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The reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and its equivalent equalities

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Abstract

This paper collects 26 conditions for the reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ to hold for the Moore-Penrose inverse of matrix.

Throughout, the symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, the rank and the range (column space) of a complex matrix A, respectively; the symbol $[\,A,\,B\,]$ denotes a row block matrix consisting of A and B.

For a general $m \times n$ complex matrix A, the Moore-Penrose inverse $A^\dagger$ of A is the unique $n \times m$ matrix X that satisfies the following four Penrose equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^* = AX, \quad \text{(iv) } (XA)^* = XA,$$

cf. Penrose [8]. For simplicity, denote $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$. A matrix X is called an outer inverse of A, if it satisfies $XAX = X$. General properties of the Moore-Penrose inverse can be found in [1], [2], [7].

Let A and B be a pair of matrices such that AB exists. Because $A^\dagger A$, $BB^\dagger$ and $BB^\dagger A^\dagger A$ are not necessarily identity matrices, the reverse-order law $(AB)^\dagger = B^\dagger A^\dagger$ for the matrix product AB does not necessarily hold. Greville [5] showed that $(AB)^\dagger = B^\dagger A^\dagger$ holds true if and only if

$$\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B) \ \text{ and } \ \mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*).$$

Many other necessary and sufficient conditions for $(AB)^\dagger = B^\dagger A^\dagger$ to hold were also given in the literature. If $(AB)^\dagger = B^\dagger A^\dagger$ does not hold, $(AB)^\dagger$ can be written as either

$$(AB)^\dagger = B^\dagger XA^\dagger \ \text{ or } \ (AB)^\dagger = B^\dagger A^\dagger + Y,$$

where X and Y are some matrices consisting of A and B. A possible expression of $(AB)^\dagger$ is

$$(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger. \tag{1}$$

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This expression is derived from writing AB as $AB = A(A^{\dagger}ABB^{\dagger})B$ and applying the reverse-order law $(PNQ)^{\dagger} = Q^{\dagger}N^{\dagger}Q^{\dagger}$ to it. Some previous work on (1) can be found in [4], [6], [11], [12].

When investigating various reverse-order laws for $(AB)^{\dagger}$, we notice that some of them are in fact equivalent. For instance,

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow (AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger} \quad \text{and} \quad (A^{\dagger}ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}A,$$

which is shown in Tian [14]. When revisiting (1), we also find that many other matrix equalities consisting of A and B are equivalent to (1). These equalities are summarized in the following theorem.

Theorem 1. *Let A and B be two $m \times n$ and $n \times p$ matrices, respectively. Then the following 27 statements are equivalent*:

