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On the stability of the tangent bundle of a hypersurface in a Fano variety

By

Indranil BISWAS and Georg SCHUMACHER

Abstract

Let M be a complex projective Fano manifold whose Picard group is isomorphic to $\mathbb{Z}$ and the tangent bundle TM is semistable. Let $Z \subset M$ be a smooth hypersurface of degree strictly greater than degree$(TM)(\dim_{\mathbb{C}} Z - 1)/(2\dim_{\mathbb{C}} Z - 1)$ and satisfying the condition that the inclusion of Z in M gives an isomorphism of Picard groups. We prove that the tangent bundle of Z is stable. A similar result is proved also for smooth complete intersections in M. The main ingredient in the proof of it is a vanishing result for the top cohomology of the twisted holomorphic differential forms on Z.

1. Introduction

Let M be an irreducible smooth projective variety defined over the field $\mathbb{C}$ of complex numbers such that the anti–canonical line bundle K_M^{-1} is ample. We fix an indivisible ample line bundle ξ over M with the property that K_M^{-1} is a tensor power of ξ. The degree of any torsionfree coherent sheaf on M will be defined using ξ. The degree is so normalized that degree$(\xi) = 1$. For any positive integer d, an effective divisor on M in the complete linear system defined by $\xi^{\otimes d}$ will be called a hypersurface of degree d on M.

We prove the following (see Proposition 2.1):

Proposition 1.1. *Let $Z \subset M$ be an irreducible smooth hypersurface of degree d and $\ell \in \mathbb{Z}$. Then*

$$H^{m-1}(Z, \Omega_Z^k(\ell)) = 0,$$

where $m := \dim_{\mathbb{C}} M$, if the following three conditions are valid:

1. $\ell + d > 0$,
2. $0 < k < \frac{m(\ell+\text{degree}(TM))}{\text{degree}(TM)} - 1$, *and*
3. *the tangent bundle TM is semistable.*

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(The above integer degree(TM) is also called the *index* of M.)

If the canonical line bundle of a projective manifold X is ample or trivial, then the tangent bundle TX is polystable. However, if the anti–canonical line bundle of X is ample (which means that X is Fano) then TX in general is not polystable.

We use Proposition 1.1 to investigate stability of the tangent bundle of Fano hypersurfaces and, more generally, of Fano complete intersections.

We prove the following theorem (see Theorem 3.1):

Theorem 1.1. *Let M be a Fano projective manifold of dimension m and with* $\mathrm{Pic}(M) = \mathbb{Z}$ *such that the tangent bundle TM is semistable. Let $Z \subset M$ be an irreducible smooth Fano hypersurface of degree d with the following property: the homomorphism* $\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(Z)$ *defined by the inclusion of Z in M is an isomorphism* (*this condition is automatically satisfied if* $\dim_{\mathbb{C}} Z \geq 3$)*. Then the tangent bundle TZ is stable provided the inequality*

$$d > \frac{\mathrm{degree}(TM)(m-2)}{2m-3}$$

holds.

An example of M satisfying the conditions in Theorem 1.1 would be the quotient space G/P, where P is a proper maximal parabolic subgroup of a simple linear algebraic group G defined over $\mathbb{C}$.

Using an inductive argument, the above theorem can be extended to smooth Fano complete intersections; see Corollary 3.1 for the details.

We note that in [PW], Peternell and Wiśniewski proved that if M is a Fano n–fold with $b_2(M) = 1$ and index at least $n-1$, then TM is stable. They also proved that for any Fano 4–fold M with $b_2(M) = 1$, the tangent bundle TM is stable (see [PW, page 364, Theorem 3]).

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2. A vanishing result

Let M be a connected complex projective manifold such that the anti–canonical line bundle K_M^{-1} is ample. A holomorphic line bundle L over M is called *indivisible* if there is no holomorphic line bundle L_1 over M with $L_1^{\otimes i}$ holomorphically isomorphic to L for some integer $i \geq 2$.

