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ON AMICABLE TUPLES

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ABSTRACT. For an integer $k \geq 2$, a tuple of k positive integers $(M_i)_{i=1}^{k}$ is called an *amicable k-tuple* if the equation

$$\sigma(M_1) = \cdots = \sigma(M_k) = M_1 + \cdots + M_k$$

holds. This is a generalization of amicable pairs. An *amicable pair* is a pair of distinct positive integers each of which is the sum of the proper divisors of the other. Gmelin (Über vollkommene und befreundete Zahlen (1917) Heidelberg University) conjectured that there is no relatively prime amicable pairs and Artjuhov (*Acta Arith.* **27** (1975) 281–291) and Borho (*Math. Ann.* **209** (1974) 183–193) proved that for any fixed positive integer K, there are only finitely many relatively prime amicable pairs (M, N) with $\omega(MN) = K$. Recently, Pollack (*Mosc. J. Comb. Number Theory* **5** (2015), 36–51) obtained an upper bound

$$MN < (2K)^{2^{K^2}}$$

for such amicable pairs. In this paper, we improve this upper bound to

$$MN < \frac{\pi^2}{6} 2^{4^K - 2 \cdot 2^K}$$

and generalize this bound to some class of general amicable tuples.

1. Introduction

For an integer $k \geq 2$, a tuple of k positive integers

$$(M_1, \ldots, M_k)$$

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is called an *amicable k-tuple* [6] if the equation

$$\sigma(M_1) = \cdots = \sigma(M_k) = M_1 + \cdots + M_k \tag{1}$$

holds, where $\sigma(n)$ is the usual divisor summatory function. This is one of generalizations of amicable pairs. An *amicable pair* is a pair of distinct positive integers each of which is the sum of the proper divisors of the other. For a pair of distinct positive integers (M, N), this definition of amicable pair can be rephrased as

$$\sigma(M) - M = N \quad \text{and} \quad \sigma(N) - N = M,$$

which is equivalent to

$$\sigma(M) = \sigma(N) = M + N$$

so (M, N) is an amicable pair if and only if (M, N) is an amicable 2-tuple provided $M \neq N$.

Although the history of amicable pairs can be traced back more than 2000 years ago to Pythagoreans, their nature is still shrouded in mystery. In 1917, Gmelin [8] noted that there is no amicable pairs (M, N) on the list at that time for which M and N are relatively prime. Based on this observation, he conjectured that there is no such relatively prime amicable pair.

As for this problem, Artjuhov [1] and Borho [2] proved that for any fixed integer $K \geq 1$, there are only finitely many relatively prime amicable pairs (M, N) with $\omega(MN) = K$ where $\omega(n)$ denote the number of distinct prime factors of n. Recently, Pollack [15] obtained an explicit upper bound

$$MN < (2K)^{2^{K^2}} \tag{2}$$

for relatively prime amicable pairs (M, N), where $K = \omega(MN)$. The main aim of this paper is to improve and generalize this result of Pollack.

Actually, not only for amicable pairs as in Gmelin's conjecture, members of each amicable tuple seem to share relatively many common factors. Although there are not so many examples of large amicable k-tuples for $k \geq 4$, it seems further that there is no amicable tuple whose greatest common divisor is 1. For some technical and artificial reason, we consider a much stronger condition on amicable tuples. We define this condition not only for amicable tuples but for general tuples of integers.

Definition 1 (*Anarchic tuple*). For an integer $k \geq 2$, we say a tuple of distinct positive integers $(M_i)_{i=1}^{k}$ is *anarchic* if

$$\bigl(M_i, M_j\sigma(M_j)\bigr) = 1 \quad \text{for all distinct } i \text{ and } j,$$

where (a, b) denotes the greatest common divisor of a and b.

We also introduce a new class of tuples of integers, which has already been introduced essentially by Kozek, Luca, Pollack and Pomerance [11].

Definition 2 (*Harmonious tuple*). For an integer $k \geq 2$, we say a tuple of positive integers $(M_i)_{i=1}^k$ is a *harmonious k-tuple* if the equation

$$\frac{M_1}{\sigma(M_1)} + \cdots + \frac{M_k}{\sigma(M_k)} = 1 \tag{3}$$

holds.

In this paper, we generalize and improve Pollack's bound (2) as follows.

Theorem 1. *For any anarchic harmonious tuple $(M_i)_{i=1}^k$, we have*

$$M_1 \cdots M_k < \frac{\pi^2}{6} 2^{4^K - 2 \cdot 2^K},$$

where $K = \omega(M_1 \cdots M_k)$.

Note that if $(M_i)_{i=1}^k$ is an amicable tuple, then

$$1 = \frac{M_1}{M_1 + \cdots + M_k} + \cdots + \frac{M_k}{M_1 + \cdots + M_k} = \frac{M_1}{\sigma(M_1)} + \cdots + \frac{M_k}{\sigma(M_k)}$$

so every amicable tuple is harmonious. Also, for any amicable tuple $(M_i)_{i=1}^k$,

$$\bigl(M_i, M_j \sigma(M_j)\bigr) = \bigl(M_i, M_j(M_1 + \cdots + M_k)\bigr)$$

so an amicable tuple $(M_i)_{i=1}^k$ is anarchic if and only if M_i are pairwise coprime and $M_1 \cdots M_k$ is coprime to $M_1 + \cdots + M_k$. Moreover, an amicable pair (M, N) is anarchic if and only if M and N are relatively prime since

$$\bigl(M(M+N), N\bigr) = 1 \iff (M, N) = 1 \iff \bigl(M, N(M+N)\bigr) = 1.$$

Thus Theorem 1 leads to the following corollaries.

Theorem 2. *For any relatively prime amicable pair (M, N), we have*

$$MN < \frac{\pi^2}{6} 2^{4^K - 2 \cdot 2^K},$$

where $K = \omega(MN)$.

Theorem 3. *For any amicable tuple $(M_i)_{i=1}^k$ satisfying*

$$M_1, \ldots, M_k \text{ are pairwise coprime} \quad \text{and} \quad (M_1 \cdots M_k, M_1 + \cdots + M_k) = 1, \tag{4}$$

we have

$$M_1 \cdots M_k < \frac{\pi^2}{6} 2^{4^K - 2 \cdot 2^K},$$

where $K = \omega(M_1 \cdots M_k)$.

Therefore, Theorem 2 improves the upper bound (2).

