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Note on discrete phenomena in uniqueness in doubly characteristic Cauchy problems

In memory of Professor S. Mizohata

By

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Abstract

Concerning the uniqueness in the Cauchy problem at doubly characteristic points, discrete conditions on lower order terms are known. In this paper, uniqueness is studied when those conditions are not satisfied.

1. Introduction

In Holmgren's uniqueness theorem, the initial surface is assumed to be non-characteristic. We already have an important extension to simply characteristic points. Namely, let $P(x, \partial)$ be a partial differential operator of order m with analytic coefficients in an open set Ω in $\mathbf{R}^n$, the coefficients of its principal part $P_m(x, \partial)$ be real-valued and $F(x)$ be a real-valued C^2 function with $F'(x) \neq 0$. Then the following theorem is known (see [3] and [10]).

Theorem 1.1. *Let $x^o \in \Omega$ and denote $\nu = F'(x^o)$. Suppose P is simply characteristic at (x^o, ν) and let $(x(t), \xi(t))$ be the bicharacteristic strip with $(x(0), \xi(0)) = (x^o, \nu)$. Also, suppose there exists a distribution solution $u(x)$ to the equation $Pu = 0$ with $x^o \in \operatorname{supp}[u] \subset \Omega^+ = \{x \in \Omega; F(x) \geq F(x^o)\}$. Then $x(t) \in \Omega^+$ in a neighborhood of $t = 0$.*

If P is doubly characteristic at (x^o, ν) and there are two bicharacteristic strip issued from (x^o, ν), how does the uniqueness depend on them? This paper is concerned with this problem, however our consideration is restricted to the following equation for simplicity.

$$Pu := \{\partial_1^2 - x_1^2 \partial_2^2 + a(x)\partial_1 + b(x)\partial_2 + c(x)\}u = 0, \tag{1.1}$$

where $x = (x_1, x_2) \in \mathbf{R}^2$, $\partial_i = \partial/\partial x_i$ $(i = 1, 2)$ and the coefficients are all analytic in an open set Ω containing the origin. Let $F(x) = x_2 - \rho(x_1)$ and $\rho(x_1)$ be a real- valued C^1 function satisfying $\rho(0) = \rho'(0) = 0$. This operator

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P is doubly characteristic at $(0, \nu)$ with $\nu = F'(0) = (0, 1)$ and it has two characteristic curves $\varphi(x) := x_2 + x_1^2/2 = 0$ and $\psi(x) := x_2 - x_1^2/2 = 0$ which satisfy $\varphi'(0) = \psi'(0) = \nu$. We denote $x_\varphi(t) = (t, -t^2/2)$ and $x_\psi(t) = (t, t^2/2)$. Then we have the following theorem and its corollary.

Theorem 1.2. *Suppose there exists a solution* $u(x) \in \mathcal{D}'(\Omega)$ *to the equation* (1.1) *with* $0 \in \mathrm{supp}[u] \subset \Omega^+ = \{x \in \Omega; F(x) \geq 0\}$. *Then* i) $x_\psi(t) \in \Omega^+$ *in a neighborhood of* $t = 0$. *Moreover,* ii) *when* $b(0) \notin \{1, 3, 5, \cdots\}$, $x_\varphi(t) \in \Omega^+$ *in at least one of one-sided neighborhoods of* $t = 0$, *i.e. one of two intervals* $-\delta < t \leq 0$ *and* $0 \leq t < \delta$ *with some* $\delta > 0$.

Corollary 1.1. *Let* G *be a closed subset of* Ω *and* σ *be a real constant. Suppose* $K = G \cap \{x \in \Omega; x_2 - \sigma x_1^2 \leq 0\}$ *be compact and* $u(x)$ *be a distribution solution to* (1.1) *in* Ω *with* $\mathrm{supp}[u] \subset G$. *Then* $u(x)$ *vanishes in a neighborhood of* K *if i)* $\sigma \geq 1/2$ *or if ii)* $b(0, x_2) \notin \{1, 3, 5, \cdots\}$ *for* $(0, x_2) \in \Omega$ *and* $1/2 > \sigma \geq -1/2$.