Let ξ be an indivisible ample line bundle over M such that K_M^{-1} is isomorphic to $\xi^{\otimes i}$ for some integer i. The *degree* of any torsionfree coherent sheaf V over F will be defined to be

$$\mathrm{degree}(V) := \frac{\int_M c_1(V) c_1(\xi)^{\dim_{\mathbb{C}} M - 1}}{\int_M c_1(\xi)^{\dim_{\mathbb{C}} M}}.$$

The above definition of degree is so normalized that the degree of $\xi^{\otimes j}$ is j. We recall that a vector bundle V over M is called *stable* (respectively, *semistable*) if the inequality

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} < \frac{\mathrm{degree}(V)}{\mathrm{rank}(V)}$$

(respectively, $\mathrm{degree}(F)/\mathrm{rank}(F) \leq \mathrm{degree}(V)/\mathrm{rank}(V)$) holds for every nonzero coherent subsheaf $F \subset V$ with $\mathrm{rank}(F) < \mathrm{rank}(V)$.

For any vector bundle V over M, and any integer i, the vector bundle $V \otimes \xi^{\otimes i}$ over M will be denoted by $V(i)$. Similarly, for any subvariety $Z \subset M$ and any coherent sheaf V' on Z, the coherent sheaf $V' \otimes (\xi^{\otimes i}|_Z)$ on Z will be denoted by $V'(i)$. We will identify a coherent sheaf on Z with the coherent sheaf on M obtained by taking its direct image using the inclusion map of Z in M. The restriction to Z of a vector bundle V over M will be denoted by $V|_Z$. By a hypersurface of degree d on M we will mean an effective divisor on M in the complete linear system defined by the line bundle $\xi^{\otimes d}$.

Set

$$\tau := \mathrm{degree}(K_M^{-1}) \geq 1, \tag{2.1}$$

so $\xi^{\otimes \tau} = K_M^{-1}$ (recall that K_M^{-1} is a tensor power of ξ); and set $m := \dim_{\mathbb{C}} M$. We will assume that $m \geq 2$.

Proposition 2.1. *Let $Z \subset M$ be an irreducible smooth hypersurface of degree d and $\ell \in \mathbb{Z}$. Then*

$$H^{m-1}(Z, \Omega_Z^k(\ell)) = 0$$

(*recall that* $m := \dim_{\mathbb{C}} M$) *if all the following three conditions hold*:

1. $\ell + d > 0$,
2. $0 < k < \frac{m(\ell+\tau)}{\tau} - 1$, *where τ is defined in* (2.1), *and*
3. *the tangent bundle TM is semistable.*

Proof. Since the hypersurface Z is in the linear system defined by $\xi^{\otimes d}$, the normal bundle to Z is isomorphic to $\xi^{\otimes d}|_Z$ (this is the Poincaré adjunction formula). Hence we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_Z(-d) \longrightarrow \Omega_M^1|_Z \longrightarrow \Omega_Z^1 \longrightarrow 0$$

on Z (in our notation $(\xi^{\otimes d})^*|_Z = \mathcal{O}_Z(-d)$). The homomorphism $\Omega_M^{k+1}|_Z \longrightarrow \Omega_Z^{k+1}$ between exterior powers induced by the above projection fits in a short exact sequence

$$0 \longrightarrow \Omega_Z^k(-d) \longrightarrow \Omega_M^{k+1}|_Z \longrightarrow \Omega_Z^{k+1} \longrightarrow 0.$$

Tensoring the above short exact sequence with $\mathcal{O}_Z(\ell+d)$ we have

$$0 \longrightarrow \Omega_Z^k(\ell) \longrightarrow \Omega_M^{k+1}(\ell+d)|_Z \longrightarrow \Omega_Z^{k+1}(\ell+d) \longrightarrow 0. \tag{2.2}$$

For notational convenience, we set $n := m - 1 = \dim_{\mathbb{C}} Z$.

The short exact sequence of sheaves in (2.2) gives a long exact sequence

$$\longrightarrow H^{n-1}(Z,\, \Omega_Z^{k+1}(\ell + d)) \longrightarrow H^n(Z,\, \Omega_Z^k(\ell)) \longrightarrow H^n(Z,\, \Omega_M^{k+1}(\ell + d)|_Z) \longrightarrow \tag{2.3}$$

of cohomologies.

To prove the proposition it suffices to show that both $H^{n-1}(Z,\, \Omega_Z^{k+1}(\ell + d))$ and $H^n(Z,\, \Omega_M^{k+1}(\ell + d)|_Z)$ vanish under the three conditions given in the proposition.