Even without conditions on divisibility similar to being relatively prime or being anarchic, Borho [3] proved an upper bound for amicable pairs in terms of $\Omega(n)$, the number of prime factors of n counted with multiplicity. He also dealt with *unitary amicable pairs*. A unitary amicable pair is a pair of positive integers (M, N) satisfying the equation

$$\sigma^*(M) = \sigma^*(N) = M + N, \qquad \sigma^*(n) := \sum_{d \parallel n} d,$$

where $d \parallel n$ means $d \mid n$ and $(d, n/d) = 1$. Borho's upper bound is

$$MN < L^{2^L}$$

for any amicable or unitary amicable pairs (M, N) with $L = \Omega(MN)$ for amicable pairs and $L = \omega(M) + \omega(N)$ for unitary amicable pairs. Indeed, we can see a prototype of our Lemmas 6 and 7 on Diophantine equation in Satz 1 of Borho [3]. By introducing our lemmas in Borho's argument, we can improve and generalize Borho's theorem. As Kozek, Luca, Pollack and Pomerance [11] remarked, what we can deal with are not only amicable tuples but harmonious tuples. Thus, we first introduce the unitary analogue of harmonious tuples.

Definition 3 (*Unitary harmonious tuple*). For an integer $k \geq 2$, we say a tuple of positive integers $(M_i)_{i=1}^k$ is a *unitary harmonious tuple* if the equation

$$\frac{M_1}{\sigma^*(M_1)} + \cdots + \frac{M_k}{\sigma^*(M_k)} = 1 \tag{5}$$

holds.

Then our theorem of the Borho-type is the following.

Theorem 4. *For any harmonious tuple $(M_i)_{i=1}^k$, we have*

$$M_1 \cdots M_k \leq k^{-k}\left(2^{2^L} - 2^{2^{L-1}}\right),$$

where $L = \Omega(M_1 \cdots M_k)$. For any unitary harmonious tuple $(M_i)_{i=1}^k$, we again have the same upper bound but L is replaced by $L = \omega(M_1) + \cdots + \omega(M_k)$.

We prove Theorem 4 in Section 4.

The above problems on the upper bound of harmonious or amicable tuples are analogues of a similar problem in the context of odd perfect numbers, which has been studied since a long time ago. A positive integer N is called a *perfect number* if its sum of all proper divisors is equal to N itself, that is, if $\sigma(N) = 2N$. It is also a long-standing mystery whether or not there is an odd perfect number. The finiteness theorem like the Artjuhov–Borho theorem was proved for odd perfect numbers by Dickson [7] in 1913. However, it took more than 60 years to get the explicit upper bound. The first explicit upper bound for odd perfect numbers was achieved by Pomerance [16] in 1977. Pomerance obtained an upper bound

$$N < (4K)^{(4K)^{2^{K^2}}}$$

for an odd perfect number with $K = \omega(N)$. Note that by modifying the method of Pomerance slightly, we may improve this upper bound to

$$N < (4K)^{2^{K^2}}$$

as remarked in Lemma 2 and Remark of [15]. Further improvement on this bound was given by Heath-Brown [10] by using a new method. Heath-Brown's upper bound is

$$N < 4^{4^K}.$$

Heath-Brown's method has been further developed by several authors. The list of the world records of upper bounds based on Heath-Brown's method is here:

$$\begin{aligned} &C^{4^K} && \text{(Cook [5])}, && \bigl(C = (195)^{1/7} = 2.123\ldots\bigr),\\ &2^{4^K} && \text{(Nielsen [12])}, &&\\ &2^{4^K - 2^K} && \text{(Chen and Tang [4])}, &&\\ &2^{(2^K-1)^2} && \text{(Nielsen [13])}, && \end{aligned}$$

where the last result of Nielsen is the current best result.

The method of Pollack [15] was mainly based on the method of Pomerance. In the same paper, Pollack [15, page 38] suggested that it would be interesting to find whether the method of Heath-Brown is available in the context of amicable pairs. Indeed, the method of this paper mainly follows a version of Heath-Brown's method given by Nielsen [13] and Theorem 2 corresponds to Nielsen's latest bound on odd perfect numbers. Thus, this paper gives one possible answer to Pollack's suggestion.

Also, Pollack [14, page 680] asked to find a suitable condition to obtain a finiteness theorem for amicable tuples or sociable numbers. Our condition (4) in Theorem 3 can be, though it seems too strong, one of partial answers to his question. However, the current author have no idea for the same problem with sociable numbers.

It would be interesting that there are many examples of harmonious pairs which are relatively prime but not anarchic. In order to list up such pairs, we used a `C` program, which is based on a program provided by Yuki Yoshida [17]. By using this program, we can list up all 2566 relatively prime harmonious pairs among all 49929 harmonious pairs (M, N) up to 10^8 in the sense $M \leq N \leq 10^8$, and we find none of these pairs are anarchic. For interested readers, we list up all 30 relatively prime harmonious pairs up to 10^5 in Table 1 and the number of harmonious and relatively prime harmonious pairs in several ranges in Table 2. This numerical search shows that Theorem 1 captures more pairs in its scope than Gmelin's conjecture. Therefore, it is natural to ask: are there anarchic harmonious tuples? Surprisingly, by continuing the numerical

Table 1. All coprime harmonious pairs (M, N) with $M \leq N \leq 10^5$

M	N	Factorization of M	and N	$(M, \sigma(N))$	$(\sigma(M), N)$
135	3472	$3^3 \times 5$	$2^4 \times 7 \times 31$	1	16
135	56896	$3^3 \times 5$	$2^6 \times 7 \times 127$	1	16
285	45136	$3 \times 5 \times 19$	$2^4 \times 7 \times 13 \times 31$	1	16
315	51088	$3^2 \times 5 \times 7$	$2^4 \times 31 \times 103$	1	16
345	38192	$3 \times 5 \times 23$	$2^4 \times 7 \times 11 \times 31$	3	16
868	1485	$2^2 \times 7 \times 31$	$3^3 \times 5 \times 11$	4	1
1204	4455	$2^2 \times 7 \times 43$	$3^4 \times 5 \times 11$	4	11
1683	3500	$3^2 \times 11 \times 17$	$2^2 \times 5^3 \times 7$	3	4
1683	62000	$3^2 \times 11 \times 17$	$2^4 \times 5^3 \times 31$	3	8
2324	9945	$2^2 \times 7 \times 83$	$3^2 \times 5 \times 13 \times 17$	28	3
3556	63855	$2^2 \times 7 \times 127$	$3^3 \times 5 \times 11 \times 43$	4	1
4455	21328	$3^4 \times 5 \times 11$	$2^4 \times 31 \times 43$	11	8
4845	7084	$3 \times 5 \times 17 \times 19$	$2^2 \times 7 \times 11 \times 23$	3	4
5049	65968	$3^3 \times 11 \times 17$	$2^4 \times 7 \times 19 \times 31$	1	16
6244	43875	$2^2 \times 7 \times 223$	$3^3 \times 5^3 \times 13$	28	1
6244	90675	$2^2 \times 7 \times 223$	$3^2 \times 5^2 \times 13 \times 31$	28	1
6675	33488	$3 \times 5^2 \times 89$	$2^4 \times 7 \times 13 \times 23$	3	8
7155	13244	$3^3 \times 5 \times 53$	$2^2 \times 7 \times 11 \times 43$	3	4
9945	41168	$3^2 \times 5 \times 13 \times 17$	$2^4 \times 31 \times 83$	3	8
12124	84825	$2^2 \times 7 \times 433$	$3^2 \times 5^2 \times 13 \times 29$	28	1
13275	81424	$3^2 \times 5^2 \times 59$	$2^4 \times 7 \times 727$	1	4
13965	23312	$3 \times 5 \times 7^2 \times 19$	$2^4 \times 31 \times 47$	3	16
24327	75460	$3^3 \times 17 \times 53$	$2^2 \times 5 \times 7^3 \times 11$	9	20
31724	61335	$2^2 \times 7 \times 11 \times 103$	$3^2 \times 5 \times 29 \times 47$	4	3
32835	92456	$3 \times 5 \times 11 \times 199$	$2^3 \times 7 \times 13 \times 127$	15	8
34485	37492	$3 \times 5 \times 11^2 \times 19$	$2^2 \times 7 \times 13 \times 103$	1	28
52700	68211	$2^2 \times 5^2 \times 17 \times 31$	$3^2 \times 11 \times 13 \times 53$	4	9
55341	58900	$3^2 \times 11 \times 13 \times 43$	$2^2 \times 5^2 \times 19 \times 31$	1	4
60515	78864	$5 \times 7^2 \times 13 \times 19$	$2^4 \times 3 \times 31 \times 53$	1	48
62992	63855	$2^4 \times 31 \times 127$	$3^3 \times 5 \times 11 \times 43$	16	1

