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Existence of solutions to initial value problems for the second order mixed monotone type of impulsive differential inclusions

By

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Abstract

In this paper we consider the initial value problems for the second order mixed monotone type of impulsive differential inclusions. By introducing a special partial order, we present the existence of maximal and minimal fixed points for mixed monotone multivalued operators in Banach spaces. Applying the results, we establish the existence and uniqueness of above impulsive differential inclusions.

1. Introduction

Recently, the theory of impulsive differential equations and inclusions has been emerging as an important area of investigation (see [1]). The first- and second-order problems in this area has been widely discussed (see [1–6]). In the earlier paper [2], the Banach fixed point theorem was used to investigate the existence and uniqueness of solutions to second order impulsive integro-differential equations in Banach spaces. In [3], the coupled fixed point theorem for mixed monotone condensing operators was used to discuss the initial value problems for the second order mixed monotone type of impulsive differential equations. In this paper, we shall use the coupled fixed point theorem for multivalued operators which is derived in the present paper to investigate the existence and uniqueness of solutions for the initial value problems of the second order mixed monotone type of impulsive differential inclusions (IDI) in a special partial order Banach space.

Our paper has two main sections. In Section 2, we shall derive some coupled fixed point results for multivalued operators on special partially ordered sets by means of monotone sequence of iterations of such operators. Coupled fixed point theorems for mixed monotone operators have been considered in [7–9]. For instance, in [7] the existence and iterative approximation of coupled fixed points for multivalued operators are proved by applying the set-condensing condition, in [8] the existence result for single-valued operators is given by using

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the completely continuous property of operators. Then, in Section 3, offer some applications to the initial value problems for the second order mixed monotone type of IDI.

Let $(E, |\cdot|)$ be a Banach space, $(Y, |\cdot|_Y)$ a ordered Banach space. For the sake of convenience, we first recall some definitions.

Definition 1. A function $p : E \to Y$ is said to be of class $\mathcal{B}$, denoted by $p \in \mathcal{B}$, if p is uniformly continuous on E and $p(x) = p(y)$ if and only if $x = y$.

Given a $p \in \mathcal{B}$, introduce a partial ordering $\leq$ in E as follows:
$x \leq y$ if and only if $p(x) \leq p(y)$ and $x < y$ if and only if$x \leq y$ and$x \neq y$. Here $x, y \in E$.

$C(J, E)$ is a Banach space consisting of all continuous functions from $J = [0, 1]$ into E with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$. For any $x, y \in C(J, E)$, define $x \leq y$ if and only if $x(t) \leq y(t)$ for each $t \in J$, $x < y$ if and only if $x \leq y$ and there exists some $t \in J$ such that $x(t) \neq y(t)$.

For two subsets A, B of E we write $A \leq B$ if

$$\forall a \in A \ \ \exists\, b \in B \text{ such that } a \leq b.$$

Definition 2. Let D be a subset of E, and $A : D \times D \to 2^E$ a multi-valued operator. A is said to be mixed monotone, if $A(x, y)$ is increasing in x and decreasing in y, that is,

(a$_1$) for each $y \in D$ and any $x_1, x_2 \in D$ with $x_1 \leq x_2$ $(x_1 \geq x_2)$, if $u_1 \in A(x_1, y)$ then there exists a $u_2 \in A(x_2, y)$ such that $u_1 \leq u_2$ $(u_1 \geq u_2)$;

(a$_2$) for each $x \in D$ and any $y_1, y_2 \in D$ with $y_1 \leq y_2$ $(y_1 \geq y_2)$ if $v_1 \in A(x, y_1)$ then there exists a $v_2 \in A(x, y_2)$ such that $v_1 \geq v_2$ $(v_1 \leq v_2)$.

Definition 3. Let D be a subset of E. Point $(x, y) \in D \times D$ is called a coupled fixed point of A, if $x \leq y$ and

$$x \in A(x, y), \quad y \in A(y, x).$$

Point (x^*, y^*) is said to be a coupled minimax fixed point if it is a coupled fixed point and satisfies that $x^* \leq x \leq y \leq y^*$ for any coupled fixed point (x, y) of A.

Definition 4. Let D be a subset of E. $A : D \times D \to 2^E$ is called p-continuous at point $(x_0, y_0) \in D \times D$, if for any sequences $\{x_n\}$, $\{y_n\} \subset D$, $p(x_n) \to p(x_0)$, $p(y_n) \to p(y_0)$ and any weak neighbourhood W of $A(x_0, y_0)$, there exists a positive integer N such that for $n \geq N$, we have $p \circ A(x_n, y_n) \subset p(W)$. If A is p-continuous at each point of $D \times D$, then A is called p-continuous on $D \times D$.

2. Existence of coupled minimax fixed points

Throughout this section we always assume that E is partially ordered by a given $p \in \mathcal{B}$. Take $u_0, v_0 \in E$ with $u_0 \leq v_0$ and denote by $D = [u_0, v_0] =$

$\{u \in E : u_0 \le u \le v_0\}$ the ordered interval of E which is bounded with regard to the norm.