The Akizuki–Nakano vanishing theorem says that $H^{n-1}(Z,\, \Omega_Z^{k+1}(\ell+d)) = 0$, if $\mathcal{O}_Z(\ell + d)$ is ample (that is, $\ell + d > 0$) and $n - 1 + k + 1 > \dim_{\mathbb{C}} Z = n$ [Kob, page 74, Theorem 3.11], [Kob, page 68, (3.2)].

Therefore, we have proved the following:

Statement 1. If all the three conditions in the proposition hold, then

$$H^{n-1}(Z,\, \Omega_Z^{k+1}(\ell + d)) = 0.$$

Since Z is a hypersurface in M of degree d, we have the following short exact sequence

$$0 \longrightarrow \Omega_M^{k+1}(\ell) \longrightarrow \Omega_M^{k+1}(\ell + d) \longrightarrow \Omega_M^{k+1}(\ell + d)|_Z \longrightarrow 0$$

of sheaves on M which is obtained by tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_M(-d) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M|_Z = \mathcal{O}_Z \longrightarrow 0$$

with $\Omega_M^{k+1}(\ell + d)$. The above short exact sequence of sheaves gives a long exact sequence of cohomologies

$$\begin{aligned} \longrightarrow H^n(M,\, \Omega_M^{k+1}(\ell + d)) \longrightarrow\ & H^n(Z,\, \Omega_M^{k+1}(\ell + d)|_Z) \\ \longrightarrow\ & H^{n+1}(M,\, \Omega_M^{k+1}(\ell)) \longrightarrow . \end{aligned} \tag{2.4}$$

The Akizuki–Nakano vanishing theorem theorem says $H^n(M,\, \Omega_M^{k+1}(\ell + d)) = 0$, if $\mathcal{O}_M(\ell+d)$ is ample (that is, $\ell+d > 0$) and $n+k+1 > n+1 = \dim_{\mathbb{C}} M$.

So we have proved the following:

Statement 2. $H^n(M,\, \Omega_M^{k+1}(\ell + d)) = 0$ if the three conditions in the proposition hold.

The Serre duality gives

$$H^{n+1}(M,\, \Omega_M^{k+1}(\ell)) = H^0(M,\, (\wedge^{k+1} TM)(-\ell - \tau))^*, \tag{2.5}$$

where τ is defined in (2.1).

Henceforth, we will assume the tangent bundle TM to be semistable. Since any exterior power of a semistable vector bundle is semistable [RR, page 285, Theorem 3.18], we conclude that the vector bundle $\bigwedge^{k+1} TM$ is semistable. Therefore, the vector bundle

$$(\wedge^{k+1}TM)(-\ell - \tau) := \xi^{\otimes(-\ell-\tau)} \otimes \wedge^{k+1}TM$$

in (2.5) is semistable.

A semistable vector bundle of negative degree does not admit any nonzero sections. Indeed, a nonzero section of a vector bundle F over M gives an injective homomorphism from $\mathcal{O}_M$ to F. Therefore, the image of this homomorphism, which is of degree zero, contradicts the semistability condition for F if the degree of F is negative.

For any vector bundle V over M, the quotient degree(V)/rank$(V) \in \mathbb{Q}$ will be denoted by $\mu(V)$. Note that

$$\mu((\wedge^{k+1}TM)(-\ell - \tau)) = \frac{(k+1)\tau}{n+1} - \ell - \tau; \tag{2.6}$$

this follows immediately from the fact that $\mu(TM) = \frac{\tau}{n+1}$ (for any vector bundle V and any integer $i \in [1, \mathrm{rank}(V)]$, we have $\mu(\bigwedge^i V) = i \cdot \mu(V)$, and $\mu(V \otimes W) = \mu(V) + \mu(W)$ for any vector bundle W). Here we have assumed that $k + 1 \leq n + 1 = \dim_{\mathbb{C}} M$; if $k > n$, then $\Omega_Z^k = 0$, and the proposition is automatically valid. We note that

$$\frac{(k+1)\tau}{n+1} - \ell - \tau < 0$$

if and only if

$$k < \frac{(n+1)(\ell + \tau)}{\tau} - 1. \tag{2.7}$$

Therefore, combining (2.5) and (2.6) with the above remark that a semistable vector bundle of negative degree does not admit any nonzero sections we conclude that

$$H^{n+1}(M, \Omega_M^{k+1}(\ell)) = 0$$

if TM is semistable and the inequality (2.7) holds. (Note that (2.7) is one of the conditions in the statement of the proposition.)