(a1) $(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$.
(a2) $(AB)^{\dagger} = B^{*}(A^{*}ABB^{*})^{\dagger}A^{*}$.
(a3) $(AB)^{\dagger} = B^{\dagger}A^{\dagger} - B^{\dagger}(E_BF_A)^{\dagger}A^{\dagger}$.
(b1) $[\,(A^{\dagger})^{*}B\,]^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{*}$.
(b2) $[\,(A^{\dagger})^{*}B\,]^{\dagger} = B^{*}[\,(A^{*}A)^{\dagger}BB^{*}\,]^{\dagger}A^{\dagger}$.
(b3) $[\,(A^{\dagger})^{*}B\,]^{\dagger} = B^{\dagger}A^{*} - B^{\dagger}(E_BF_A)^{\dagger}A^{*}$.
(c1) $[\,A(B^{\dagger})^{*}\,]^{\dagger} = B^{*}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$.
(c2) $[\,A(B^{\dagger})^{*}\,]^{\dagger} = B^{\dagger}[\,A^{*}A(BB^{*})^{\dagger}\,]^{\dagger}A^{*}$.
(c3) $[\,A(B^{\dagger})^{*}\,]^{\dagger} = B^{*}A^{\dagger} - B^{*}(E_BF_A)^{\dagger}A^{\dagger}$.
(d1) $(B^{\dagger}A^{\dagger})^{\dagger} = A(BB^{\dagger}A^{\dagger}A)^{\dagger}B$.
(d2) $(B^{\dagger}A^{\dagger})^{\dagger} = (A^{\dagger})^{*}[\,(BB^{*})^{\dagger}(A^{*}A)^{\dagger}\,]^{\dagger}(B^{\dagger})^{*}$.
(d3) $(B^{\dagger}A^{\dagger})^{\dagger} = AB - A(F_AE_B)^{\dagger}B$.
(e1) $(A^{\dagger}AB)^{\dagger}A^{\dagger} = B^{\dagger}(ABB^{\dagger})^{\dagger}$.
(e2) $(A^{\dagger}AB)^{\dagger}A^{*} = B^{\dagger}[\,(A^{\dagger})^{*}BB^{\dagger}\,]^{\dagger}$.
(e3) $[\,A^{\dagger}A(B^{\dagger})^{*}\,]^{\dagger}A^{\dagger} = B^{*}(ABB^{\dagger})^{\dagger}$.
(e4) $(BB^{\dagger}A^{\dagger})^{\dagger}B = A(B^{\dagger}A^{\dagger}A)^{\dagger}$.
(e5) $(A^{*}AB)^{\dagger}A^{*} = B^{*}(ABB^{*})^{\dagger}$.
(e6) $[\,(A^{*}A)^{\dagger}B\,]^{\dagger}A^{\dagger} = B^{*}[\,(A^{\dagger})^{*}BB^{*}\,]^{\dagger}$.
(e7) $[\,A^{*}A(B^{\dagger})^{*}\,]^{\dagger}A^{*} = B^{\dagger}[\,A(BB^{*})^{\dagger}\,]^{\dagger}$.
(e8) $B^{\dagger}[\,(A^{*})^{\dagger}(BB^{*})^{\dagger}\,]^{\dagger} = [\,(A^{*}A)^{\dagger}(B^{*})^{\dagger}\,]^{\dagger}A^{\dagger}$.
(e9) $(AA^{*}ABB^{*}B)^{\dagger} = B^{\dagger}(A^{*}ABB^{*})^{\dagger}A^{\dagger}$.
(f1) $(A^{\dagger}AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}$ *and* $(ABB^{\dagger})^{\dagger} = (A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$.
(f2) $(A^{\dagger}AB)^{\dagger} = B^{*}(A^{\dagger}ABB^{*})^{\dagger}$ *and* $(ABB^{\dagger})^{\dagger} = (A^{*}ABB^{\dagger})^{\dagger}A^{*}$.
(f3) $(A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A - B^{\dagger}(E_BF_A)^{\dagger}A^{\dagger}A$ *and*
$(ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger} - BB^{\dagger}(E_BF_A)^{\dagger}A^{\dagger}$.
(g1) $\mathscr{R}[\,(AB)^{\dagger}\,] = \mathscr{R}[\,B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}\,]$ *and*
$\mathscr{R}\{[\,(AB)^{\dagger}\,]^{*}\} = \mathscr{R}\{[\,B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}\,]^{*}\}$.
(g2) $\mathscr{R}[\,(AB)^{\dagger}\,] = \mathscr{R}(B^{\dagger}A^{\dagger})$ *and* $\mathscr{R}[\,(B^{*}A^{*})^{\dagger}\,] = \mathscr{R}[\,(A^{*})^{\dagger}(B^{*})^{\dagger}\,]$.
(g3) $\mathscr{R}(\,AA^{*}AB\,) = \mathscr{R}(AB)$ *and* $\mathscr{R}[\,B^{*}B(AB)^{*}\,] = \mathscr{R}[(AB)^{*}]$.

The results in Theorem 1 bring a great convenience for using reverse-order laws in different situations. In order to show Theorem 1, we use a rank formula for the difference of two outer inverses.

Lemma 2 ([9]). *Let* X_1 *and* X_2 *be a pair of outer inverses of a matrix* A, *that is,* $X_1AX_1 = X_1$ *and* $X_2AX_2 = X_2$. *Then*

$$r(X_1 - X_2) = r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + r[X_1,\, X_2] - r(X_1) - r(X_2). \tag{2}$$

Hence, the equality $X_1 = X_2$ *holds if and only if*

$$r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = r(X_1) = r(X_2) \quad \text{and} \quad r[X_1,\, X_2] = r(X_1) = r(X_2),$$

i.e., $\mathscr{R}(X_1) = \mathscr{R}(X_2)$ *and* $\mathscr{R}(X_1^*) = \mathscr{R}(X_2^*)$.

Some other simple results on ranks and Moore-Penrose inverses of matrices are given below

$$\text{if } PX = Y \text{ and } X = QY, \text{ then } r(X) = r(Y), \tag{3}$$

$$\begin{gathered}(A^\dagger)^* = (A^*)^\dagger,\ (A^\dagger)^*A^* = AA^\dagger,\ A^* = A^*A(A^\dagger)^* = (A^\dagger)^*AA^*, \\ (A^\dagger)^*A^\dagger(A^\dagger)^* = (A^*AA^*)^\dagger,\end{gathered} \tag{4}$$

$$r(B^\dagger A^\dagger) = r[(A^*)^\dagger B] = r[A(B^*)^\dagger] = r[(AB)^\dagger] = r(AB). \tag{5}$$

