde{U}_2$. Therefore the realification of F, which is trivial, is isomorphic to one of $\widetilde{L}_\chi^\pm$.

(4) In this case $\rho(G) = D_n$ by Table 2 and the fiber H-modules of the generators are U. By a similar argument to case (2) there are two trivial generators, L_χ^{++} and $L_\chi^{--} \cong L_\chi^{++} \otimes L_1^{--}$. It suffices to show that the other two generators $L_\chi^{+-} \cong L_\chi^{++} \otimes L_1^{+-}$ and $L_\chi^{-+} \cong L_\chi^{++} \otimes L_1^{-+}$ are nontrivial. Consider the following isomorphisms

(**)
$$\mathrm{Hom}_H(L_\chi^{++}, L_\chi^{+-}) \cong \mathrm{Hom}_H(L_\chi^{++}, L_\chi^{++} \otimes L_1^{+-}) \cong \mathrm{Hom}_H(L_\chi^{++}, L_\chi^{++}) \otimes L_1^{+-}.$$

Note that $\mathrm{Hom}_H(L_\chi^{++}, L_\chi^{++})$ is isomorphic to the product bundle $S(\rho) \times \mathbb{R}^k$, where $k = 1, 2,$ or 4 according to the type of χ. It follows that $\mathrm{Hom}_H(L_\chi^{++}, L_\chi^{+-}) \cong kL_1^{+-}$.

Claim. Both kL_1^{+-} and kL_1^{-+} are nontrivial for all $k > 0$.

Proof of Claim. Note that the fiber G_1-module of L_1^{+-} at $1 \in S(\rho)$ is the trivial G_1-module $\mathbb{R}_1^+$, while the fiber G_μ-module at μ is the nontrivial G_μ-module $\mathbb{R}_\mu^-$ by the remark after Lemma 6.1. Then b (resp. ab) acts on $\mathbb{R}_1^+$ (resp. $\mathbb{R}_\mu^-$) as multiplication by 1 (resp. -1). Recall that H acts on both $\mathbb{R}_1^+$ and $\mathbb{R}_\mu^-$ trivially, i.e., as multiplication by 1.

Assume that kL_1^{+-} is trivial. Then there exists a G-module W such that $\mathrm{res}_{G_1} W \cong k\mathbb{R}_1^+$ and $\mathrm{res}_{G_\mu} W \cong k\mathbb{R}_\mu^-$. Thus b and ab act on W as multiplication by 1 and -1, respectively. Hence a acts on W as multiplication by -1, and since n is odd, a^n also acts on W as multiplication by -1. But this contradicts that $a^n \in H$ acts trivially on W. In the same way we can prove that kL_1^{-+} is also nontrivial. □

Since L_χ^{++} is trivial, L_χ^{+-} must be nontrivial by the equation (**) and the claim above. Replacing L_χ^{+-} by L_χ^{-+} we can similarly prove that L_χ^{-+} is nontrivial.

(5) *Case* A. In this case $\widetilde{\widetilde{L}}_\chi^{\pm}$ are $N_\chi^{\pm}$ in Lemma 4.1. Since n is odd, U has a G-extension by Lemma 2.4 (1). So we may assume that one generator, say N_χ^+, is trivial. Then the following isomorphisms

$$\mathrm{Hom}_H(N_\chi^+, N_\chi^-) \cong \mathrm{Hom}_H(N_\chi^+, N_\chi^+ \otimes N_1^-) \cong \mathrm{Hom}_H(N_\chi^+, N_\chi^+) \otimes N_1^- \cong N_1^-$$

imply that N_χ^- is nontrivial since N_1^- is nontrivial by Lemma 6.1 (1).

Case B. In this case $\widetilde{\widetilde{L}}_\chi^{\pm}$ are $M_\chi^{\pm}$ in Lemma 4.5. Remember that $M_\chi^+ = \Phi(L_\chi^{++}) = \Phi(L_\chi^{--})$ and $M_\chi^- = \Phi(L_\chi^{+-}) = \Phi(L_\chi^{-+})$ from the proof of Lemma 4.5. Since $L_\chi^{++} \in \mathrm{Vect}_G^{\mathbb{C}}(S(\rho), \chi)$ is trivial by Theorem C in [CKMS01], M_χ^+ is also trivial. In the following we shall prove that M_χ^- is nontrivial.

Assume that M_χ^- is trivial, i.e., it is isomorphic to the product bundle $S(\rho) \times W$ for some G-extension W of the fiber H-module $2U$.

Claim. W is of real type.

Proof of Claim. If W is not of real type, then we may view M_χ^- as the realification of a complex product bundle $S(\rho)\times W$, but this contradicts that M_χ^- is the realification of the nontrivial bundles L_χ^{+-} and L_χ^{-+} in $\mathrm{Vect}_G^{\mathbb{C}}(S(\rho), \chi)$. □

Denote by χ_W the character of W. Every fiber G_z-module of M_χ^-, which is $\mathrm{res}_{G_z} W$, is isomorphic to $\mathrm{ind}_H^{G_z} U$ and it is irreducible of complex type by Lemma 2.1 (2). It is well known in representation theory that the character of $\mathrm{res}_{G_z} W \cong \mathrm{ind}_H^{G_z} U$ is zero on $G_z \setminus H$. Thus χ_W is always zero on $\bigcup_{z\in S(\rho)} G_z \setminus H = G \setminus P$, where $P = \rho^{-1}(Z_n)$. It follows that we have

$$1 = \int_G \chi_W(g)^2 dg = \frac{1}{2}\int_P \chi_W(p)^2 dp + \frac{1}{2}\int_{G\setminus P} \chi_W(p)^2 dp = \frac{1}{2}\int_P \chi_W(p)^2 dp,$$

(see [BtD85, Exercises 6.10 (3), Chapter II] for the first equality) so $\int_P \chi_W(p)^2 dp = 2$. This implies that $\mathrm{res}_P W$ is either irreducible of complex type or reducible with different direct summands of real type. In the sequel we show that neither case occurs.

It is easy to see that the latter case does not occur because if it does, then each summand of $\mathrm{res}_P W$ is a P-extension of U which contradicts the uniqueness of the P-extension of U by Lemma 2.4 (1).

Now, suppose $\mathrm{res}_P W$ is irreducible and of complex type. We claim that the set $\mathcal{J}(W)^G$ of G-invariant complex structures on W is not empty. Then it contradicts that W is of real type. Since $\mathrm{res}_{G_1} W$ is irreducible and of complex type, there exists a G_1-invariant complex structure on W, i.e., $(\mathcal{J}(W)^H)^{G_1/H} = \mathcal{J}(W)^{G_1} \neq \varnothing$. This means that each connected component of $\mathcal{J}(W)^H \cong \mathcal{J}(\mathbb{R}^2)$ is invariant under the action of G_1/H because $\mathcal{J}(W)^H \cong \mathcal{J}(\mathbb{R}^2)$ has two connected components and G_1/H is of order two. On the other hand, since the order n of P/H is odd and the number of connected components of $\mathcal{J}(W)^H$ is two, each connected component is also invariant under the P/H-action. Therefore, it is invariant under the G/H-action because P and G_1 generate G. Now we note that each connected component of $\mathcal{J}(W)^H \cong \mathcal{J}(\mathbb{R}^2)$ is homeomorphic to $\mathbb{R}^2$ (see [MS98, Exercise 2.57]) and that any smooth action of a finite group on $\mathbb{R}^2$ is linear, so the G/H-action on $\mathcal{J}(W)^H$ has a fixed point, i.e., $\mathcal{J}(W)^G \neq \varnothing$. $\square$

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