Table 2. Number of harmonious pairs (M, N) with $M \leq N \leq 10^k$

	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
Harmonious	1	10	55	252	983	3666	13602	49929	176453
Coprime harmonious	0	0	0	6	30	133	631	2566	10013

search, we find an *anarchy in the harmony*, i.e., an anarchic harmonious pair

$$(M,N) = (64, 173369889),$$

which is the only anarchic harmonious pair with $M \le N \le 10^9$. Note that

$$64 = 2^6, \qquad 173369889 = 3^4 \times 7^2 \times 11^2 \times 19^2,$$
$$\sigma(64) = 127, \qquad \sigma(173369889) = 3^2 \times 7 \times 11^2 \times 19^2 \times 127.$$

This observation may indicate that our proof of Theorem 2 still overlooks some essential obstruction for the existence of relatively prime amicable pairs.

2. Notation

We denote the greatest common divisor of positive integers a and b by (a,b), which we may distinguish from the notation for a pair of integers (M,N) by the context. For positive integers d and n, we write $d \parallel n$ if $d \mid n$ and $(d, n/d) = 1$.

For any finite set $\mathcal{S}$ of integers, we let

$$\Pi(\mathcal{S}) = \prod_{m\in\mathcal{S}} m, \qquad \Phi(\mathcal{S}) = \prod_{m\in\mathcal{S}} (m-1), \qquad \Psi(\mathcal{S}) = \Pi(\mathcal{S})\Phi(\mathcal{S}).$$

Following the notation of Nielsen [13], we let

$$F_r(x) = x^{2^r} - x^{2^{r-1}}$$

for an integer $r \ge 1$ and a real number $x \ge 1$ and we let $F_0(x) = x - 1$ for the case $r = 0$ and $x \ge 1$. Actually, we will not use the full power of this function $F_r(x)$ for the proof of Theorem 1, but we introduce them for trying to give a better bound in lemmas on Diophantine inequalities. We prepare three lemmas on $F_r(x)$.

LEMMA 1. *For any integer $r \ge 0$, $F_r(x)$ is increasing as a function of $x \ge 1$.*

Proof. This is obvious for $r = 0$ and also obvious for $r \ge 1$ by

$$F_r(x) = x^{2^{r-1}}\left(x^{2^{r-1}} - 1\right) \quad (r \ge 1) \tag{6}$$

since $x^{2^{r-1}}$ and $(x^{2^{r-1}} - 1)$ are increasing and non-negative for $x \ge 1$. $\square$

LEMMA 2. *For any integer $r \ge 1$ and real numbers $\alpha, x \ge 1$, we have*

$$F_r(x) \le \alpha^{-2^{r-1}} F_r(\alpha x) \le \alpha^{-1} F_r(\alpha x).$$

Proof. By (6), we have

$$\alpha^{-2^{r-1}} F_r(\alpha x) = x^{2^{r-1}}\left((\alpha x)^{2^{r-1}} - 1\right) \ge x^{2^{r-1}}\left(x^{2^{r-1}} - 1\right) = F_r(x).$$

This completes the proof. $\square$

LEMMA 3. *For any integers $r, s \ge 0$ and a real number $x \ge 2$, we have*

$$F_r\big(F_s(x)\big) \le F_{r+s}(x).$$

Proof. We first note that the assumption $x \geq 2$ assures $F_s(x) \geq 1$, which is obvious for $s = 0$ and follows from (6) for $s \geq 1$. For the case $r = 0$, we have

$$F_0\big(F_s(x)\big) = F_s(x) - 1 \leq F_s(x)$$

so the lemma holds. For the case $r \geq 1$, by Lemma 1,

$$F_r\big(F_s(x)\big) \leq F_r\big(x^{2^s}\big) = \big(x^{2^s}\big)^{2^r} - \big(x^{2^s}\big)^{2^{r-1}} = x^{2^{r+s}} - x^{2^{r+s-1}} = F_{r+s}(x).$$

This completes the proof. $\square$

3. Lemmas on Diophantine inequalities

In this section, we prove variants of Heath-Brown's lemma [10, Lemma 1] on Diophantine inequalities related to the equation (3). We need to introduce some modification suitable for applications to amicable tuples. Our proof of Theorem 1 heavily relies on the equation (3), or its generalization

$$\frac{b_1}{a_1}\frac{M_1}{\sigma(M_1)} + \cdots + \frac{b_k}{a_k}\frac{M_k}{\sigma(M_k)} = 1, \tag{7}$$

where $a_i, b_i \geq 1$ are integers. This equation is not so flexible as the equation

$$\frac{\sigma(M)}{M} = \frac{a}{b}, \tag{8}$$

which is used in the context of perfect numbers.

Actually, in the induction steps of Heath-Brown's method, there are two point to use such Diophantine equation or corresponding inequalities. For the first point, we use Diophantine inequality with its original form. On the other hand, in the second point, we need to take the "reciprocal" of the same Diophantine inequality. For odd perfect numbers, we can take the reciprocal of (8) without any big change of its shape. However, for amicable pairs, we need to take the reciprocal of each terms in (7), which transforms the equation into slightly different shape. Thus, we prepare two different lemmas.

We start with Lemma 2 of Cook [5] in its refined form. This refinement was indicated by Goto [9, Lemma 2.4]. Nielsen [13, Lemma 1.2] also proved this refinement in a stronger form, which allows us to have some equalities between x_i. We also need a variant of Cook's lemma given by Goto [9, Lemma 2.5]. For completeness, we give a proof of these lemmas following the argument of Nielsen [13].