Proof of Theorem 1. The reverse-order law in (1) was first studied by Galperin and Waksman [4], and then by Izumino [6] for a product of two linear operators. It is easy to verify that $B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ is an outer inverse of AB. From this fact and (2), Tian [11] showed that

$$\begin{aligned} r[(AB)^\dagger - B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger] &= r\begin{bmatrix} (AB)^\dagger \\ B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger \end{bmatrix} \\ &\quad + r[(AB)^\dagger,\ B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger] \\ &\quad - r[(AB)^\dagger] - r[B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger] \qquad (6) \\ &= r\begin{bmatrix} (AB)^\dagger \\ B^\dagger A^\dagger \end{bmatrix} + r[(AB)^\dagger,\ B^\dagger A^\dagger] - 2r[(AB)^\dagger] \qquad (7) \\ &= r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} + r[AB,\, AA^*AB] - 2r(AB). \qquad (8) \end{aligned}$$

Recall that a matrix is zero matrix if and only if the rank of the matrix is zero. Let the right-hand sides of (6), (7) and (8) be zero. Then we obtain the equivalence of (a1), (g1) and (g3). It is also easy to show

$$\begin{aligned} r[(AB)^\dagger - B^*(A^*ABB^*)^\dagger A^*] &= r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} + r[AB,\, AA^*AB] \\ &\quad - 2r(AB), \end{aligned} \tag{9}$$

see [11], [12]. Note that the right-hand sides of (8) and (9) are the same. Hence (a1) and (b1) are equivalent.

The rank formula associated with (a3) is

$$r[(AB)^\dagger - B^\dagger A^\dagger + B^\dagger (E_B F_A)^\dagger A^\dagger] = r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} + r[AB,\, AA^*AB] - 2r(AB), \tag{10}$$

which is shown in [12]. Because the right-hand sides of (8) and (10) are identical, the equivalence of (a1) and (a3) follows from (8) and (10). Replacing A in (8) with $(A^\dagger)^*$ gives

$$r\{[(A^\dagger)^*B]^\dagger - B^\dagger (A^\dagger ABB^\dagger)^\dagger A^*\} = r\begin{bmatrix} (A^\dagger)^*B \\ (A^\dagger)^*BB^*B \end{bmatrix} + r[(A^\dagger)^*B,\, (A^*AA^*)^\dagger B] - 2r[(A^\dagger)^*B]. \tag{11}$$

It is easy to derive from (3) and (4) that

$$r\begin{bmatrix} (A^\dagger)^*B \\ (A^\dagger)^*BB^*B \end{bmatrix} = r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix}, \quad r[(A^\dagger)^*B,\, (A^*AA^*)^\dagger B] = r[AA^*AB,\, AB]. \tag{12}$$

Substituting (5) and (12) into (11) gives

$$r\{[(A^\dagger)^*B]^\dagger - B^\dagger (A^\dagger ABB^\dagger)^\dagger A^*\} = r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} + r[\,AB,\, AA^*AB\,] - 2r(AB). \tag{13}$$

Comparing (8) and (13) yields

$$r\{[(A^\dagger)^*B]^\dagger - B^\dagger (A^\dagger ABB^\dagger)^\dagger A^*\} = r[(AB)^\dagger - B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger]. \tag{14}$$

This implies the equivalence of (a1) and (b1). Similarly, we can show that

$$\begin{aligned} r\{[A(B^\dagger)^*]^\dagger - B^*(A^\dagger ABB^\dagger)^\dagger A^\dagger\} &= r[(B^\dagger A^\dagger)^\dagger - A(BB^\dagger A^\dagger A)^\dagger B] \\ &= r[(AB)^\dagger - B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger]. \end{aligned}$$

Hence, the equivalence of (a1), (c1) and (d1) follows. The following three formulas

$$\begin{aligned} & r\{[(A^\dagger)^*B]^\dagger - B^*[(A^*A)^\dagger BB^*]^\dagger A^\dagger\} \\ &\quad = r\{[A(B^\dagger)^*]^\dagger - B^\dagger [A^*A(BB^*)^\dagger]^\dagger A^*\} \\ &\quad = r\{(B^\dagger A^\dagger)^\dagger - (A^\dagger)^*[(BB^*)^\dagger (A^*A)^\dagger]^\dagger (B^\dagger)^*\} \\ &\quad = r[(AB)^\dagger - B^*(A^*ABB^*)^\dagger A^*] \end{aligned}$$

are derived from (3), (4), (5) and (9). Thus, we obtain the equivalence of (a2), (b2), (c2) and (d2). The following three formulas