When $a = c = 0$ and b is a real constant, Corollary 1.1 with $\sigma = 0$ is due to F. Trèves [12] and B. Birkeland and J. Persson [1], and Theorem 1.2 to S. Nakane [9]. The author ([5]) has extended it in a slightly different form to more general equations including (1.1), which will be explained later (see **3.1** and **3.2**). Concerning the uniqueness and non-uniqueness in the characteristic Cauchy problem, see also [4], [7], [8], [11] and their references.

The discrete condition $b(0) \notin \{1, 3, 5, \cdots\}$ still interests us much. When $a = c = 0$ and b is a constant, we know it is necessary (cf. [12]), however the author knows no other results. In general, is it essential for the uniqueness? Our aim is to investigate this problem. The obtained results will be stated in the following section. We will see there the uniqueness may hold in the case $b(0) \in \{1, 3, 5, \cdots\}$,too, namely Theorem 1.2 remains true even if the condition $b(0) \notin \{1, 3, 5, \cdots\}$ is replaced by the one that $b(0, x_2) \equiv 2\mu + 1 \in \{1, 3, 5, \cdots\}$ and $c(0)$ differs from a certain number (Theorem 2.1). Especially, when $a = 0$ and b, c are constants, we will have a necessary and sufficient condition that $(b, c) \neq (2\mu + 1, 0)$, $\mu = 0, 1, 2, \cdots$ (Corollary 2.2).

2. Results

First, we suppose

$$b(0, x_2) \equiv 2\mu + 1 \in \{1, 3, 5, \cdots\}. \tag{2.1}$$

To state the results, we need a new quantity. Let ${}^tP$ stand for the transposed operator of P and λ be a parameter. Since φ is a phase function of ${}^tP$, one can write $\varphi^{-\lambda+1}\ {}^tP(\varphi^\lambda v) = \lambda M v + O(\varphi)$, where

$$M = 2x_1(\partial_1 - x_1\partial_2) - b - ax_1 + 1. \tag{2.2}$$

This is a Fuchsian partial differential operator with characteristic exponent μ.

Proposition 2.1. *Concerning the operator M, the following* i) *and* ii) *hold under the condition* (2.1).

i) *There exists a unique sequence $\ell_j(\xi)$ $(0 \le j \le \mu)$ with $\ell_\mu = 1$ such that $\pi_0 LM = 0$ with $L = \sum_{j=0}^{\mu} \ell_j(\varphi)(\partial_1 - x_1\partial_2)^j$, where π_0 denotes the operator which restricts functions onto $x_1 = 0$.*

ii) *There exists a unique sequence $r_j(\xi)$ $(j \ge \mu)$ with $r_\mu = 1$ such that $MR = 0$ with $R = \sum_{j=\mu}^{\infty} r_j(\varphi)x_1^j/j!$.*

Setting $Q = {}^tP - M\partial_2$, we define

$$q(x_2) = \pi_0 LQR. \tag{2.3}$$

Then the following theorem and its corollary hold.

Theorem 2.1. *Suppose* (2.1) *and $q(0) \ne 0$. Also, suppose there exists a solution $u(x) \in \mathcal{D}'(\Omega)$ to the equation* (1.1) *with $0 \in \operatorname{supp}[u] \subset \Omega^+$. Then $x_\varphi(t) \in \Omega^+$ in at least one of one-sided neighborhoods of $t = 0$, i.e. one of two intervals $-\delta < t \le 0$ and $0 \le t < \delta$ with some $\delta > 0$.*

Corollary 2.1. *Let G be a closed subset of Ω and σ be a real constant with $1/2 > \sigma \ge -1/2$. Suppose* (2.1) *and $q(x_2) \ne 0$. Also, suppose $K = G \cap \{x \in \Omega; x_2 - \sigma x_1^2 \le 0\}$ be compact and $u(x)$ be a distribution solution to* (1.1) *in Ω with $\operatorname{supp}[u] \subset G$. Then $u(x)$ vanishes in a neighborhood of K.*

Remark 1. Since $\pi_0 Lc(x)R = c(0, x_2)$, one may take $q(0) \ne 0$ a condition on the value $c(0)$.

As a special case, consider the equation

$$Pu = \{\partial_1^2 - x_1^2\partial_2^2 + b\partial_2 + c\}u = 0, \quad b, c \text{ constants.} \tag{2.4}$$

When $b = 2\mu + 1 (\mu = 0, 1, 2, \cdots)$, we see easily $L = (\partial_1 - x_1\partial_2)^\mu$, $R = x_1^\mu/\mu!$ and so $q = c$. Therefore the following corollary follows from Theorem 1.2 and Theorem 2.1. (One can also get a corresponding result to Corollary 2.1.)