LEMMA 4. *For real numbers $0 < x_1 \leq x_2$ and $0 < \alpha < 1$, we have*

$$\left(1 - \frac{1}{x_1}\right)\left(1 - \frac{1}{x_2}\right) > \left(1 - \frac{1}{x_1\alpha}\right)\left(1 - \frac{1}{x_2\alpha^{-1}}\right)$$

and

$$\left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) < \left(1 + \frac{1}{x_1\alpha}\right)\left(1 + \frac{1}{x_2\alpha^{-1}}\right).$$

Proof. By expanding both sides of the inequalities, we find that it suffices to prove the inequality

$$\frac{1}{x_1} + \frac{1}{x_2} < \frac{1}{x_1\alpha} + \frac{1}{x_2\alpha^{-1}}.$$

This is equivalent to

$$\left(\frac{x_2}{x_1} - \alpha\right)(1-\alpha) > 0.$$

Since $\alpha < 1 \leq x_2/x_1$, the last inequality holds. This completes the proof. $\square$

LEMMA 5. *Let*

$$1 < x_1 \leq x_2 \leq \cdots \leq x_k, \qquad 1 < y_1 \leq y_2 \leq \cdots \leq y_k \tag{9}$$

be sequences of real numbers satisfying

$$\prod_{i=1}^{s} x_i \leq \prod_{i=1}^{s} y_i \tag{10}$$

for every s with $1 \leq s \leq k$. Then we have

$$\prod_{i=1}^{k}\left(1 - \frac{1}{x_i}\right) \leq \prod_{i=1}^{k}\left(1 - \frac{1}{y_i}\right), \qquad \prod_{i=1}^{k}\left(1 + \frac{1}{x_i}\right) \geq \prod_{i=1}^{k}\left(1 + \frac{1}{y_i}\right),$$

where each of two equalities holds if and only if $x_i = y_i$ for every $i \geq 1$.

Proof. We first fix the tuple $\mathbf{x} = (x_i)$. Let us identify the tuple $\mathbf{y} = (y_i)$ with a point in the Euclidean space $\mathbb{R}^k$ and let

$$\mathcal{R} = \left\{ \mathbf{y} \in \mathbb{R}^k \;\middle|\; y_1 \leq \cdots \leq y_k \text{ and } \prod_{i=1}^{s} x_i \leq \prod_{i=1}^{s} y_i \text{ for all } 1 \leq s \leq k \right\},$$

$$G(\mathbf{y}) = \prod_{i=1}^{k}\left(1 - \frac{1}{y_i}\right), \qquad H(\mathbf{y}) = \prod_{i=1}^{k}\left(1 + \frac{1}{y_i}\right).$$

Note that the condition $y_1 > 1$ in (9) is assured by the case $s = 1$ of (10) since $y_1 \geq x_1 > 1$. Thus what we have to prove is that the minimum value of $G(\mathbf{y})$ and the maximum value of $H(\mathbf{y})$ for $\mathbf{y} \in \mathcal{R}$ is taken only at $\mathbf{y} = \mathbf{x}$.

Note that $G(\mathbf{y})$ is increasing in every variable y_i and $H(\mathbf{y})$ is decreasing in every variable y_i. and that if $\mathbf{y} \in \mathcal{R}$, then

$$\left(\min\left(y_1, \prod_{i=1}^{k} x_i\right), \ldots, \min\left(y_k, \prod_{i=1}^{k} x_i\right)\right) \in \mathcal{R}.$$

Thus, the minimum value of $G(\mathbf{y})$ and the maximum value of $H(\mathbf{y})$ for $\mathbf{y} \in \mathcal{R}$ exists and is taken in the closed set $\mathcal{R} \cap [1, \prod_{i=1}^{k} x_i]^k$.

Take $\mathbf{y} \in \mathcal{R}$ with $\mathbf{y} \neq \mathbf{x}$ arbitrarily. Since we proved the existence of the minimum and maximum values of $G(\mathbf{y})$ and $H(\mathbf{y})$, it suffices to prove that we can modify $\mathbf{y}$ to $\tilde{\mathbf{y}} \in \mathcal{R}$ such that $G(\mathbf{y}) > G(\tilde{\mathbf{y}})$ and $H(\mathbf{y}) < H(\tilde{\mathbf{y}})$. Take the smallest index t with $x_t \neq y_t$ and $1 \leq t \leq k$. Then we have

$$x_i = y_i \quad \text{for all } 1 \leq i < t \tag{11}$$

so that

$$\prod_{i=1}^{t-1} x_i = \prod_{i=1}^{t-1} y_i. \tag{12}$$

By (10), (12) and $x_t \neq y_t$, we have

$$x_t < y_t, \quad \text{so} \quad \prod_{i=1}^{t} x_i < \prod_{i=1}^{t} y_i. \tag{13}$$

If t is the last index, that is, $t = k$, then by (11) and (13) we have

$$G(\mathbf{y}) = \prod_{i=1}^{k-1} \left(1 - \frac{1}{x_i}\right)\left(1 - \frac{1}{y_k}\right) > \prod_{i=1}^{k-1} \left(1 - \frac{1}{x_i}\right)\left(1 - \frac{1}{x_k}\right) = G(\mathbf{x}),$$
$$H(\mathbf{y}) = \prod_{i=1}^{k-1} \left(1 + \frac{1}{x_i}\right)\left(1 + \frac{1}{y_k}\right) < \prod_{i=1}^{k-1} \left(1 + \frac{1}{x_i}\right)\left(1 + \frac{1}{x_k}\right) = H(\mathbf{x}).$$

Thus, $\tilde{\mathbf{y}} = \mathbf{x}$ satisfies the conditions. Hence, we may assume $1 \leq t \leq k-1$.

We next take the largest index v with $y_v = y_{t+1}$ and $t < v \leq k$. Then

$$y_{t+1} = y_{t+2} = \cdots = y_v < y_{v+1}, \tag{14}$$

where we use a convention $y_{k+1} = 2y_k$. Now we prove

$$\prod_{i=1}^{s} x_i < \prod_{i=1}^{s} y_i \quad \text{for all } t \leq s < v. \tag{15}$$

Assume to the contrary that there is s with $t \leq s < v$ and

$$\prod_{i=1}^{s} x_i \geq \prod_{i=1}^{s} y_i.$$

By (10), we find that

$$\prod_{i=1}^{s} x_i = \prod_{i=1}^{s} y_i. \tag{16}$$

Thus, by the second inequality in (13),

$$\prod_{i=t+1}^{s} x_i = \left(\prod_{i=1}^{t} x_i\right)^{-1} \prod_{i=1}^{s} x_i > \left(\prod_{i=1}^{t} y_i\right)^{-1} \prod_{i=1}^{s} y_i = \prod_{i=t+1}^{s} y_i.$$

Combined with (9) and (14), this gives

$$x_{s+1} \geq x_s \geq \left(\prod_{i=t+1}^{s} x_i\right)^{1/(s-t)} > \left(\prod_{i=t+1}^{s} y_i\right)^{1/(s-t)} = y_s = y_{s+1}$$

since $s+1 \leq v$. By multiplying both sides by (16), we obtain

$$\prod_{i=1}^{s+1} x_i > \prod_{i=1}^{s+1} y_i,$$

which contradicts to the assumption (10). Thus, we obtain (15).