$$\begin{aligned} & r[(AB)^{\dagger} - B^{\dagger}A^{\dagger} + B^{\dagger}(E_B F_A)^{\dagger}A^{\dagger}] \\ & \quad = r\{[\,(A^{\dagger})^{*}B\,]^{\dagger} - B^{\dagger}A^{*} + B^{\dagger}(E_B F_A)^{\dagger}A^{*}\} \\ & \quad = r\{[\,A(B^{\dagger})^{*}\,]^{\dagger} - B^{*}A^{\dagger} + B^{*}(E_B F_A)^{\dagger}A^{\dagger}\} \\ & \quad = r[\,(B^{\dagger}A^{\dagger})^{\dagger} - AB + A(F_A E_B)^{\dagger}B\,] \end{aligned}$$

are derived from (3), (4), (5) and (10). Thus, the equivalence of (a3), (b3), (c3) and (d3) follows.

Replacing A and B in (8), (9) and (10) with $A^{\dagger}A$ and $BB^{\dagger}$ respectively and simplifying yield

$$\begin{aligned} & r[(A^{\dagger}AB)^{\dagger} - B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}] \\ & \quad = r[(A^{\dagger}AB)^{\dagger} - B^{*}(A^{\dagger}ABB^{*})^{\dagger}] \\ & \quad = r[\,(A^{\dagger}AB)^{\dagger} - B^{\dagger}A^{\dagger}A + B^{\dagger}(E_B F_A)^{\dagger}A^{\dagger}A] \\ & \quad = r\begin{bmatrix} AB \\ ABB^{*}B \end{bmatrix} - r(AB) \end{aligned}$$

and

$$\begin{aligned} r[(ABB^{\dagger})^{\dagger} - (A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}] &= r[(ABB^{\dagger})^{\dagger} - (A^{*}ABB^{\dagger})^{\dagger}A^{*}] \\ &= r[(ABB^{\dagger})^{\dagger} - BB^{\dagger}A^{\dagger} + BB^{\dagger}(E_B F_A)^{\dagger}A^{\dagger}] \\ &= r[\,AB,\ AA^{*}AB\,] - r(AB). \end{aligned}$$

The equivalence of (a1), (f1), (f2) and (f3) follows from these rank formulas.

Note that

$$(A^{\dagger}AB)^{\dagger}A^{\dagger}AB(A^{\dagger}AB)^{\dagger}A^{\dagger} = (A^{\dagger}AB)^{\dagger}A^{\dagger}$$

and

$$B^{\dagger}(ABB^{\dagger})^{\dagger}ABB^{\dagger}(ABB^{\dagger})^{\dagger} = B^{\dagger}(ABB^{\dagger})^{\dagger}.$$

Thus, both $(A^{\dagger}AB)^{\dagger}A^{\dagger}$ and $B^{\dagger}(ABB^{\dagger})^{\dagger}$ are outer inverses of AB. In these cases, applying (2) gives

$$\begin{aligned} r[(A^{\dagger}AB)^{\dagger}A^{\dagger} - B^{\dagger}(ABB^{\dagger})^{\dagger}] &= r\begin{bmatrix} (A^{\dagger}AB)^{\dagger}A^{\dagger} \\ B^{\dagger}(ABB^{\dagger})^{\dagger} \end{bmatrix} \\ &\quad + r[(A^{\dagger}AB)^{\dagger}A^{\dagger},\ B^{\dagger}(ABB^{\dagger})^{\dagger}] \\ &\quad - r[A^{\dagger}AB)^{\dagger}A^{\dagger}] - r[B^{\dagger}(ABB^{\dagger})^{\dagger}]. \end{aligned} \tag{15}$$

It is also easy to find by (3), (4) and (5) that

$$r\begin{bmatrix}(A^\dagger AB)^\dagger A^\dagger\\ B^\dagger(ABB^\dagger)^\dagger\end{bmatrix} = r\begin{bmatrix}B^*A^\dagger\\ BB^\dagger A^*\end{bmatrix} = r\begin{bmatrix}B^*A^*\\ B^*A^*AA^*\end{bmatrix} = r[AB,\ AA^*AB],$$

$$\begin{aligned} r[(A^\dagger AB)^\dagger A^\dagger,\ B^\dagger(ABB^\dagger)^\dagger\,] &= r[B^*A^*,\ B^\dagger A^*] \\ &= r[B^*BB^*A^*,\ B^*A^*] = r\begin{bmatrix}AB\\ ABB^*B\end{bmatrix}, \end{aligned}$$

$$r[(A^\dagger AB)^\dagger A^\dagger] = r[B^\dagger(ABB^\dagger)^\dagger] = r(AB).$$