Theorem 1. *Let $A : D \times D \to 2^D$ be a p-continuous mixed monotone miltivalued operator with nonempty weakly closed values. Suppose that A satisfies the conditions*:

(*H*) *Let $C_1 = \{x_n\}$ and $C_2 = \{y_n\}$ be countable and totally ordered subsets that satisfy $C_1 \subset cl(\{x_1\} \cup A(C_1, C_2))$ and $C_2 \subset cl(\{y_1\} \cup A(C_2, C_1))$, respectively, then C_1 and C_2 both are relatively compact.*

Then A has a coupled fixed point $(x^*, y^*) \in D \times D$ and

$$p(x^*) = \lim_{n\to\infty} p(u_n), \quad p(y^*) = \lim_{n\to\infty} p(v_n),$$

where $u_n \in A(u_{n-1}, v_{n-1})$ and $v_n \in A(v_{n-1}, u_{n-1})$ for $n = 1, 2, \ldots$ satisfy the following condition:

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0,$$

and if $u_{n+1} = u_n$, $v_{n+1} = v_n$ then $u_{n+k} = u_n$, $v_{n+k} = v_n$ for $k = 1, 2, \ldots$. Moreover, (x^*, y^*) is the coupled minimax fixed point of A in $D \times D$.

Proof. We first prove that A has a coupled mninmax fixed point if A has at least a coupled fixed point. In order to do this, we define

$$B = D_1 \times D_2 = \{(x, y) : x \in A(x, y) \cap D, y \in A(y, x) \cap D\}.$$

Then B is nonempty under this hypothesis. Let us introduce a partial order in B by

$$(x_1, y_1) \le (x_2, y_2) \Leftrightarrow x_2 \le x_1 \le y_1 \le y_2$$

for any $(x_1, y_1), (x_2, y_2) \in B$. We are now in a position to prove the existence of maximal element of B. In order to apply Zorn's Lemma, we consider any given totally ordered subset $M = D_1^M \times D_2^M$ of B. It is sufficient to show that M an upper bound. First we prove that any sequence $\{(x_n', y_n')\}$ of M there is a convergent subsequence. For each $n = 1, 2, \ldots$, let

$$(x_n, y_n) = \max\{(x_1', y_1'), (x_2', y_2'), \ldots, (x_n', y_n')\},$$

then $\{(x_n, y_n)\}$ is increasing. Let $C_1 = \{x_n\}$ and $C_2 = \{y_n\}$, then C_1 and C_2 satisfy the condition (H), therefore, they are relatively compact, which shows that $\{(x_n', y_n')\}$ there is a convergent subsequence. Moreover, this implies that M is relatively compact. Note that $p(M) = p(D_1^M) \times p(D_2^M)$ is also relatively compact, hence it is separable, i.e., there exists a countable subset $\{(x_n', y_n')\}$ of M such that $\{(p(x_n'), p(y_n'))\}$ is dense in $P(M)$. Take $(x_n, y_n) = \max\{(x_1', y_1'), (x_2', y_2'), \ldots, (x_n', y_n')\}$ for $n = 1, 2, \ldots$, then $\{(x_n, y_n)\}$ is an increasing sequence and $\{(p(x_n), p(y_n))\}$ is dense in $P(M)$. We claim that there exists $(x', y') \in B$ such that

$$p(x') = \inf_{x \in D_1^M} p(x), \quad p(y') = \sup_{y \in D_2^M} p(y). \tag{2.1}$$

Indeed, if there exists some $\{(x_n, y_n)\}$ such that (2.1) is satisfied, then our claim holds. Otherwise, since $p(M)$ is relatively compact, there exists a subsequence $\{(x_{n_i}, y_{n_i})\}$ of $\{(x_n, y_n)\}$ and a point $(x', y') \in E$ such that

$$p(x_{n_i}) \to p(x'),\ p(y_{n_i}) \to p(y')$$

for $i \to \infty$. It is easy to prove $p(x_n) \to p(x'),\ p(y_n) \to p(y')$ for $n \to \infty$ since $\{x_n\}$ is decreasing and $\{y_n\}$ is increasing. In virtue of the density of $\{(p(x_n), p(y_n))\}$ we conclude that (x', y') satisfies (2.1).

It remains to prove that $(x', y') \in B$. Obviously, $(x', y') \in D \times D$. It is enough to show that (x', y') is a coupled fixed point of A. In fact, by the p-continuity of A at point (x', y'), for any weakly closed neighbourhood W of $A(x', y')$, there exists a positive integer N such that for $n \geq N$, we have

$$p(x_n) \in p \circ A(x_n, y_n) \subset p(W).$$

Let n tend to infinity, from the continuity of p it follows that $p(x') \in p(W)$. The definition of p guarantees that $x' \in W$. By the arbitrariness of W, we obtain that x' is a weak cluster of $A(x', y')$. Since $A(x', y')$ is weakly closed, $x' \in A(x', y')$. Similarly we can prove that $y' \in A(y', x')$. Hence, (x', y') is an upper bound and $(x', y') \in B$. This implies that B has a maximal element (x^*, y^*). It is easy to see that (x^*, y^*) is a coupled minimax fixed point of A.