By $x_t < y_t$, $y_v < y_{v+1}$ and (15), we find that

$$0 < \alpha := \max\left(x_t/y_t, y_v/y_{v+1}, \prod_{i=1}^{t} x_i/y_i, \ldots, \prod_{i=1}^{v-1} x_i/y_i\right) < 1. \tag{17}$$

We now define $\tilde{\mathbf{y}} = (\tilde{y}_i) \in \mathbb{R}^k$ by

$$\tilde{y}_i = \begin{cases} y_i & (\text{for } i \neq t, v), \\ y_i\alpha & (\text{for } i = t), \\ y_i\alpha^{-1} & (\text{for } i = v), \end{cases} \tag{18}$$

and check that this $\tilde{\mathbf{y}}$ satisfies the desired conditions.

First, we check

$$\tilde{y}_1 \leq \cdots \leq \tilde{y}_k, \quad \text{i.e.,} \quad \tilde{y}_i \leq \tilde{y}_{i+1} \quad \text{for all } 1 \leq i < k. \tag{19}$$

This is obvious for $i \notin \{t-1, t, v-1, v\}$ since in this case, $\tilde{y}_i$ and $\tilde{y}_{i+1}$ coincide with y_i and y_{i+1} respectively. For the case $i \in \{t-1, t, v-1, v\}$ and $1 \leq i < k$, we use (17) to check

$$\begin{gathered} \tilde{y}_{t-1} = y_{t-1} = x_{t-1} \leq x_t = y_t \cdot (x_t/y_t) \leq y_t\alpha = \tilde{y}_t, \\ \tilde{y}_t = y_t\alpha \leq y_t \leq y_{t+1} \leq \tilde{y}_{t+1}, \qquad \tilde{y}_{v-1} \leq y_{v-1} \leq y_v \leq y_v\alpha^{-1} = \tilde{y}_v, \\ \tilde{y}_v = y_v\alpha^{-1} \leq y_v \cdot (y_v/y_{v+1})^{-1} = y_{v+1} = \tilde{y}_{v+1}. \end{gathered}$$

Thus the condition (19) holds.

Second, we check

$$\prod_{i=1}^{s} x_i \leq \prod_{i=1}^{s} \tilde{y}_i \quad \text{for all } 1 \leq s \leq k. \tag{20}$$

This is obvious for $1 \leq s < t$ and $v \leq s \leq k$ since in these cases, we have

$$\prod_{i=1}^{s} \tilde{y}_i = \prod_{i=1}^{s} y_i$$

by our choice (18). For $t \le s < v$, we see that

$$\prod_{i=1}^{s} x_i = \left(\prod_{i=1}^{s} x_i/y_i\right)\left(\prod_{i=1}^{s} y_i\right) \le \alpha \prod_{i=1}^{s} y_i = \prod_{i=1}^{s} \tilde{y}_i$$

by (17). Thus the condition (20) also holds, that is, $\tilde{\mathbf{y}} \in \mathcal{R}$.

Finally, we check that $G(\mathbf{y}) > G(\tilde{\mathbf{y}})$ and $H(\mathbf{y}) < H(\tilde{\mathbf{y}})$. Since $0 < \alpha < 1$ and $y_t \le y_v$, by recalling (18), we can apply Lemma 4 to obtain

$$\begin{aligned} G(\mathbf{y}) &= \prod_{\substack{i=1\\ i\ne t,v}}^{k} \left(1 - \frac{1}{\tilde{y}_i}\right)\left(1 - \frac{1}{y_t}\right)\left(1 - \frac{1}{y_v}\right) \\ &> \prod_{\substack{i=1\\ i\ne t,v}}^{k} \left(1 - \frac{1}{\tilde{y}_i}\right)\left(1 - \frac{1}{y_t\alpha}\right)\left(1 - \frac{1}{y_v\alpha^{-1}}\right) = G(\tilde{\mathbf{y}}), \\ H(\mathbf{y}) &= \prod_{\substack{i=1\\ i\ne t,v}}^{k} \left(1 + \frac{1}{\tilde{y}_i}\right)\left(1 + \frac{1}{y_t}\right)\left(1 + \frac{1}{y_v}\right) \\ &< \prod_{\substack{i=1\\ i\ne t,v}}^{k} \left(1 + \frac{1}{\tilde{y}_i}\right)\left(1 + \frac{1}{y_t\alpha}\right)\left(1 + \frac{1}{y_v\alpha^{-1}}\right) = H(\tilde{\mathbf{y}}). \end{aligned}$$

Thus, our $\tilde{\mathbf{y}}$ satisfies the desired conditions. This completes the proof. □

We next prove the first lemma on Diophantine inequality. The Diophantine inequality in the next lemma seems to be a natural generalization of the Diophantine inequality (2) of [10] to the linear form with several summands.

LEMMA 6. *Let k and R be positive integers. Consider a sequence of integers*

$$\mathcal{M} = (m_j)_{j=1}^{R}, \quad 1 < m_1 \le m_2 \le \cdots \le m_R,$$

a decomposition of the index set

$$J = \{1, \ldots, R\}, \qquad J = \bigcup_{i=1}^{k} J_i, \qquad J_1, \ldots, J_k\colon \textit{disjoint}$$

and a tuple of integers

$$a_1, \ldots, a_k, b_1, \ldots, b_k \ge 1$$

satisfying $a_i \ge b_i$ for all i. If a pair of inequalities

$$\sum_{i=1}^{k} \frac{b_i}{a_i} \prod_{\substack{j=1\\ j\in J_i}}^{R} \left(1 - \frac{1}{m_j}\right) \le 1, \tag{21}$$

$$\sum_{i=1}^{k} \frac{b_i}{a_i} \prod_{\substack{j=1 \\ j \in J_i}}^{R-1} \left(1 - \frac{1}{m_j}\right) > 1 \tag{22}$$

holds, then we have

$$a \prod_{j=1}^{R} m_j \leq F_R(a+1), \tag{23}$$

where $a = a_1 \cdots a_k$.

Proof. We first give a preliminary remark. By (22), we always have

$$\sum_{i=1}^{k} \frac{b_i}{a_i} > 1 \tag{24}$$

since each of $\Pi(1 - 1/m)$ is ≤ 1 even if it is an empty product. Since the left-hand side of (24) is a rational fraction with denominator a,

$$\sum_{i=1}^{k} \frac{b_i}{a_i} \geq \frac{a+1}{a}. \tag{25}$$

We use this inequality several times below.