Hence (15) is reduced to

$$(16)\quad r[(A^\dagger AB)^\dagger A^\dagger - B^\dagger(ABB^\dagger)^\dagger] = r\begin{bmatrix}AB\\ ABB^*B\end{bmatrix} + r[AB,\ AA^*AB] - 2r(AB).$$

Comparing (8) and (16) results in the equivalence of (a1) and (e1). Replacing A and B in (16) with $(A^\dagger)^*$ and $(B^\dagger)^*$ respectively and simplifying by (3), (4) and (5), we also obtain

$$\begin{aligned} r\{(A^\dagger AB)^\dagger A^* - B^\dagger[(A^\dagger)^*BB^\dagger]^\dagger\} &= r\{[A^\dagger A(B^\dagger)^*]^\dagger A^\dagger - B^*(ABB^\dagger)^\dagger\} \\ &= r[(BB^\dagger A^\dagger)^\dagger B - A(B^\dagger A^\dagger A)^\dagger] \\ &= r[(A^\dagger AB)^\dagger A^\dagger - B^\dagger(ABB^\dagger)^\dagger]. \end{aligned}$$

Thus the equivalence of Theorem 1 (e1)–(e4) follows. Also by (2)

$$\begin{aligned} r[(A^*AB)^\dagger A^* - B^*(ABB^*)^\dagger] &= r\{[(A^*A)^\dagger B]^\dagger A^\dagger - B^*[(A^\dagger)^*BB^*]^\dagger\} \\ &= r\{[A^*A(B^\dagger)^*]^\dagger A^* - B^\dagger[A(BB^*)^\dagger]^\dagger\} \\ &= r\{B^\dagger[(A^*)^\dagger(BB^*)^\dagger]^\dagger - [(A^*A)^\dagger B^\dagger]^\dagger A^\dagger\} \\ &= r\begin{bmatrix}AB\\ ABB^*B\end{bmatrix} + r[AB,\ AA^*AB] - 2r(AB). \end{aligned}$$

Hence, (a1) and (e1)–(e4) are equivalent. The equivalence of (e5) and (a1) is derived from (8) and

$$r[(A^*AB)^\dagger A^* - B^*(ABB^*)^\dagger] = r\begin{bmatrix}AB\\ ABB^*B\end{bmatrix} + r[AB,\ AA^*AB] - 2r(AB).$$

The proof of this formula is left for the reader. The equivalence of (a1), (e6), (e7) and (e8) is derived from (e5) by the previous replacement method.

The equivalence of (a1) and (e9) follows from (8) and

$$\begin{aligned} &r[B^*(A^*ABB^*)^\dagger A^* - B^*B(AA^*ABB^*B)^\dagger AA^*] \\ &= r\begin{bmatrix}AB\\ ABB^*B\end{bmatrix} + r[AB,\ AA^*AB] - 2r(AB). \end{aligned}$$

The proof is also left for the reader. $\square$

A square matrix A is called an orthogonal projector if $A^* = A = A^2$. Obviously, $A^\dagger = A$ if A is an orthogonal projector. Suppose A and B are two orthogonal projectors of order m. Then they satisfy the two range equalities in Theorem 1 (g3). By Theorem 1 (a3), AB satisfies the following identity

$$(AB)^\dagger = BA - B[(I_m - B)(I_m - A)]^\dagger A.$$

Pre- and post-multiplying A and B gives

$$(AB)^2 = AB + AB[(I_m - B)(I_m - A)]^\dagger AB.$$

These two results can be used to establish various equalities for $(AB)^k$, $(A-B)^\dagger$ and $(AB - BA)^\dagger$. For more details, see [3], [12], [15].

The results in Theorem 1 can be extended to the weighted Moore-Penrose inverse of a matrix product. Suppose M and N are two $m \times m$ and $n \times n$ Hermitian positive definite matrices, respectively. The weighted Moore-Penrose inverse of an $m \times n$ matrix A with respect to M and N is defined to be the unique $n \times m$ matrix X of satisfying the following four matrix equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X,$$
$$\text{(iii) } (MAX)^* = MAX, \quad \text{(iv) } (NXA)^* = NXA,$$

and is denoted as $X = A^\dagger_{M,N}$. When $M = I_m$ and $N = I_n$, $A^\dagger_{I_m,I_n}$ is the standard Moore-Penrose inverse $A^\dagger$ of A. Reverse-order laws for the weighted Moore-Penrose inverse of matrix products have also been studied; see, e.g., [10], [13]. It is well known (see, e.g., [1]) that the weighted Moore-Penrose inverse $A^\dagger_{M,N}$ of A can be rewritten as

$$A^\dagger_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}AN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}, \tag{17}$$

where $M^{\frac{1}{2}}$ and $N^{\frac{1}{2}}$ are the positive definite square roots of M and N, respectively. It turns out from (17) that