Next, we prove the existence of coupled fixed points of A on $D \times D$ and etceteras of Theorem 1 hold. If $u_0 \in A(u_0, v_0)$ and $v_0 \in A(v_0, u_0)$, take $x^* = u_n = u_0,\ y^* = v_n = v_0$ for $n = 1, 2, \ldots$, then the conclusion of Theorem 1 is proved. Otherwise, from the mixed monotonicity of A, for any $u_1' \in A(u_0, v_0)$, there exists $\bar{v}_1 \in A(u_1', v_0)$ with $u_1' \leq \bar{v}_1$ and there exists $\tilde{v}_1 \in A(u_1', u_0)$ such that $\bar{v}_1 \leq \tilde{v}_1$. Moreover, there exists $v_1' \in A(v_0, u_0)$ such that $\tilde{v}_1 \leq v_1' \leq v_0$. On the other hand, it is obvious that $u_0 < x^* \leq y^* < v_0$, thus there exist $v_1^* \in A(v_0, u_0)$ $u_1^* \in A(u_0, v_0)$ such that $u_0 \leq u_1^* \leq x^* \leq y^* \leq v_1^* \leq v_0$. We now take $u_1 = \min\{u_1', u_1^*\}, v_1 = \max\{v_1', v_1^*\}$, then

$$u_0 \leq u_1 \leq x^* \leq y^* \leq v_1 \leq v_0.$$

Repeating this process, we can inductively get sequences $\{u_n\}$ and $\{v_n\}$ such that

$$u_k \in A(u_{k-1}, v_{k-1}), \quad v_k \in A(v_{k-1}, u_{k-1})$$

and

$$u_0 \leq u_{k-1} \leq u_k \leq x^* \leq y^* \leq v_k \leq v_{k-1} \leq v_0 \text{ for } k = 1, 2, \ldots.$$

It is clear that $C_1 = \{u_n\}$ and $C_2 = \{v_n\}$ both satisfy the condition (H), hence, both are relatively compact. By the same process as the above, there exist $x, y \in E$ such that $p(x) = \lim_{n\to\infty} p(u_n)$ and $p(y) = \lim_{n\to\infty} p(v_n)$. Clearly, $(x, y) \in D \times D$. By the same way as the above proof we can verify that (x, y) is a coupled fixed point of A on $D \times D$ and

$$x \leq x^* \leq y^* \leq y. \tag{2.2}$$

On the other hand, we have proved that (x^*, y^*) is a coupled minimax fixed point, which, together with $(x, y) \in B$, implies that

$$x^* \leq x \leq y \leq y^*.$$

This, combining (2.2), yields $x = x^*, y = y^*$. This completes the proof of Theorem 1. □

Corollary 1. *Suppose that A satisfies all conditions in Theorem 1 except for (H), then the results of Theorem 1 hold if $p \in \mathcal{B}$ is completely continuous.*

If E is a partially ordered Banach space, then the results of Theorem 1 hold when we take p to be identity mapping. In addition, we have

Corollary 2. *Let E be a real Banach space with a partial order introduced by a normal cone of E, $A : D \times D \to 2^D$ a p-continuous mixed monotone multivalued operator with nonempty weakly closed values. Then the results of Theorem 1 hold if one of the following hypotheses holds.*

(h1) *A is completely continuous.*

(h2) *Assume that $A(D, D)$ is weakly sequentially compact in E.*

(h3) *For any $D_1, D_2 \subset D$ if either $\gamma(D_1)$ or $\gamma(D_2)$ is greater than 0, then we have*

$$\gamma(A(D_1, D_2)) < \max\{\gamma(D_1), \gamma(D_2)\},$$

where γ stands for Kuratowski noncompactness measure.

Remark 1. (h1) and (h2) are the main conditions of [8] and [9], respectively, for single-valued operators, (h3) is the main condition of [7]. Hence, the results presented here extend and improve the corresponding results of the above cited references.

Theorem 2. *Let all assumptions in Theorem* 1 *be satisfied. For any $x, y \in D$, suppose that there exists* $0 < L < 1$ *such that*

$$|p(u) - p(v)|_Y \leq L|p(x) - p(y)|_Y$$

for all $u \in A(x, y)$, $v \in A(y, x)$. Then A has a unique fixed point u^* in D and $x^* = u^* = y^*$, where (x^*, y^*) is the coupled minimax fixed point of A in D.

The proof of this theorem we refer to [8].