We use induction on R. If $R = 1$, by symmetry we may assume without loss of generality that $J_1 = \cdots = J_{k-1} = \emptyset$ and $J_k = \{1\}$. Then (21) implies

$$\sum_{i=1}^{k} \frac{b_i}{a_i} - \frac{b_k}{a_k} \cdot \frac{1}{m_1} \leq 1.$$

By (25), we find that

$$\frac{b_k}{a_k} \cdot \frac{1}{m_1} \geq \sum_{i=1}^{k} \frac{b_i}{a_i} - 1 \geq \frac{1}{a}$$

so that

$$am_1 \leq a^2 \leq a(a+1) = F_1(a+1)$$

since $a_k \geq b_k$. This completes the proof of the case $R = 1$.

We next assume that the assertion holds for any sequence $\mathcal{M}$ of length $\leq R - 1$ and prove the assertion for the case in which $\mathcal{M}$ has the length R. We use a special sequence

$$1 < x_1 < \cdots < x_R,$$

which is defined by

$$x_j = \begin{cases} (a+1)^{2^{j-1}} + 1 & (\text{for } 1 \leq j < R), \\ (a+1)^{2^{R-1}} & (\text{for } j = R). \end{cases}$$

We first consider the case

$$\prod_{j=1}^{r} x_j > \prod_{j=1}^{r} m_j,$$

for some $1 \le r < R$. Then we have

$$am_1 \cdots m_r < ax_1 \cdots x_r = (a+1)^{2^r} - 1. \tag{26}$$

By using notations

$$a_i' = a_i \prod_{\substack{j=1 \\ j \in J_i}}^{r} m_j, \qquad b_i' = b_i \prod_{\substack{j=1 \\ j \in J_i}}^{r} (m_j - 1), \qquad a' = a_1' \cdots a_k',$$

we can rewrite (21) and (22) as

$$\sum_{i=1}^{k} \frac{b_i'}{a_i'} \prod_{\substack{j=r+1 \\ j \in J_i}}^{R} \left(1 - \frac{1}{m_j}\right) \le 1, \qquad \sum_{i=1}^{k} \frac{b_i'}{a_i'} \prod_{\substack{j=r+1 \\ j \in J_i}}^{R-1} \left(1 - \frac{1}{m_j}\right) > 1.$$

Note that the condition $r < R$ is necessary for rewriting the inequality (22) as above. By the induction hypothesis, we obtain

$$a \prod_{j=1}^{R} m_j = a' \prod_{j=r+1}^{R} m_j \le F_{R-r}\big(a'+1\big) = F_{R-r}(am_1 \cdots m_r + 1).$$

By (26) and Lemma 1, this implies

$$a \prod_{j=1}^{R} m_j \le F_{R-r}\big((a+1)^{2^r}\big) = F_R(a+1)$$

so the assertion follows. Thus, we may assume

$$\prod_{j=1}^{r} x_j \le \prod_{j=1}^{r} m_j \tag{27}$$

for all $1 \le r < R$. We may also assume (27) for the case $r = R$ since otherwise

$$a \prod_{j=1}^{R} m_j \le a \prod_{j=1}^{R} x_j = F_R(a+1) \tag{28}$$

and the assertion follows. Thus, for the remaining case, we have (27) for every $1 \le r \le R$. Then we can apply Lemma 5 to obtain

$$\prod_{j=1}^{R} \left(1 - \frac{1}{m_j}\right) \ge \prod_{j=1}^{R} \left(1 - \frac{1}{x_j}\right) = \frac{a}{a+1}. \tag{29}$$

Then by (21) and (25), we have

$$1 \geq \sum_{i=1}^{k} \frac{b_i}{a_i} \prod_{\substack{j=1 \\ j \in J_i}}^{R} \left(1 - \frac{1}{m_j}\right) \geq \prod_{j=1}^{R} \left(1 - \frac{1}{m_j}\right) \sum_{i=1}^{k} \frac{b_i}{a_i} \geq \frac{a}{a+1} \sum_{i=1}^{k} \frac{b_i}{a_i} \geq 1.$$

Thus we must have the equality in (29). By Lemma 5, we find that

$$m_1 = x_1, \ldots, m_R = x_R.$$

By using (28), we have the assertion again. This completes the proof. $\square$

We next prove the second lemma on Diophantine inequality.

LEMMA 7. *Let k and R be positive integers. Consider a sequence of integers*

$$\mathcal{M} = (m_j)_{j=1}^{R}, \quad 1 < m_1 \leq m_2 \leq \cdots \leq m_R,$$

a decomposition of the index set

$$J = \{1, \ldots, R\}, \qquad J = \bigcup_{i=1}^{k} J_i, \qquad J_1, \ldots, J_k\colon \textit{disjoint}$$

and a tuple of integers

$$a_1, \ldots, a_k, b_1, \ldots, b_k \geq 1.$$

If a pair of inequalities

$$\sum_{i=1}^{k} \frac{b_i}{a_i} \prod_{\substack{j=1 \\ j \in J_i}}^{R} \left(1 - \frac{1}{m_j}\right)^{-1} \geq 1, \tag{30}$$

$$\sum_{i=1}^{k} \frac{b_i}{a_i} \prod_{\substack{j=1 \\ j \in J_i}}^{R-1} \left(1 - \frac{1}{m_j}\right)^{-1} < 1 \tag{31}$$

holds, then we have

$$a \prod_{j=1}^{R} (m_j - 1) \leq F_R(a), \tag{32}$$

where $a = a_1 \cdots a_k$.

REMARK 1. We assumed $a_i \geq b_i$ in Lemma 6, but we do not assume this condition in Lemma 7 above. Actually, the assumption (31) implies $a_i > b_i$.

Proof. By (31), we always have

$$\frac{1}{a} \leq \sum_{i=1}^{k} \frac{b_i}{a_i} \leq 1 - \frac{1}{a} \tag{33}$$

as in the proof of (25). Again this is a key in the argument below.

We use induction on R. If $R=1$, by symmetry we may assume without loss of generality that $J_1=\cdots=J_{k-1}=\emptyset$ and $J_k=\{1\}$. Then (30) implies

$$1\le\sum_{i=1}^{k-1}\frac{b_i}{a_i}+\frac{b_k}{a_k}\left(1-\frac{1}{m_1}\right)^{-1}=\sum_{i=1}^{k}\frac{b_i}{a_i}+\frac{b_k}{a_k}\cdot\frac{1}{m_1-1}.$$

By (33), we find that

$$\frac{b_k}{a_k}\cdot\frac{1}{m_1-1}\ge 1-\sum_{i=1}^{k}\frac{b_i}{a_i}\ge\frac{1}{a}$$

so that

$$a(m_1-1)\le a(a-1)=F_1(a)$$

since $a_k>b_k$. This completes the proof of the case $R=1$.