Corollary 2.2. *Suppose there exists a distribution solution $u(x)$ to the equation* (2.4) *in Ω with $0 \in \operatorname{supp}[u] \subset \Omega^+$. It then follows that $x_\varphi(t) \in \Omega^+$ in at least one of one-sided neighborhoods of $t = 0$, if and only if $(b, c) \notin \{(2\mu + 1, 0); \mu = 0, 1, 2, \cdots\}$.*

Lastly, by an example, we point out a different aspect of the problem appearing in the case $b(0) = 2\mu + 1$ $(\mu = 0, 1, 2, \cdots)$ but $b(0, x_2) \not\equiv 2\mu + 1$.

$$Pu = \{\partial_1^2 - x_1^2\partial_2^2 + (2\mu + 1 + \psi)\partial_2 + c\}u = 0, \quad c \text{ a constant,} \tag{2.5}$$

where $\psi = x_2 - x_1^2/2$. For this equation, the result is opposite to the above ones, namely, the following proposition holds, which means the consequence in Theorem 2.1 and that in Corollary 2.1 are not true for this equation.

Proposition 2.2. *There exists a distribution solution $u(x)$ to the equation* (2.5) *in a neighborhood of the origin such that* $0 \in \mathrm{supp}[u] \subset \{x; \psi(x) \geq 0\}$.

3. Proofs

Though Theorem 1.2 and Corollary 1.1 are essentially not new, we begin with their proofs for the paper to be self-contained.

3.1. Proof of Theorem 1.2

Take $\delta_i > 0$ $(i = 1, 2)$ such that $D := \{x; |x_i| < \delta_i, i = 1, 2\} \subset \Omega$, $|\rho(x_1)| < \delta_2$ and $x_1^2/2 < \delta_2$ for $|x_1| < \delta_1$ and suppose there exists $|\alpha| < \delta_1$ such that $x_\psi(\alpha) \notin \Omega^+$, i.e. $\rho(\alpha) > \alpha^2/2$. If $\alpha > 0$, there exists $\epsilon > 0$ such that $\rho(\alpha) > \epsilon\alpha + \alpha^2/2$. The curve $G(x) := x_2 - \epsilon x_1 - x_1^2/2 = C$ with a parameter C is non-characteristic for the equation (1.1) when $x_1 > -\epsilon/2$. Hence, noting that $u = 0$ when $x_2 < \rho(x_1)$ and $\rho(0) = \rho'(0) = 0$, we see $u = 0$ near the origin by the well-known sweeping-out method with Holmgren's uniqueness theorem, which contradicts the assumption $0 \in \mathrm{supp}[u]$. One can derive the same contradiction in the case $\alpha < 0$, too, which proves i).

Next, let D be the same one as defined above and suppose there exist α and β with $-\delta_1 < \alpha < 0 < \beta < \delta_1$ such that $\rho(\alpha) > -\alpha^2/2$ and $\rho(\beta) > -\beta^2/2$. Then there exists $0 < \epsilon < 1$ such that $\rho(\alpha) > (\epsilon - 1/2)\alpha^2$ and $\rho(\beta) > (\epsilon - 1/2)\beta^2$. Since the curve $x_2 - (\epsilon - 1/2)x_1^2 = C$ with a parameter C is non-characteristic except for $x_1 = 0$ for the equation (1.1), one can see $u = 0$ for $x_2 - (\epsilon - 1/2)x_1^2 < 0$ by the sweeping-out method with Holmgren's uniqueness theorem. Besides, by the following theorem, we see $u = 0$ near the origin. It contradicts the assumption $0 \in \mathrm{supp}[u]$, which proves ii).