3. Solvability of impulsive differential inclusions

In this section, as an application of Theorem 1, we shall discuss the initial value problems for the second order mixed monotone type of impulsive differential inclusions (IDI) in the partial ordered Banach space E,

$$\begin{cases} u'' \in F(t, u, u) & t \in J,\ t \neq t_i, \\ \Delta u|_{t=t_i} = I_i(u(t_i)) & \\ \Delta u'|_{t=t_i} = \bar{I}_i(u(t_i)) & (i = 1, 2, \ldots, m), \\ u(0) = w_0, \quad u'(0) = w_1 & \end{cases} \tag{3.1}$$

where $F : J \times E \times E \to W(E)$ is p-continuous multivalued map, $W(E)$ is the family of all nonempty weakly closed subsets of E, $J = [0,1]$, $I_i,\ \bar{I}_i \in C(E,E)$ and all $I_k, \bar{I}_k$ are monotone operators for $i = 1,2,\dots,m$, $0 < t_1 < t_2 < \dots < t_m < 1$, $\Delta u|_{t=t_i} = u(t_i^+) - u(t_i^-)$ with $u(t_i^+)$ and $u(t_i^-)$ representing the right and left limits of $u(t)$ at $t = t_i$, respectively, and $\Delta u'|_{t=t_i}$ has a similar meaning for $u'(t)$, $w_0, w_1 \in E$.

Let $PC[J,E] = \{x : x$ is a function from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+)$ exist for $k = 1,2,\dots,m\}$, $PC^1[J,E] = \{x : x$ is a function from J into E such that $x(t)$ is continuously differentiable at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+),\ x'(t_k^+),\ x'(t_k^-)$ exist for $k = 1,2,\dots,m\}$. Throughout this section, $u'(t_i)$ is understood as $u'_-(t_i)$ (see [2]). Evidently, $PC[J,E]$ and $PC^1[J,E]$ both are Banach spaces with norm $|x|_{PC} = \sup\limits_{t\in J} |x(t)|$ and $|x|_{PC^1} = \max\{|x(t)|_{PC},\ |x'|_{PC}\}$.

Given $p \in \mathcal{B}$, and let E be partially ordered as in section 1, then $Q = \{u \in PC^1[J,E] : u(t) \geq 0, u'(t) \geq 0, t \in J\}$ is a cone in $PC^1[J,E]$. Denote $J' = J/\{t_1.t_2,\dots,t_m\}$. $u, v \in PC^1[J,E] \cap C^2[J',E]$ with $u \leq v$ is called a coupled solution of IDI(3.1) if

$$\begin{cases} u'' \in F(t,u,v) & t \in J,\ t \neq t_i, \\ \Delta u|_{t=t_i} = K_i(u(t_i), v(t_i)) & \\ \Delta u'|_{t=t_i} = \bar{K}_i(u(t_i), v(t_i)) & (= 1,2,\dots,m), \\ u(0) = w_0, \quad u'(0) = w_1 & \end{cases}$$

$$\begin{cases} v'' \in F(t,v,u) & t \in J,\ t \neq t_i, \\ \Delta u|_{t=t_i} = K_i(v(t_i), u(t_i)) & \\ \Delta u'|_{t=t_i} = \bar{K}_i(v(t_i), u(t_i)) & (i = 1,2,\dots,m), \\ u(0) = w_0, \quad u'(0) = w_1 & \end{cases}$$

where

$$K_i(x,y) = \begin{cases} I_i(x) & \text{if } I_i \text{ is increasing} \\ I_i(y) & \text{if } I_i \text{ is decreasing,} \end{cases}$$

$$\bar{K}_i(x,y) = \begin{cases} \bar{I}_i(x) & \text{if } \bar{I}_i \text{ is increasing} \\ \bar{I}_i(y) & \text{if } \bar{I}_i \text{ is decreasing.} \end{cases}$$

If $u = v = x$, then x is called a solution of IDI(3.1).

For any $x, y \in PC[J,E]$, the set of L^1- selections $S_{F,x,y}$ of the multivalued map F defined by

$$S_{F,x,y} := \{f_{x,y} \in L^1(J,E) : f_{x,y}(t) \in F(t,x(t),y(t)) \text{ a. e. for } t \in J\}.$$

This may be empty. It is nonempty if and only if the function $z : J \to \mathbf{R}$ defined by

$$z(t) = \inf\{|v| : v \in F(t,x(t),y(t))\}$$

belongs to $L^1(J, \mathbf{R})$ (see [10]).

Throughout this paper we always assume that the multivalued map F has nonempty, weakly closed values and L^1−selections $S_{F,x,y}$ is nonempty.

Let us list the following hypotheses which are crucial in the proof of our main theorems.