Combining the above assertion with the exact sequence (2.4) and Statement 2 the following is obtained:

Statement 3. $H^n(Z, \Omega_M^{k+1}(\ell + d)|_Z) = 0$ if the three conditions in the proposition hold.

In view of Statement 1 and Statement 3, from (2.3) we conclude the following:

$$H^n(Z, \Omega_Z^k(\ell)) = 0$$

if all the three conditions in the proposition hold. This completes the proof of the proposition. □

3. Stability of tangent bundle

Let M be a connected complex projective manifold with $\mathrm{Pic}(M) = \mathbb{Z}$ such that the anti–canonical line bundle K_M^{-1} is ample. Since $\mathrm{Pic}(M) = \mathbb{Z}$, the indivisible line bundle ξ considered in the beginning of Section 2 is the ample generator of $\mathrm{Pic}(M)$.

Let $Z \subset M$ be a smooth hypersurface and V a torsionfree coherent sheaf defined on Z. Then we define the *degree* of V as follows:

$$\mathrm{degree}(V) := \frac{\int_Z c_1(V) c_1(\xi|_Z)^{\dim_{\mathbb{C}} Z - 1}}{\int_Z c_1(\xi|_Z)^{\dim_{\mathbb{C}} Z}}.$$

Stable and semistable vector bundles over Z are defined using the above definition of degree.

Let $Z \subset M$ be an irreducible smooth hypersurface of degree d. If $d > \tau$, where τ is the integer defined in (2.1), then the canonical line bundle K_Z of Z is ample. If $d = \tau$, then the canonical line bundle K_Z is trivial. Hence if $d \geq \tau$, then Z admits a Kähler–Einstein metric [Au], [Ya]. In particular, the tangent bundle TZ is polystable if $d \geq \tau$.

If $d < \tau$, then Z is a Fano manifold, that is, the anti–canonical line bundle K_Z^{-1} is ample. There are obstructions for a Fano manifold to admit a Kähler–Einstein metric [Fu], [Ti]. As an application of Proposition 2.1, we have the following theorem on the stability of TZ.

Theorem 3.1. *Let M be a connected complex projective manifold of dimension at least three with* $\mathrm{Pic}(M) = \mathbb{Z}$ *such that*

1. *the anti–canonical line bundle K_M^{-1} is ample, and*
2. *the tangent bundle TM is semistable.*

Let $Z \subset M$ be an irreducible smooth Fano hypersurface of degree d (so $d <$ $\mathrm{degree}(TM)$*) with the following property: the homomorphism* $\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(Z)$ *defined by the inclusion of Z in M is an isomorphism (this condition is automatically satisfied if* $\dim_{\mathbb{C}} Z \geq 3$*; see Remark* 1*). Then the tangent bundle TZ is stable if the inequality*

$$d > \frac{\mathrm{degree}(TM)(\dim_{\mathbb{C}} Z - 1)}{2 \dim_{\mathbb{C}} Z - 1} \tag{3.1}$$

holds.

Proof. Take any hypersurface $Z \subset M$ satisfying the above conditions. Assume that TZ is not stable. Let

$$F \subset TZ$$

be a coherent subsheaf which contradicts the stability condition for TZ. Set $\delta := \mathrm{degree}(F)$ and $k := \mathrm{rank}(F)$. Since F contradicts the stability condition for TZ, we have $1 \leq k < n$, where $n = \dim_{\mathbb{C}} Z$, and

$$\frac{\delta}{k} \geq \frac{\tau - d}{n} \tag{3.2}$$

with τ defined in (2.1) (recall that the degree of TZ is $\tau - d$).

Since the homomorphism $\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(Z)$ given by the inclusion map $Z \hookrightarrow M$ is an isomorphism, and $\mathrm{Pic}(M) = \mathbb{Z}$, the fact that the degree of F is δ implies that $\bigwedge^k F = \mathcal{O}_Z(\delta)$.