We next assume that the assertion holds for any sequence $\mathcal{M}$ of length $\le R-1$ and prove the assertion for the case in which $\mathcal{M}$ has the length R. We use a special sequence

$$1<x_1<\cdots<x_R,$$

which is defined by

$$x_j=\begin{cases}a^{2^{j-1}}+1 & (\text{for } 1\le j<R),\\ a^{2^{R-1}} & (\text{for } j=R).\end{cases}$$

We first consider the case

$$\prod_{j=1}^{r}(x_j-1)>\prod_{j=1}^{r}(m_j-1),$$

for some $1\le r<R$. Then we have

$$a(m_1-1)\cdots(m_r-1)<a(x_1-1)\cdots(x_r-1)=a^{2^r}. \tag{34}$$

By using notations

$$a_i''=a_i\prod_{\substack{j=1\\ j\in J_i}}^{r}(m_j-1),\qquad b_i''=b_i\prod_{\substack{j=1\\ j\in J_i}}^{r}m_j,\qquad a''=a_1''\cdots a_k'',$$

we can rewrite (30) and (31) as

$$\sum_{i=1}^{k}\frac{b_i''}{a_i''}\prod_{\substack{j=r+1\\ j\in J_i}}^{R}\left(1-\frac{1}{m_j}\right)^{-1}\ge 1,\qquad \sum_{i=1}^{k}\frac{b_i''}{a_i''}\prod_{\substack{j=r+1\\ j\in J_i}}^{R-1}\left(1-\frac{1}{m_j}\right)^{-1}<1.$$

By the induction hypothesis and (34), we obtain

$$a\prod_{j=1}^{R}(m_j-1)=a''\prod_{j=r+1}^{R}(m_j-1)\le F_{R-r}\big(a''\big)\le F_R(a)$$

so the assertion follows. Thus we may assume

$$\prod_{j=1}^{r}(x_j-1)\le\prod_{j=1}^{r}(m_j-1) \tag{35}$$

for all $1\le r<R$. We may also assume (35) for the case $r=R$ since otherwise

$$a\prod_{j=1}^{R}(m_j-1)\le a\prod_{j=1}^{R}(x_j-1)=F_R(a) \tag{36}$$

and the assertion follows. Thus, for the remaining case, we have (35) for every $1\le r\le R$. Note that by (33),

$$1<a=(x_1-1).$$

Thus we can apply Lemma 5 to obtain

$$\prod_{j=1}^{R}\left(1-\frac{1}{m_j}\right)^{-1}=\prod_{j=1}^{R}\left(1+\frac{1}{m_j-1}\right)\le\prod_{j=1}^{R}\left(1+\frac{1}{x_j-1}\right)=\frac{a}{a-1}. \tag{37}$$

Then by (30) and (33), we have

$$1\le\sum_{i=1}^{k}\frac{b_i}{a_i}\prod_{\substack{j=1\\ j\in J_i}}^{R}\left(1-\frac{1}{m_j}\right)^{-1}\le\prod_{j=1}^{R}\left(1-\frac{1}{m_j}\right)^{-1}\sum_{i=1}^{k}\frac{b_i}{a_i}\le\frac{a}{a-1}\sum_{i=1}^{k}\frac{b_i}{a_i}\le 1.$$

Thus we must have the equality in (37). By Lemma 5, we find that

$$m_1=x_1,\dots,m_R=x_R.$$

By using (36), we have the assertion again. This completes the proof. □

4. Upper bounds à la Borho

In this section, we prove Theorem 4, which gives upper bounds of the Borho-type for harmonious tuples and unitary harmonious tuples.

Proof of Theorem 4. We first consider a harmonious tuple $(M_i)_{i=1}^{k}$. Note that

$$\frac{M}{\sigma(M)}=\prod_{p^e\|M}\frac{p^e}{1+\dots+p^e}=\prod_{p^e\|M}\prod_{f=1}^{e}\left(1-\frac{1}{1+\dots+p^f}\right)$$

as it is mentioned in the proof of Satz 3 of [3]. Thus, (3) is rewritten as

$$\sum_{i=1}^{k}\prod_{p^e\|M_i}\prod_{f=1}^{e}\left(1-\frac{1}{1+\dots+p^f}\right)=1.$$

If we remove any factor from any summand, then the left-hand side becomes larger. Thus, we can apply Lemma 6 and obtain

$$\sigma(M_1)\cdots\sigma(M_k) \le \prod_{i=1}^{k} \prod_{p^e \| M_i} \prod_{f=1}^{e} \big(1+\cdots+p^f\big) \le F_L(2) = 2^{2^L} - 2^{2^{L-1}}, \tag{38}$$

where L is given by the number of factors in the product above, so

$$L = \sum_{i=1}^{k} \sum_{p^e \| M_i} e = \sum_{i=1}^{k} \Omega(M_i) = \Omega(M_1\cdots M_k).$$

Note that M_i can share some common factor since we do not assume anything on the divisibility. However, this does not affect the above arguments. By using the inequality of the arithmetic and geometric mean in (3), we find that

$$1 = \frac{M_1}{\sigma(M_1)} + \cdots + \frac{M_k}{\sigma(M_k)} \ge k\left(\frac{M_1\cdots M_k}{\sigma(M_1)\cdots\sigma(M_k)}\right)^{1/k}$$

so

$$\sigma(M_1)\cdots\sigma(M_k) = \left(\frac{\sigma(M_1)\cdots\sigma(M_k)}{M_1\cdots M_k}\right) M_1\cdots M_k \ge k^k M_1\cdots M_k.$$

On inserting this into (38), we arrive at the assertion for harmonious tuples.

We next consider a unitary harmonious tuple $(M_i)_{i=1}^k$. By using

$$\frac{M}{\sigma^*(M)} = \prod_{p^e\|M} \frac{p^e}{1+p^e} = \prod_{p^e\|M}\left(1 - \frac{1}{1+p^e}\right),$$

we can rewrite (5) as

$$\sum_{i=1}^{k} \prod_{p^e\|M_i}\left(1-\frac{1}{1+p^e}\right) = 1.$$

Applying Lemma 6 as above, we see that

$$\sigma^*(M_1)\cdots\sigma^*(M_k) = \prod_{i=1}^{k}\prod_{p^e\|M_i}\big(1+p^e\big) \le F_L(2) = 2^{2^L} - 2^{2^{L-1}}, \tag{39}$$

where L is given by

$$L = \sum_{i=1}^{k}\sum_{p^e\|M_i} 1 = \omega(M_1) + \cdots + \omega(M_k).$$

By using the inequality of the arithmetic and geometric mean in (5), we find

$$\sigma^*(M_1)\cdots\sigma^*(M_k) = \left(\frac{\sigma^*(M_1)\cdots\sigma^*(M_k)}{M_1\cdots M_k}\right) M_1\cdots M_k \ge k^k M_1\cdots M_k.$$

Combining with (39), this gives the assertion for unitary harmonious tuples. $\square$

5. The induction lemma

In this section, we prove an induction lemma. We start with a lemma on the divisibility, whose special case is also used in the proof of Lemma 4 of [15].