(i) All $I_k, \bar{I}_k$ $(k = 1, 2, \ldots, m)$ satisfy

$$\sup\{|I_k(x(t_k))| : x \in E, 1 \le k \le m\} < \infty,$$
$$\sup\{|\bar{I}_k(x(t_k))| : x \in E, 1 \le k \le m\} < \infty;$$

(ii) $F(t, x, y)$ is increasing in $x \in E$ for each fixed $t \in J$ and $y \in E$, decreasing in $y \in E$ for each fixed $x \in E$ and $t \in J$;

(iii) There exist functions $u_0, v_0 \in PC^1[J, E] \cap C^2[J', E]$ with $u_0 \le v_0$ and $D = [u_0, v_0]$ is bounded with regard to the norm in $PC^1[J, E]$ such that

$$\begin{cases} \{u_0''(t)\} \le F(t, (u_0), v_0(t)), & t \in J', \\ \triangle u_0|_{t=t_k} \le \min\{I_k(u_0(t_k)), I_k(v_0(t_k)), & \\ \triangle u_0'|_{t=t_k} \le \min\{\bar{I}_k(u_0(t_k)), \bar{I}_k(v_0(t_k)) & k = 1, 2, \ldots, m \\ u_0(0) \le w_0, \quad u_0'(0) \le w_1 & \end{cases}$$

$$\begin{cases} \{v_0''(t)\} \ge F(t, (v_0), u_0(t)), & t \in J', \\ \triangle v_0|_{t=t_k} \ge \max\{I_k(u_0(t_k)), I_k(v_0(t_k)), & \\ \triangle v_0'|_{t=t_k} \ge \max\{\bar{I}_k(u_0(t_k)), \bar{I}_k(v_0(t_k)) & k = 1, 2, \ldots, m \\ v_0(0) \ge w_0, \quad v_0'(0) \ge w_1 & \end{cases}$$

(iv) $\sup\{|w(t)| : w(t) \in F(t, x, y)\} \le \alpha(t)$ a.e. on J, for all $x, y \in E$. Here the function α satisfies that $\beta(t) := \int_0^t \alpha(s)ds \in L^1(J, \mathbf{R}_+)$.

(v) There exists a function $\omega : J \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, $\omega(t, \cdot, \cdot)$ being increasing for given $t \in J$, such that

$$\gamma(F(t, M_1, M_2)) \le \omega(t, \gamma(M_1), \gamma(M_2)) \text{ a.e. on } J.$$

for every set $M_j \subset D$ satisfying $\sup\{|x| : x \in M_j\} \le \alpha(t)$ $(j = 1, 2)$ with $\alpha(t)$ given as in (iv). In addition $\rho(t) = 0$ is the unique solution in $L^1(J, \mathbf{R}_+)$ to the inequality

$$\rho(t) \le 2 \int_0^t \omega(t, \rho(s), \rho(s))ds \text{ a.e. on } J.$$

Theorem 3. *If conditions* (i)–(v) *are satisfied, then IDI*(3.1) *has the coupled minimax solution* (u^*, v^*) *with* $u^*, v^* \in D \cap PC^1[J, E] \cap C^2[J', E]$. *Moreover, we construct the iterative sequences*

$$
\begin{aligned}
u_n(t) = w_0 + w_1 t &+ \int_0^t (t-s) f_{u_{n-1},v_{n-1}}(s)ds \\
&+ \sum_{0<t_{i'}<t} I_{i'}(u_{n-1}(t_{i'})) + \sum_{0<t_{i''}<t} I_{i''}(v_{n-1}(t_{i''})) \\
&+ \sum_{0<t_{i_1}<t} (t-t_{i_1})\bar{I}_{i_1}(u_{n-1}(t_{i_1})) \\
&+ \sum_{0<t_{i_2}<t} (t-t_{i_2})\bar{I}_{i_2}(v_{n-1}(t_{i_2})),
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
v_n(t) = w_0 + w_1 t &+ \int_0^t (t-s) f_{v_{n-1},u_{n-1}}(s)ds \\
&+ \sum_{0<t_{i'}<t} I_{i'}(v_{n-1}(t_{i'})) + \sum_{0<t_{i''}<t} I_{i''}(u_{n-1}(t_{i''})) \\
&+ \sum_{0<t_{i_1}<t} (t-t_{i_1})\bar{I}_{i_1}(v_{n-1}(t_{i_1})) \\
&+ \sum_{0<t_{i_2}<t} (t-t_{i_2})\bar{I}_{i_2}(u_{n-1}(t_{i_2}))
\end{aligned} \tag{3.3}
$$

for $n = 1, 2, \ldots$. We have $p(u_n(t)) \to p(u^*(t))$ and $p(v_n(t)) \to p(v^*(t))$ $(n \to \infty)$, where $I_{i'}, \bar{I}_{i_1}$ are increasing, $I_{i''}, \bar{I}_{i_2}$ are decreasing, $f_{u_{n-1},v_{n-1}} \in S_{F,u_{n-1},v_{n-1}}$ and $f_{v_{n-1},u_{n-1}} \in S_{F,v_{n-1},u_{n-1}}$.