Theorem 3.1 ([5]). *Let $0 \in \Omega$ and suppose $b(0) \notin \{1, 3, 5, \cdots\}$. Also, let u be a distribution solution to* (1.1) *with* $\mathrm{supp}[u] \subset \{0\} \cup \{x; x_2 + x_1^2/2 > 0\}$. *Then u vanishes in a neighborhood of $x = 0$.*

3.2. Proof of Corollary 1.1

Suppose $\mathrm{supp}[u] \cup \{x; x_2 - \sigma x_1^2 \leq 0\} \neq \emptyset$. Denoting by $\kappa(C)$ the curve $x_2 - \sigma x_1^2 = C$, let C_{min} be the minimum of C such that $\mathrm{supp}[u] \cup \{x; x_2 - \sigma x_1^2 = C\} \neq \emptyset$. Since the curve $\kappa(C)$ is non-characteristic for the equation (1.1) when $\sigma \neq \pm 1/2$ and simply characteristic when $\sigma = \pm 1/2$ both for $x_1 \neq 0$ and $\mathrm{supp}[u] \cap \kappa(C_{min})$ is compact, we see $\mathrm{supp}[u] \cap \kappa(C_{min}) = \{(0, C_{min})\}$ by Holmgren's theorem and Theorem 1.1. Lastly, by applying Theorem 1.2 at $(0, C_{min})$, we see $u = 0$ near this point. It contradicts the assumption, which proves Corollary 1.1.

3.3. Proof of Proposition 2.1

We change the variables by $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = x_2 + x_1^2/2$ and denote $(\tilde{x}_1, \tilde{x}_2)$ by (x_1, x_2) again for simplicity. Then P, φ, ψ and F are respectively written as

$$\text{(3.1)} \qquad \begin{cases} P = \partial_1^2 + 2x_1\partial_1\partial_2 + a\partial_1 + (b + ax_1 + 1)\partial_2 + c, \\ \varphi = x_2,\ \psi = x_2 - x_1^2,\ F = x_2 - \rho^*(x_1) \end{cases}$$

with $\rho^*(x_1) = \rho(x_1) + x_1^2/2$. Besides, ${}^tP$ and M are done as

$$\text{(3.2)} \qquad \begin{cases} {}^tP = \partial_1^2 + 2x_1\partial_1\partial_2 - a\partial_1 - (b + ax_1 - 1)\partial_2 \\ \qquad + c - \partial_1 a - \partial_2 b - x_1\partial_2 a, \\ M = 2x_1\partial_1 - b - ax_1 + 1. \end{cases}$$

Denote $\tilde{b} = b + ax_1 - 1$, then

$$\begin{aligned} \pi_0 LM &= \pi_0 \sum_{j=0}^{\mu} \ell_j(x_2)\partial_1^j(2x_1\partial_1 - \tilde{b}(x)) \\ &= \pi_0 \sum_{j=0}^{\mu} \ell_j \left\{ 2x_1\partial_1^{j+1} + 2j\partial_1^j - \sum_{0\le k\le j} \binom{j}{k} (\partial_1^k\tilde{b})\partial_1^{j-k} \right\} \\ &= \sum_{j=0}^{\mu} \left\{ 2j\ell_j - \sum_{0\le k\le \mu-j} \binom{j+k}{k} \ell_{j+k}\partial_1^k\tilde{b}(0,x_2) \right\} \pi_0\partial_1^j. \end{aligned}$$

Therefore $\pi_0 LM = 0$ means that

$$\begin{aligned} &(2\mu - \tilde{b}(0,x_2))\ell_\mu = 0, \\ &(2j - \tilde{b}(0,x_2))\ell_j = \sum_{1\le k\le \mu-j} \binom{j+k}{k} \ell_{j+k}\partial_1^k\tilde{b}(0,x_2) \text{ for } 0 \le j < \mu. \end{aligned}$$

Since $\tilde{b}(0,x_2) = 2\mu$ by the assumption (2.1), we see ℓ_μ is free, and so we may put $\ell_\mu = 1$. For each $0 \le j < \mu$, ℓ_j is determined uniquely by the second relations successively. Thus the proof of i) has been completed. The second part ii) is a well-known fact in the theory of Fuchsian partial differential equations.