$$(AB)^\dagger_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}ABN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}, \tag{18}$$

$$A^\dagger_{M,I_n} = (M^{\frac{1}{2}}A)^\dagger M^{\frac{1}{2}}, \quad B^\dagger_{I_n,N} = N^{-\frac{1}{2}}(BN^{-\frac{1}{2}})^\dagger, \tag{19}$$

$$(BN^{-\frac{1}{2}})(BN^{-\frac{1}{2}})^\dagger = BB^\dagger_{I_n,N}, \quad (M^{\frac{1}{2}}A)^\dagger(M^{\frac{1}{2}}A) = A^\dagger_{M,I_n}A. \tag{20}$$

Theorem 3. *Let A and B be two $m\times n$ and $n\times p$ matrices, respectively, and let M and N be two $m \times m$ and $p \times p$ Hermitian positive definite matrices, respectively. Then the following* 27 *statements are equivalent*:

(a1) $(AB)^\dagger_{M,N} = B^\dagger_{I_n,N}(A^\dagger_{M,I_n}ABB^\dagger_{I_n,N})^\dagger A^\dagger_{M,I_n}$.

(a2) $(AB)^\dagger_{M,N} = N^{-1}B^*(A^*MABN^{-1}B^*)^\dagger A^*M$.

(a3) $(AB)^\dagger_{M,N} = B^\dagger_{I_n,N}A^\dagger_{M,I_n} - B^\dagger_{I_n,N}[(I_n - BB^\dagger_{I_n,N})(I_n - A^\dagger_{M,I_n}A)]^\dagger A^\dagger_{M,I_n}$.

(b1) $[\,(A^*)^\dagger_{I_n,M^{-1}}B\,]^\dagger_{M^{-1},N} = B^\dagger_{I_n,N}(A^\dagger_{M,I_n}ABB^\dagger_{I_n,N})^\dagger A^*$.

(b2) $[\,(A^*)^\dagger_{I_n,M^{-1}}B\,]^\dagger_{M^{-1},N} = N^{-1}B^*[\,(A^*MA)^\dagger(BN^{-1}B^*)\,]^\dagger A^\dagger_{M,I_n}M^{-1}$.

(b3) $[\,(A^*)^{\dagger}_{I_n,M^{-1}}B\,]^{\dagger}_{M^{-1},N} = B^{\dagger}_{I_n,N}A^* - B^{\dagger}_{I_n,N}[\,(\,I_n - BB^{\dagger}_{I_n,N}\,)(\,I_n - A^{\dagger}_{M,I_n}A\,)\,]^{\dagger}A^*.$

(c1) $[\,A(B^*)^{\dagger}_{N^{-1},I_n}\,]^{\dagger}_{M,N^{-1}} = B^*(A^{\dagger}_{M,I_n}ABB^{\dagger}_{I_n,N})^{\dagger}A^{\dagger}_{M,I_n}.$

(c2) $[\,A(B^*)^{\dagger}_{N^{-1},I_n}\,]^{\dagger}_{M,N^{-1}} = NB^{\dagger}_{I_n,N}[\,(A^*MA)(BN^{-1}B^*)^{\dagger}\,]^{\dagger}A^*M.$

(c3) $[\,A(B^*)^{\dagger}_{N^{-1},I_n}\,]^{\dagger}_{M,N^{-1}} = B^*[\,(\,I_n - BB^{\dagger}_{I_n,N}\,)(\,I_n - A^{\dagger}_{M,I_n}A\,)\,]^{\dagger}A^{\dagger}_{M,I_n}.$

(d1) $(B^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n})^{\dagger}_{N,M} = A(BB^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n}A)^{\dagger}B.$

(d2) $(B^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n})^{\dagger}_{N,M} = M^{-1}(A^*)^{\dagger}_{I_n,M^{-1}}[\,(BN^{-1}B^*)^{\dagger}(A^*MA)^{\dagger}\,]^{\dagger}(B^*)^{\dagger}_{N^{-1},I_n}N.$

(d3) $(B^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n})^{\dagger}_{N,M} = AB - A[\,(\,I_n - A^{\dagger}_{M,I_n}A\,)(\,I_n - BB^{\dagger}_{I_n,N}\,)\,]^{\dagger}B.$

(e1) $(A^{\dagger}_{M,I_n}AB)^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n} = B^{\dagger}_{I_n,N}(ABB^{\dagger}_{I_n,N})^{\dagger}_{M,I_n}.$

(e2) $(A^{\dagger}_{M,I_n}AB)^{\dagger}_{I_n,N}A^* = B^{\dagger}_{I_n,N}[\,(A^*)^{\dagger}_{I,M^{-1}}BB^{\dagger}_{I_n,N}\,]^{\dagger}_{M^{-1},I_n}.$