Proof. We denote by Q the set $D \cap PC^1[J,E] \cap C^2[J',E]$. In virtue of the references [2] and [3] we can prove that $(u,v) \in Q \times Q$ is a coupled solution of IDI(3.1) if and only if (u,v) is a coupled fixed point of A defined by

$$
\begin{aligned}
A(u,v)(t) = w_0 + w_1 t &+ \int_0^t (t-s) f_{u,v}(s)ds \\
&+ \sum_{0<t_{i'}<t} I_{i'}(u(t_{i'})) + \sum_{0<t_{i''}<t} I_{i''}(v(t_{i''})) \\
&+ \sum_{0<t_{i_1}<t} (t-t_{i_1})\bar{I}_{i_1}(u(t_{i_1})) + \sum_{0<t_{i_2}<t} (t-t_{i_2})\bar{I}_{i_2}(v(t_{i_2})),
\end{aligned}
$$

where $I_{i'}, \bar{I}_{i_1}$ are increasing, $I_{i''}, \bar{I}_{i_2}$ are decreasing and $f_{u,v} \in S_{F,u,v}$. To prove the existence of coupled solutions for problem IDI(3.1) is thus sufficient to see that are satisfied the hypotheses in Theorem 1.

It has been proved in [3] that A is the multivalued operator from $D \times D$ into 2^D, by the condition (ii) we know easily A is mixed monotone and p-continuous, and clearly, A has nonempty weakly closed values.

It remains to prove that the condition (H) is satisfied. Suppose that the sets $C_1 = \{x_n\} \subset D, C_2 = \{y_n\} \subset D$ both are countable and totally ordered and satisfy $C_1 \subset cl(\{x_1\} \cup A(C_1, C_2))$, $C_2 \subset cl(\{y_1\} \cup A(C_2, C_1))$, we have to prove that the set C_1, C_2 are relatively compact. Similar to [4], we can obtain that there exist $\varphi, \psi \in L^1[J, \mathbf{R}_+]$ such that

$$|x_n(t)| \le \varphi(t), \quad |y_n(t)| \le \psi(t)$$

for a.e. all $t \in J$, where $n = 1, 2, \ldots$. Hence, by virtue of [10] we see that $\gamma(\{f_{x_n,y_n}(t) : n \geq 1\}) \in L^1(J, \mathbf{R}_+)$, $\gamma(\{f_{y_n,x_n}(t) : n \geq 1\}) \in L^1(J, \mathbf{R}_+)$ for given $t \in J$ and

$$\begin{aligned}\gamma(C_1(t)) &= \gamma\left(\left\{\int_0^t (t-s) f_{x_n,y_n}(s) ds : n \geq 1\right\}\right)\\ &\leq 2\int_0^t \gamma(\{f_{x_n,y_n}(s) : n \geq 1\})\, ds\\ \gamma(C_2(t)) &= \gamma\left(\left\{\int_0^t (t-s) f_{y_n,x_n}(s) ds : n \geq 1\right\}\right)\\ &\leq 2\int_0^t \gamma(\{f_{y_n,x_n}(s) : n \geq 1\})\, ds.\end{aligned}$$

for each $t \in J_0 = [0, t_1]$. While by means of (v) we have

$$\begin{aligned}\gamma(\{f_{x_n,y_n}(s) : n \geq 1\}) &\leq \gamma(F(s, C_1(s), C_2(s)))\\ &\leq \omega(s, \gamma(C_1(s)), \gamma(C_2(s))) \leq \omega(s, \mu(s), \mu(s)),\\ \gamma(\{f_{y_n,x_n}(s) : n \geq 1\}) &\leq \gamma(F(s, C_2(s), C_1(s)))\\ &\leq \omega(s, \gamma(C_2(s)), \gamma(C_1(s))) \leq \omega(s, \mu(s), \mu(s)),\end{aligned}$$

where $\mu(s) = \max\{\gamma(C_1(s)), \gamma(C_2(s))\}$. It yields

$$\mu(t) \leq 2\int_0^t \omega(s, \mu(s), \mu(s)) ds. \tag{3.4}$$

By means of (v) again we obtain that $\mu(t) = 0$ for all $t \in J_0$ (in especial, $\mu(t_1) = 0$, i.e., $C_1(t_1)$, $C_2(t_1)$ is relatively compact).

For each $t \in J_1 = (t_1, t_2]$, in view of (3.6), one has

$$\begin{aligned}\mu(t) \leq 2\int_0^t \omega(s, \mu(s), \mu(s)) ds &+ \max\{\gamma(I_1(C_1(t_1))), \gamma(I_1(C_2(t_1)))\}\\ &+ \max\{\gamma(\bar{I}_1(C_1(t_1))), \gamma(\bar{I}_1(C_2(t_1)))\}.\end{aligned} \tag{3.5}$$

Since I_1, $\bar{I}_1 \in C(E, E)$ and $C_i(t_1)$ $(i = 1, 2)$ is relatively compact in E, we have that

$$\gamma(I_1(C_i(t_1))) = \gamma(\bar{I}_1(C_i(t_1)) = 0 \quad (i = 1, 2).$$

Load this into (3.7), we obtain that (3.6) holds for each $t \in J_1$. From (v) it follows that $\mu(t) = 0$ for all $t \in J_1$ (in especial, $\mu(t_2) = 0$, i.e., $C_1(t_2)$, $C_2(t_2)$ is relatively compact).