3.4. A characteristic Cauchy problem

Employing the same coordinate system as in **3.3**, we consider the following characteristic Cauchy problem, whose solvability will be used to prove Theorem 2.1.

$$\text{(3.3)} \qquad {}^tPv = M\partial_2 v + Qv = f,\ v = O(x_2 - h)$$

where $Q = \partial_1^2 - a\partial_1 + \tilde{c}$, $\tilde{c} = c - \partial_1 a - \partial_2 b - x_1\partial_2 a$ and h is a parameter. Because $\pi_0 LM = 0$, wee see the right hand side needs to satisfy

$$\text{(3.4)} \qquad \pi_0 Lf = 0 \quad \text{at } x_2 = h.$$

We denote by $A_{\hat{x}}^r$ the set of all analytic functions $f(x)$ at $x = \hat{x}$ whose Taylor expansion $f(x) = \sum_\alpha \partial^{(\alpha)} f(\hat{x})(x - \hat{x})^\alpha/\alpha!$ converge in $\{x; |x_j - \hat{x}_j| < r, j = 1,2\}$. One may suppose all the coefficients of P belong to A_0^r for some $r > 0$. Then the following theorem holds.

Theorem 3.2 ([6]). *Let $r > 0$ and assume all the coefficients of P belong to A_0^r. Also, assume* (2.1) *and $q(0) \neq 0$. Then there exists $\delta > 0$ such that for any $f(x) \in A_0^r$ satisfying* (3.4) *and any $|h| < \delta$ there exists a unique solution $u(x) \in A_{(0,h)}^\delta$ to the Cauchy problem* (3.3).

This theorem is a special case of Theorem 3.1 in our preceding paper [6], and so we only explain how the condition $q(0) \neq 0$ works.

Let $M_h(x_1, \partial_1)$ stand for $M(x_1, h, \partial_1)$, and so do $L_h(x_1, \partial_1)$, $Q_h(x_1, \partial_1)$ and $R_h(x_1)$. We set $\tilde{v} = \partial_2 v$ and define $\partial_2^{-1}\tilde{v} = \int_h^{x_2} \tilde{v}(x_1, x_2)dx_2$. Then the Cauchy problem (3.3) is written as

$$M_h\tilde{v} + Q^*\partial_2^{-1}\tilde{v} = f,$$

where $Q^* = (M - M_h)\partial_2 + Q$. If we put

$$M_h\tilde{v} = w, \quad \pi_0\partial_1^\mu\tilde{v} = p(x_2), \tag{3.5}$$

then we have $\pi_0 L_h w = 0$ because $\pi_0 L_h M_h = 0$. Oppositely, given analytic $w(x)$ with $\pi_0 L_h w = 0$ and analytic $p(x_2)$ arbitrarily, one can verify easily that there exists a unique analytic function $\tilde{v}$ satisfying (3.5). We set

$$M_h^{-1}w = \tilde{v} - p(x_2)R_h(x_1).$$

Then, since $M_h R_h = 0$, it follows that $M_h M_h^{-1}w = w$ and $\pi_0\partial_1^\mu M_h^{-1}w = 0$. Hence we have

$$w + Q^*\partial_2^{-1}M_h^{-1}w + Q^*\partial_2^{-1}pR_h = f. \tag{3.6}$$

Next, we operate $\pi_0 L_h$ from the left. Then, by noting $\pi_0 L_h w = 0$, $\pi_0 L_h Q_h R_h = q(h)$ and $\partial_2^{-1}pR_h = R_h\partial_2^{-1}p$, we have

$$q(h)\partial_2^{-1}p + \pi_0 L_h Q^*\partial_2^{-1}M_h^{-1}w + \pi_0 L_h Q^{**}R_h\partial_2^{-1}p = \pi_0 L_h f, \tag{3.7}$$

where $Q^{**} = (M - M_h)\partial_2 + Q - Q_h$. If $q(h) \neq 0$, one can solve the equations (3.6) and (3.7) with respect to $w(x)$ and $\partial_2^{-1}p(x_2)$ by the contraction principle. (See [6] for the details.)

3.5. Proof of Theorem 2.1 and Corollary 2.1

In the same coordinate system as in **3.3** and **3.4**, take $\delta_i > 0$ $(i = 1, 2)$ such that $D := \{x; |x_i| < \delta_i, i = 1, 2\} \subset \Omega$, $|\rho^*(x_1)| < \delta_2$ and $x_1^2 < \delta_2$ for $|x_1| < \delta_1$, and suppose there exist $-\delta_1 < \alpha < 0 < \beta < \delta_1$ such that $\rho^*(\alpha) > 0$ and $\rho^*(\beta) > 0$. Then there exists $0 < \sigma < 1$ such that $\sigma\alpha^2 < \rho^*(\alpha)$ and $\sigma\beta^2 < \rho^*(\beta)$. We set