(e3) $[\,A^{\dagger}_{M,I_n}A(B^*)^{\dagger}_{N^{-1},I_n}\,]^{\dagger}_{I_n,N^{-1}}A^{\dagger}_{M,I_n} = B^*(ABB^{\dagger}_{I_n,N})^{\dagger}_{M,I_n}.$

(e4) $(BB^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n})^{\dagger}_{I_n,M}B = A(B^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n}A)^{\dagger}_{N,I_n}.$

(e5) $N(A^*MAB)^{\dagger}_{I_n,N}A^*M = B^*(ABN^{-1}B^*)^{\dagger}_{M,I_n}.$

(e6) $N[\,(A^*MA)^{\dagger}B\,]^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n} = B^*[\,(A^*)^{\dagger}_{I_n,M^{-1}}BN^{-1}B^*\,]^{\dagger}_{M^{-1},I_n}M.$

(e7) $[\,A^*MA(B^*)^{\dagger}_{N^{-1},I_n}\,]^{\dagger}_{I_n,N^{-1}}A^*M = NB^{\dagger}_{I_n,N}[\,A(BN^{-1}B^*)^{\dagger}\,]^{\dagger}_{M,I_n}.$

(e8) $NB^{\dagger}_{I_n,N}[\,(A^*)^{\dagger}_{I_n,M^{-1}}(BN^{-1}B^*)^{\dagger}\,]^{\dagger}_{M^{-1},I_n}M = [\,(A^*MA)^{\dagger}(B^*)^{\dagger}_{N^{-1},I_n}\,]^{\dagger}_{I_n,N^{-1}}A^{\dagger}_{M,I_n}.$

(e9) $(AA^*MABN^{-1}B^*B)^{\dagger}_{M,N} = B^{\dagger}_{I_n,N}(A^*MABN^{-1}B^*)^{\dagger}A^{\dagger}_{M,I_n}.$

(f1) $(A^{\dagger}_{M,I_n}AB)^{\dagger}_{I_n,N} = B^{\dagger}_{I_n,N}(A^{\dagger}_{M,I_n}ABB^{\dagger}_{I_n,N})^{\dagger}$ *and* $(ABB^{\dagger}_{I_n,N})^{\dagger}_{M,I_n} = (A^{\dagger}_{M,I_n}ABB^{\dagger}_{I_n,N})^{\dagger}A^{\dagger}_{M,I_n}.$

(f2) $(A^{\dagger}_{M,I_n}AB)^{\dagger}_{I_n,N} = N^{-1}B^*(A^{\dagger}_{M,I_n}ABN^{-1}B^*)^{\dagger}$ *and* $(ABB^{\dagger}_{I_n,N})^{\dagger}_{M,I_n} = (A^*MABB^{\dagger}_{I_n,N})^{\dagger}A^*M.$

(f3) $(A^{\dagger}_{M,I_n}AB)^{\dagger}_{I_n,N} = B^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n}A - B^{\dagger}_{I_n,N}[\,(\,I_n - BB^{\dagger}_{I_n,N}\,)(\,I_n - A^{\dagger}_{M,I_n}A\,)\,]^{\dagger}A^{\dagger}_{M,I_n}A$ *and* $(ABB^{\dagger}_{I_n,N})^{\dagger}_{M,I_n} = BB^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n} - BB^{\dagger}_{I_n,N}[\,(\,I_n - BB^{\dagger}_{I_n,N}\,)(\,I_n - A^{\dagger}_{M,I_n}A\,)\,]^{\dagger}A^{\dagger}_{M,I_n}.$

(g1) $\mathscr{R}[\,(AB)^{\dagger}_{M,N}\,] = \mathscr{R}[\,B^{\dagger}_{I_n,N}(A^{\dagger}_{M,I_n}ABB^{\dagger}_{I_n,N})^{\dagger}A^{\dagger}_{M,I_n}\,]$ *and* $\mathscr{R}\{[\,(AB)^{\dagger}_{M,N}\,]^*\} = \mathscr{R}\{[\,B^{\dagger}_{I_n,N}(A^{\dagger}_{M,I_n}ABB^{\dagger}_{I_n,N})^{\dagger}A^{\dagger}_{M,I_n}\,]^*\}.$

(g2) $\mathscr{R}[\,(AB)^{\dagger}_{M,N}\,] = \mathscr{R}(B^{\dagger}_{I_n,N}A^{\dagger}_{M,I_n})$ *and* $\mathscr{R}[\,(B^*A^*)^{\dagger}_{N^{-1},M^{-1}}\,] = \mathscr{R}[\,(A^*)^{\dagger}_{I_n,M^{-1}}(B^*)^{\dagger}_{N^{-1},I_n}\,].$