Inductively assume that $\mu(t) = 0$ for all $t \in J_k = (t_k, t_{k+1}]$ and $\mu(t_{k+1}) = 0$ with $k = 1, 2, \ldots, m-1$, then, when $k = m$, the definition of operator A induces that

$$\mu(t) \leq 2\int_0^t \omega(s,\mu(s),\mu(s))ds + \sum_{k=1}^{m}\gamma(I_k(C_1(t_k)))$$
$$+\sum_{k=1}^{m}\gamma(I_k(C_2(t_k))) + \sum_{k=1}^{m}\gamma(\bar{I}_k(C_1(t_k))) + \sum_{k=1}^{m}\gamma(\bar{I}_k(C_2(t_k)))$$
$$= 2\int_0^t \omega(s,\mu(s),\mu(s))ds.$$

Similar to the above proof we have that $\mu(t) = 0$ for all $t \in J$, which implies that $C_1(t), C_2(t)$ both are relatively compact for all $t \in J$.

Now we shall prove that C_1, C_2 are equicontinuous. Indeed, since C_1, C_2 both are countable, we can find a countable set $U = \{u_n : n \geq 1\} \subset A(C_1, C_2)$, $V = \{v_n : n \geq 1\} \subset A(C_2, C_1)$ with $C_1 \subset cl(\{x_1\} \cup U), C_2 \subset cl(\{y_1\} \cup V)$. There exists $x_n \in C_1, y_n \in C_2$ and $f_{x_n,y_n} \in S_{F,x_n,y_n}$ such that

$$u_n(t) = w_0 + w_1 t + \int_0^t (t-s) f_{x_n,y_n}(s)ds$$
$$+ \sum_{0<t_{i'}<t} I_{i'}(x_n(t_{i'})) + \sum_{0<t_{i''}<t} I_{i''}(y_n(t_{i''}))$$
$$+ \sum_{0<t_{i_1}<t} (t-t_{i_1})\bar{I}_{i_1}(x_n(t_{i_1})) + \sum_{0<t_{i_2}<t} (t-t_{i_2})\bar{I}_{i_2}(y_n(t_{i_2})),$$

for $n = 1, 2, \ldots$. It is easy to see that

$$u_n'(t) = w_1 + \int_0^t f_{x_n,y_n}(s)ds$$
$$+ \sum_{0<t_{i_1}<t} \bar{I}_{i_1}(x_n(t_{i_1})) + \sum_{0<t_{i_2}<t} \bar{I}_{i_2}(y_n(t_{i_2})),$$

Set $J_k = (t_k, t_{k+1}]$ $(k = 0, 1, \ldots, m)$ with $J_0 = [0, t_1]$, $J_m = (t_m, 1]$ and take $\tau_1, \tau_2 \in J_k$ with $\tau_1 \leq \tau_2$ and $u_n \in U$, from (iv) it follows that

$$|u_n(\tau_2) - u_n(\tau_1)| \leq |w_1||\tau_2 - \tau_1|$$
$$+ \left| \int_0^{\tau_2} (\tau_2 - s) f_{x_n,y_n}(s)ds - \int_0^{\tau_1} (\tau_1 - s) f_{x_n,y_n}(s)ds \right|$$
$$\leq |w_1||\tau_2 - \tau_1| + \int_0^{\tau_1} |(\tau_2 - \tau_1) f_{x_n,y_n}(s)|ds + \int_{\tau_1}^{\tau_2} |(\tau_2 - s) f_{x_n,y_n}(s)|ds$$
$$\leq |w_1||\tau_2 - \tau_1| + (\tau_2 - \tau_1)\int_0^{\tau_1} \alpha(s)ds + \int_{\tau_1}^{\tau_2} |(\tau_2 - s)\alpha(s)|ds.$$

This inequality is also true with any $x_n \in C_1$ instead of u_n for $n \geq 1$. Hence, C_1 is equicontinuous on J_k. Similarly, we can prove C_2 is equicontinuous on J_k. This concludes that C_1, C_2 are relatively compact on J_k, for $k = 0, 1, \ldots, m$, in the light of Arzela-Ascoli's theorem. Moreover, we can prove that C_1, C_2 are relatively compact on $PC^1[J,E]$. Consequently, the condition (H) is satisfied.