$$\Sigma = \{x \in \mathrm{supp}[u]; \alpha \leq x_1 \leq \beta, x_2 \leq \sigma x_1^2\},$$

and, considering $\Sigma \neq \emptyset$, put

$$s = \min\{x_2 - \sigma x_1^2; (x_1, x_2) \in \Sigma\}.$$

If $s < 0$, the set $\Sigma_s = \Sigma \cap \{x_2 - \sigma x_1^2 = s\}$ does not contain $(0, s)$, $(\alpha, \sigma\alpha^2)$ and $(\beta, \sigma\beta^2)$. Therefore Σ_s is a compact set and the curve $x_2 - \sigma x_1^2 = s$ is non-characteristic at every point of Σ_s. Since $u = 0$ when $x_2 - \sigma x_1^2 < s$, it follows from Holmgren's uniqueness theorem that $\Sigma_s = \emptyset$. This is a contradiction, and so we may suppose $s = 0$, namely $u = 0$ in $\{\alpha < x_1 < \beta, x_2 < \sigma x_1^2\}$.

Next, we take δ, h and h' so small that $\delta < \min\{-\alpha, \beta\}$, $0 < h < h' < \sigma\delta^2 < \delta/2$, $U := \{(x_1, x_2); |x_1| < \delta, |x_2| < \sigma\delta^2\} \subset \Omega$ and Theorem 3.2 would be applicable. Let $\chi(x_2)$ be a $\mathcal{C}^\infty$ function such that $\chi(x_2) = 1$ for $x_2 \leq h$ and $\chi(x_2) = 0$ for $x_2 \geq h'$, and set $\tilde{u} = \chi u$. Then $\tilde{u} \in \mathcal{E}'(U)$ and $P\tilde{u} = 0$ in $U_h := \{(x_1, x_2) \in U; x_2 < h\}$. Besides we see there exist an positive integer ℓ, a compact set $K \subset U$ and a positive constant C such that

$$|\langle \tilde{u}, f\rangle| \leq C \sum_{|\alpha|\leq \ell} \sup_{x\in K} |\partial^\alpha f(x)| \qquad \forall f \in \mathcal{E}(U),$$

and therefore $\tilde{u}$ is extendable to a linear form on $\mathcal{E}^\ell(U)$.

Next, let $\mathcal{F}^{\ell+2}(U)$ stand for the set of all functions $f(x)$ in U such that $\partial^\alpha f$ are continuous in U for $|\alpha| \leq \ell + 2, \alpha_2 \leq \ell + 1$ and $f(x) = 0$ for $x_2 \geq h$. Then $\langle Pu, v\rangle = \langle P\tilde{u}, v\rangle = \langle \tilde{u}, {}^tPv\rangle = 0$ for any $v \in \mathcal{D}(U)$ with $\mathrm{supp}[v] \subset U_h$ which are dense in $\mathcal{F}^{\ell+2}$. Therefore we have

$$\langle \tilde{u}, {}^tPv\rangle = 0 \qquad \forall v \in \mathcal{F}^{\ell+2}(U).$$

Lastly, for $f(x) \in \mathcal{D}(U)$ with $\mathrm{supp}[f] \subset U_h$, define

$$f_\epsilon = (x_2 - h)^{\ell+2}[e_\epsilon * \{(x_2 - h)^{-\ell-2} f\}],$$

where $e_\epsilon(x) = (4\pi\epsilon)^{-1} \exp[-|x|^2/(4\epsilon)]$ and $*$ denotes the convolution. Then $f_\epsilon(x) = O\{(x_2 - h)^{\ell+2}\}$ and it is an entire function. By Theorem 3.2, there exists a solution $v_\epsilon \in A^\delta_{(0,h)}$ to the equation ${}^tPv_\epsilon = f_\epsilon$ with $v_\epsilon = O\{(x_2-h)^{\ell+2}\}$. Let f^*_ϵ denote the function such that $f^*_\epsilon = f_\epsilon$ for $x_2 < h$ and $f^*_\epsilon = 0$ for $x_2 \geq h$ and so do v^*_ϵ. Then $v^*_\epsilon \in \mathcal{F}^{\ell+2}$ and ${}^tPv^*_\epsilon = f^*_\epsilon$ and hence we have

$$\langle \tilde{u}, f^*_\epsilon\rangle = 0.$$

Now that f^*_ϵ tends to f in $\mathcal{E}^\ell(U)$, it follows that $\langle \tilde{u}, f\rangle = \langle u, f\rangle = 0$. Thus we see $u = 0$ in U_h. This contradicts the assumption $0 \in \mathrm{supp}[u]$, which completes the proof of Theorem 2.1. One can prove Corollary 2.1 in the same way as the proof of Corollary 1.1.