(g3) $\mathscr{R}(AA^*MAB) = \mathscr{R}(AB)$ *and* $\mathscr{R}[\,(ABN^{-1}B^*B)^*\,] = \mathscr{R}[\,(AB)^*\,].$

Proof. Let

$$A_1 = M^{\frac{1}{2}}A \quad \text{and} \quad B_1 = BN^{-\frac{1}{2}}. \tag{21}$$

Applying Theorem 1 to A_1 and B_1 gives the following 27 equivalent conditions

(a_1) $(A_1B_1)^{\dagger} = B_1^{\dagger}(A_1^{\dagger}A_1B_1B_1^{\dagger})^{\dagger}A_1^{\dagger}.$

(a_2) $(A_1B_1)^\dagger = B_1^*(A_1^*A_1B_1B_1^*)^\dagger A_1^*$.

(a_3) $(A_1B_1)^\dagger = B_1^\dagger A_1^\dagger - B_1^\dagger(E_{B_1}F_{A_1})^\dagger A_1^\dagger$.

$\vdots$

(g_3) $\mathscr{R}(\,A_1A_1^*A_1B_1\,) = \mathscr{R}(A_1B_1)$ and $\mathscr{R}[\,B_1^*B_1(A_1B_1)^*\,] = \mathscr{R}[\,(A_1B_1)^*\,]$.

Substituting (21) into (a_1) gives

$$(M^{\frac{1}{2}}ABN^{-\frac{1}{2}})^\dagger = (BN^{-\frac{1}{2}})^\dagger[(M^{\frac{1}{2}}A)^\dagger M^{\frac{1}{2}}ABN^{-\frac{1}{2}}(BN^{-\frac{1}{2}})^\dagger]^\dagger(M^{\frac{1}{2}}A)^\dagger.$$

Pre- and post-multiplying $N^{-\frac{1}{2}}$ and $M^{\frac{1}{2}}$ on both sides gives

$$\begin{aligned}&N^{-\frac{1}{2}}(M^{\frac{1}{2}}ABN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}\\ &\quad= N^{-\frac{1}{2}}(BN^{-\frac{1}{2}})^\dagger[(M^{\frac{1}{2}}A)^\dagger M^{\frac{1}{2}}ABN^{-\frac{1}{2}}(BN^{-\frac{1}{2}})^\dagger]^\dagger(M^{\frac{1}{2}}A)^\dagger M^{\frac{1}{2}}.\end{aligned}$$

From (18) and (19), this equality can be written as (a1). Substituting (21) into (a_2) gives

$$\begin{aligned}(M^{\frac{1}{2}}ABN^{-\frac{1}{2}})^\dagger &= (BN^{-\frac{1}{2}})^*[(M^{\frac{1}{2}}A)^*(M^{\frac{1}{2}}A)(BN^{-\frac{1}{2}})(BN^{-\frac{1}{2}})^*]^\dagger(M^{\frac{1}{2}}A)^*\\ &= N^{-\frac{1}{2}}B^*(A^*MABN^{-1}B^*)^\dagger A^*M^{\frac{1}{2}}.\end{aligned}$$

Pre- and post-multiplying $N^{-\frac{1}{2}}$ and $M^{\frac{1}{2}}$ on both sides gives (a2). Substituting (21) into (a_3) and applying (20) gives

$$\begin{aligned}&(M^{\frac{1}{2}}ABN^{-\frac{1}{2}})^\dagger\\ &\quad= (BN^{-\frac{1}{2}})^\dagger(M^{\frac{1}{2}}A)^\dagger\\ &\qquad- (BN^{-\frac{1}{2}})^\dagger\{[I_n - (BN^{-\frac{1}{2}})(BN^{-\frac{1}{2}})^\dagger][I_n - (M^{\frac{1}{2}}A)^\dagger(M^{\frac{1}{2}}A)]\}^\dagger(M^{\frac{1}{2}}A)^\dagger\\ &\quad= (BN^{-\frac{1}{2}})^\dagger(M^{\frac{1}{2}}A)^\dagger - (BN^{-\frac{1}{2}})^\dagger[(I_n - BB^\dagger_{I_n,N})(I_n - A^\dagger_{M,I_n}A)]^\dagger(M^{\frac{1}{2}}A)^\dagger.\end{aligned}$$

Pre- and post-multiplying $N^{-\frac{1}{2}}$ and $M^{\frac{1}{2}}$ on both sides gives (a3). The remaining in Theorem 3 can be shown similarly and the details are omitted here. □

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