Therefore, the above subsheaf F of TZ defines a nonzero section

$$\sigma \in H^0(Z, (\wedge^k F)^* \otimes \wedge^k TZ) = H^0(Z, (\wedge^k TZ)(-\delta)). \tag{3.3}$$

We have

$$H^0(Z, (\wedge^k TZ)(-\delta)) = H^n(Z, \Omega_Z^k(\delta - \tau + d))^*,$$

which is obtained using the Serre duality and the fact that $K_Z = \mathcal{O}_Z(d - \tau)$.

Therefore, in view of Proposition 2.1, to prove the theorem it suffices to show the following two:

$$\delta - \tau + d + d = \delta - \tau + 2d > 0 \tag{3.4}$$

and

$$k < \frac{(n+1)(\tau + \delta - \tau + d)}{\tau} - 1 = \frac{(n+1)(\delta + d)}{\tau} - 1. \tag{3.5}$$

Indeed, if (3.4) and (3.5) hold, then Proposition 2.1 says that there is no nonzero section σ as in (3.3) (recall that we have $k \geq 1$).

Using (3.2), to prove (3.4) it is enough to show that

$$\frac{k(\tau - d)}{n} > \tau - 2d.$$

On the other hand, as $k \geq 1$ and $\tau > d$, we know that

$$\frac{k(\tau - d)}{n} \geq \frac{(\tau - d)}{n}.$$

Hence the inequality (3.4) holds if

$$\frac{(\tau - d)}{n} > \tau - 2d.$$

In other words, (3.4) holds if the inequality

$$d > \frac{\tau(n-1)}{2n-1}$$

holds. But this is the inequality (3.1) in the statement of the theorem. Hence (3.4) is proved.

The inequality (3.5) holds if and only if

$$\frac{(k+1)\tau}{n+1} - d < \delta.$$

Hence, in view of (3.2), to prove (3.5) it is enough to show that

$$\frac{(k+1)\tau}{n+1} - d < \frac{k(\tau-d)}{n}. \tag{3.6}$$

It is straight–forward to see that (3.6) holds if and only if

$$(k+1)(dn+d-\tau) < (n+1)(dn+d-\tau). \tag{3.7}$$

To prove that (3.6) and (3.7) are equivalent, note that both (3.6) and (3.7) are equivalent to the inequality

$$k(\tau-d)(n+1) + d(n+1)n - (k+1)\tau n > 0.$$

Now, since we have $k < n$, the inequality (3.7) holds provided $dn+d-\tau > 0$.

A theorem due to Kobayashi and Ochiai in [KO] says that $\tau \leq n+2$ (a proof of this inequality can also be found in [Kol, page 245, Theorem 1.11]).

If $\tau \leq n$, then we have $dn+d-\tau > 0$. Also, if $d \geq 2$, then $dn+d-\tau > 0$ (as $\tau \leq n+2$). Combining these two we conclude that if the inequality $dn+d-\tau > 0$ fails to hold, then $d = 1$ and $n+1 \leq \tau$. Now we note from the inequality (3.1)

$$d > \frac{\tau(n-1)}{2n-1}$$

in the statement of the theorem that if $d = 1$ then $1 \leq \tau \leq 2$. Therefore, if the inequality $dn+d-\tau > 0$ fails, then $n \leq 1$ (recall that $n+1 \leq \tau$ if the inequality $dn+d-\tau > 0$ fails). On the other hand, we are given that $n \geq 2$. This completes the proof of the theorem. $\square$

Remark 1. If $\dim_{\mathbb{C}} Z \geq 3$, then from Grothendieck's Lefschetz theory it follows that the inclusion of the ample hypersurface Z in M gives an isomorphism of Picard groups $\mathrm{Pic}(M) \xrightarrow{\cong} \mathrm{Pic}(Z)$; see [Gr, Expose X]. Therefore, the condition in Theorem 3.1 that the inclusion of Z in M gives an isomorphism of Picard groups is redundant when $\dim_{\mathbb{C}} Z \geq 3$.