Summing up, A satisfies all the conditions of Theorem 1, thus, A has a minimax coupled fixed point and the proof of Theorem 3 is completed. □

Corollary 3. *Suppose that the conditions* (i)–(iv) *are satisfied. In addition, if there exists a constant $L \geq 0$ such that*

$$\gamma(F(t, D_1, D_2)) \leq L \max\{\gamma(D_1), \gamma(D_2)\}$$

for any $t \in J$ and $D_1,\ D_2 \subset D$. Then results of Theorem 3 *hold.*

Proof. It is enough to show that the hypothesis (v) is satisfied. Let $\omega(t, s, \tau) = L \max\{s, \tau\}$ with $s, \tau \geq 0$ and $t \in J$. Consider the multivalued operator

$$T(x(t), y(t)) = \left\{ \int_0^t (t-s) f_{x,y}(s) ds : f_{x,y} \in S_{F,x,y} \right\}.$$

For any countable set $D_1 \subset cl(T(D_1, D_2))$ and $D_2 \subset cl(T(D_2, D_1))$ with $|u(t)| \leq v(t)$ a.e. on J for all $u \in D_1 \cup D_2$ and some $v \in L^1(J, \mathbf{R}_+)$, denoting $\rho(t) = \max\{\gamma(D_1(t)), \gamma(D_2(t))\}$, from [10] it follows that

$$\begin{aligned} \rho(t) &\leq \max\{\gamma((T(D_1, D_2)(t)), \gamma((T(D_1, D_2)(t))\} \\ &= \max\left\{ \gamma\left(\left\{ \int_0^t (t-s) f_{x,y}(s) ds : (x,y) \in (D_1, D_2),\ f_{x,y} \in S_{F,x,y} \right\} \right), \right. \\ &\left. \gamma\left(\left\{ \int_0^t (t-s) f_{y,x}(s) ds : (y,x) \in (D_2, D_1),\ f_{y,x} \in S_{F,y,x} \right\} \right) \right\} \leq 2 \int_0^t \rho(s) ds. \end{aligned}$$

This implies that $\rho(t) = 0$ on J by Growall inequality. □

Corollary 4. *Let E be a ordered Banach space and $p = I$, an identity mapping. Suppose that the conditions* (ii) *and* (iii) *hold and*

(vi) *For any $t \in J, u_j, v_j \in PC^1[J,E]$ $(j = 1, 2)$, there exist constants $a \geq 0, b_i \geq 0, c_i \geq 0$ $(i = 1, 2, \ldots, m)$ satisfying $a + \sum_{i=1}^m (b_i + c_i) \leq 1$ such that*

$$\begin{aligned} \max\{|f_{u_1,v_1}(t) - f_{u_2,v_2}(t)| : f_{u_j,v_j} \in S_{F,u_j,v_j} (j = 1, 2)\} \\ \leq a\phi(\max\{|u_1 - u_2|_{PC^1}, |v_1 - v_2|_{PC^1}\}), \\ |I_i(u_1(t)) - I_i(u_2(t))| \leq b_i \phi(|u_1 - u_2|_{PC^1}), \\ |\bar{I}_i(u_1(t)) - \bar{I}_i(u_2(t))| \leq c_i \phi(|u_1 - u_2|_{PC^1}) \quad (i = 1, 2, \ldots, m), \end{aligned}$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is increasing and $\phi(\tau+) < \tau$ for $\tau > 0$.

Then IDI(3.1) *has an unique solution $x \in Q$ and the iterative sequences given by* (3.2) *and* (3.3) *satisfy that*

$$\lim_{n \to \infty} u_n(t) = x(t), \quad \lim_{n \to \infty} v_n(t) = x(t).$$

Proof. Similar to [3, Theorem 1] we can show that the conditions of Theorem 2 are satisfied. □

Corollary 5. *Let all assumptions in Theorem 3 be satisfied. For any* $x, y \in D$*, suppose that there exist constants* $a \geq 0, b_i \geq 0, c_i \geq 0$ $(i = 1, 2, \dots, m)$ *satisfying* $a + \sum_{i=1}^{m}(b_i + c_i) < 1$ *such that*

$$|p(u) - p(v)|_Y \leq a|p(x) - p(y)|_Y$$

for all $u \in F(t, x, y)$, $v \in F(t, y, x)$ *and*

$$|p(I_i(x)) - p(I_i(y))|_Y \leq b_i|p(x) - p(y)|_Y,$$
$$|p(\bar{I}_i(x)) - p(\bar{I}_i(y))|_Y \leq c_i|p(x) - p(y)|_Y \quad (i = 1, 2, \dots, m).$$

Then IDI(3.1) *has an unique solution* $x \in Q$ *and the iterative sequences given by* (3.2) *and* (3.3) *satisfy that*

$$\lim_{n\to\infty} p(u_n(t)) = p(x(t)), \quad \lim_{n\to\infty} p(v_n(t)) = p(x(t)).$$

Proof. It is easy to see that the operator A satisfies conditions of Theorem 2. □

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