3.6. Proof of Proposition 2.2

First, we change the variables by $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = \psi$ and denote the new variables by (x_1, x_2) again for simplicity. Then the transformed operator is

$$P = -2x_1\partial_1\partial_2 + (2\mu + x_2)\partial_2 + \partial_1^2 + c.$$

Next, we set $u = \partial_2^m v$ with the minimum non-negative integer m such that $-\Re c + m > -1$. Then we have $P\partial_2^m v = \partial_2^m (P - m)v$. Next, we obtain a solution v of $(P-m)v=0$ by setting $v=\sum_{k\geq 0} x_2^{\lambda+k}v_k(x_1)$ with a parameter λ. Noting that

$$\begin{aligned}(P-m)v &= \{-2x_1\partial_1\partial_2+2\mu\partial_2+x_2\partial_2+\partial_1^2+c-m\}\sum_{k\geq 0} x_2^{\lambda+k}v_k(x_1)\\ &= \sum_{k\geq 0}(-2x_1\partial_1+2\mu)v_k(\lambda+k)x_2^{\lambda+k-1}\\ &\quad+\sum_{k\geq 0}(\lambda+k+c-m+\partial_1^2)v_k x_2^{\lambda+k},\end{aligned}$$

we get a recurrence relation

$$\begin{cases}\lambda(-2x_1\partial_1+2\mu)v_0(x_1)=0,\\ (\lambda+k)(-2x_1\partial_1+2\mu)v_k(x_1)+(\lambda+k-1+c-m+\partial_1^2)v_{k-1}(x_1)=0,\\ k=1,2,\cdots.\end{cases}$$

The first relation is solved by $v_0=x_1^{\mu}$. For the second relation with $k=1$ to have a solution, it is necessary and sufficient that $\pi_0\partial_1^{\mu}(\lambda+c-m+\partial_1^2)x_1^{\mu}=(\lambda+c-m)\mu!=0$, so we set $\lambda=-c+m$. Since $-\Re c+m>-1$, we have $\lambda+k\neq 0$ for $k\geq 1$. Hence the second relation with $k=1$ is solved by a polynomial $v_1(x_1)$ of degree $\mu-2$, and successively the relation with $k\geq 2$ is done by a polynomial $v_k(x_1)$ of degree $\mu-2$. The obtained series converges in $V=\{x;|x_2|<4\}$.

Now, we set $v_+=v$ for $x_2>0$ and $v_+=0$ for $x_2\leq 0$. Noting that v_+ is locally summable, we define $u_+=\partial_2^m v_+$ in $\mathcal{D}'$ sense and prove $Pu_+=0$. For $\varphi\in\mathcal{D}(V)$, we have

$$\begin{aligned}&\langle Pu_+,\varphi\rangle=\langle v_+,{}^t(P\partial_2^m)\varphi\rangle\\ &=\lim_{\epsilon\to+0}\iint_{x_2\geq\epsilon} v\cdot{}^t(P\partial_2^m)\varphi dx=\lim_{\epsilon\to+0}\iint_{x_2\geq\epsilon} v\cdot({}^tP-m)(-\partial_2)^m\varphi dx\\ &=\lim_{\epsilon\to+0}\int_{x_2=\epsilon}(-2x_1\partial_1+2\mu+x_2)v\cdot(-\partial_2)^m\varphi dx_1\\ &\quad+\lim_{\epsilon\to+0}\iint_{x_2\geq\epsilon}(P-m)v\cdot(-\partial_2)^m\varphi dx.\end{aligned}$$

Because $(-2x_1\partial_1+2\mu+x_2)v=(-2x_1\partial_1+2\mu)v_0(x_1)x_2^{\lambda}+O(x_2^{\lambda+1})=o(1)$ as $x_2\to+0$ and $(P-m)v=0$, we see $<Pu_+,\varphi>=0$ for any $\varphi\in\mathcal{D}(V)$, that is, $Pu_+=0$, which completes the proof.

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