The homomorphism $\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(Z)$ given by the inclusion map $Z \hookrightarrow M$ is always injective (as Z is an ample hypersurface of the Fano manifold M). So the condition that the homomorphism $\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(Z)$ given by the inclusion map $Z \hookrightarrow M$ is an isomorphism is equivalent to the condition that the homomorphism is surjective. $\square$

If the inequality (3.1) holds and the inclusion of Z in M gives an isomorphism of Picard groups, then from Theorem 3.1 it follows immediately that Z satisfies all the conditions on M in Theorem 3.1. Therefore, in that case we may replace M by Z and investigate hypersurfaces in it. Theorem 3.1 now gives a criterion under which a Fano hypersurface in Z has stable tangent bundle.

Using this observation inductively, Theorem 3.1 has the following corollary:

Corollary 3.1. *Let M be a connected complex projective manifold of dimension m, with $m \geq 3$, such that*

1. $\mathrm{Pic}(M) = \mathbb{Z}$,
2. *the anti–canonical line bundle* K_M^{-1} *is ample, and*
3. *the tangent bundle* TM *is semistable.*

Let Z_i*, where* $1 \leq i \leq \ell$ *with* $m - \ell \geq 2$*, be an irreducible smooth Fano hypersurface of degree* d_i *such that* $\sum_{i=1}^{\ell} d_i < \mathrm{degree}(TM)$*. Assume that for each* $j \in [1\,,\ell]$*, the subvariety*

$$\widehat{Z}_j := \bigcap_{i=1}^{j} Z_i \subset M$$

is a smooth complete intersection of codimension j*. Also, assume the following*:

1. *the inclusion* $\widehat{Z} := \widehat{Z}_\ell \hookrightarrow M$ *induces an isomorphism of Picard groups* (*this condition is automatically satisfied when* $m - \ell \geq 3$)*, and*
2. $d_j > \frac{(\mathrm{degree}(TM) - \sum_{i=0}^{j-1} d_i)(m-j-1)}{2(m-j)-1}$ *for each* $j \in [1\,,\ell]$ *with the convention* $d_0 = 0$*.*

Then the tangent bundle of $\widehat{Z}$ *is stable.*

Proof. Since the homomorphism $\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(\widehat{Z})$ induced by the inclusion map $\widehat{Z} \hookrightarrow M$ is an isomorphism, it follows from the second part of Remark 1 that the homomorphism $\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(\widehat{Z}_j)$ induced by the inclusion map $\widehat{Z}_j \hookrightarrow M$ is an isomorphism for all $j \in [1\,,\ell]$. Therefore, by Theorem 3.1, the tangent bundle of the hypersurface Z_1 is stable. Now substituting the pair $(Z_1\,, Z_1 \cap Z_2)$ for the pair $(M\,, Z)$ in Theorem 3.1 we conclude that the tangent bundle of $Z_1 \cap Z_2$ is stable. Finally, we get inductively that the tangent bundle of $\widehat{Z}_j$ is stable for all $j \in [1\,,\ell]$. □

Let G be a simple linear algebraic group defined over $\mathbb{C}$. Let $P \subset G$ be a proper maximal parabolic subgroup. The quotient space G/P is a homogeneous Fano complex projective variety with $\mathrm{Pic}(G/P) = \mathbb{Z}$. The tangent bundle of G/P is stable, which follows from [Um, page 136, Theorem 2.4]. So G/P satisfies all the conditions for M in Theorem 3.1. A special case of G/P would the Grassmannian $\mathrm{Gr}(r, r+\ell)$ which parametrizes all linear subspaces of $\mathbb{C}^{r+\ell}$ of dimension r. The integer τ defined in (2.1) for $M = \mathrm{Gr}(r, r+\ell)$ is $r+\ell$, while the complex dimension of $\mathrm{Gr}(r, r+\ell)$ is $r\ell$. Therefore, if $r\ell \geq 4$, then for any smooth hypersurface Z in $\mathrm{Gr}(r, r+\ell)$, with $\mathrm{degree}(Z) > (r+\ell)(r\ell-2)/(2r\ell-3)$, the tangent bundle TZ of Z is stable.

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400005
India
e-mail: indranil@math.tifr.res.in

Fachbereich Mathematik und Informatik
Philipps-Universität Marburg, Lahnberge
Hans-Meerwein-Strasse, D-35032 Marburg
Germany
e-mail: schumac@mathematik.uni-marburg.de

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