LEMMA 8. *Let $k \geq 2$ be an integer, $(M_i)_{i=1}^k$ be an anarchic harmonious tuple and suppose that a tuple of decompositions*

$$M_i = U_i V_i, \qquad (U_i, V_i) = 1, \qquad U := U_1 \cdots U_k > 1$$

is given. Then we have

$$\sum_{i=1}^k \frac{V_i}{\sigma(V_i)} \prod_{\substack{p \mid U_i \\ p \in \mathcal{S}}} \left(1 - \frac{1}{p}\right) \neq 1$$

for any set $\mathcal{S}$ of prime factors of U.

Proof. Assume to the contrary that

$$\sum_{i=1}^k \frac{V_i}{\sigma(V_i)} \prod_{\substack{p \mid U_i \\ p \in \mathcal{S}}} \left(1 - \frac{1}{p}\right) = 1 \tag{40}$$

for some set $\mathcal{S}$ of prime factors of U. We first claim that $\mathcal{S}$ is non-empty. Since $U = U_1 \cdots U_k > 1$, we have $U_i/\sigma(U_i) < 1$ for some i. Thus, by (3),

$$1 = \sum_{i=1}^k \frac{U_i}{\sigma(U_i)} \frac{V_i}{\sigma(V_i)} < \sum_{i=1}^k \frac{V_i}{\sigma(V_i)}.$$

Comparing this with (40), $\mathcal{S}$ should be non-empty. By multiplying (40) by

$$\prod_{i=1}^k \sigma(V_i) \prod_{p \in \mathcal{S}} p,$$

we have

$$\sum_{i=1}^k V_i \prod_{\substack{j=1 \\ i \neq j}}^k \sigma(V_j) \prod_{\substack{p \mid U_i \\ p \in \mathcal{S}}} (p-1) \prod_{\substack{p \nmid U_i \\ p \in \mathcal{S}}} p = \prod_{i=1}^k \sigma(V_i) \prod_{p \in \mathcal{S}} p. \tag{41}$$

Let P be the largest prime in $\mathcal{S}$, which exists since $\mathcal{S}$ is non-empty. By symmetry, we may assume $P \mid U_1$. Then in (41), all terms except the case $i = 1$ on the left-hand side and the right-hand side are divisible by P since $(M_i)_{i=1}^k$ is anarchic so $U_1, \ldots, U_k$ are pairwise coprime. This implies

$$P \mid V_1 \prod_{j=2}^k \sigma(V_j) \prod_{\substack{p \mid U_1 \\ p \in \mathcal{S}}} (p-1) \prod_{\substack{p \nmid U_1 \\ p \in \mathcal{S}}} p. \tag{42}$$

Since P is the largest prime in $\mathcal{S}$ and $(M_i)_{i=1}^k$ is anarchic, we find

$$P \nmid \prod_{j=2}^{k} \sigma(V_j) \prod_{\substack{p \mid U_1 \\ p \in \mathcal{S}}} (p-1) \prod_{\substack{p \nmid U_1 \\ p \in \mathcal{S}}} p.$$

Thus by (42), $P \mid V_1$, which contradicts to $P \mid U_1$ and $(U_1, V_1) = 1$. □

By using Lemma 8, we can prove now the following variant of Heath-Brown's induction lemma, which corresponds to Lemma 1.5 of [13].

LEMMA 9. *Let $k \geq 2$ be an integer, $(M_i)_{i=1}^k$ be an anarchic harmonious tuple and suppose that a tuple of decompositions*

$$M_i = U_i V_i, \qquad (U_i, V_i) = 1, \qquad U := U_1 \cdots U_k > 1, \qquad V := V_1 \cdots V_k,$$

and a set $\mathcal{S}$ of prime factors of U are given. Then there exists a tuple of decompositions

$$M_i = U_i' V_i', \qquad \big(U_i', V_i'\big) = 1,$$
$$U' := U_1' \cdots U_k', \qquad V' := V_1' \cdots V_k', \qquad V \parallel V',$$

and a set $\mathcal{S}'$ of prime factors of U' with the following conditions:

(i) $v := |\mathcal{P}'| \geq 1$, *where* $\mathcal{P}' := \{p \text{ prime}: \ p \mid V', p \nmid V\}$,

(ii) *we have*

$$\sigma\big(V'\big)\Pi\big(\mathcal{S}'\big)\Psi\big(\mathcal{P}'\big) \leq F_{v+w}\big(\sigma(V)\Pi(\mathcal{S}) + 1\big),$$

where $w := v + |\mathcal{S}'| - |\mathcal{S}|$,

(iii) *if $w = 0$, then the inequality in* (ii) *can be improved to*

$$\sigma\big(V'\big)\Pi\big(\mathcal{S}'\big)\Psi\big(\mathcal{P}'\big) \leq F_{v+w}\big(\sigma(V)\Pi(\mathcal{S})\big).$$

Proof. We first show that there is a set $\mathcal{T}$ of prime factors of U satisfying

$$\mathcal{S} \cap \mathcal{T} = \emptyset, \tag{43}$$

$$\sum_{i=1}^{k} \frac{V_i}{\sigma(V_i)} \prod_{\substack{p \mid U_i \\ p \in \mathcal{S} \cup \mathcal{T}}} \left(1 - \frac{1}{p}\right) < 1 \tag{44}$$

and

$$\sigma(V)\Pi(\mathcal{S})\Pi(\mathcal{T}) \leq F_w\big(\sigma(V)\Pi(\mathcal{S}) + 1\big), \quad w := |\mathcal{T}|. \tag{45}$$

This w will be the same quantity as in the condition (ii). By Lemma 8,

$$H := \sum_{i=1}^{k} \frac{V_i}{\sigma(V_i)} \prod_{\substack{p \mid U_i \\ p \in \mathcal{S}}} \left(1 - \frac{1}{p}\right) \neq 1. \tag{46}$$

We consider two cases separately according to the size of H.

If $H<1$, we just take $\mathcal{T}=\emptyset$ so that $w=0$. This choice obviously satisfies the conditions (43), (44) and (45). Thus, the case $H<1$ is done.

We next consider the case $H>1$. Since $U>1$, we have

$$\frac{U_i}{\sigma(U_i)} > \prod_{p\mid U_i}\left(1-\frac{1}{p}\right)$$

for some i. Thus, by (3), we see that

$$\sum_{i=1}^{k}\frac{V_i}{\sigma(V_i)}\prod_{p\mid U_i}\left(1-\frac{1}{p}\right)<1. \tag{47}$$

By using notation

$$a_i=\sigma(V_i)\prod_{\substack{p\mid U_i\\ p\in\mathcal{S}}}p,\qquad b_i=V_i\prod_{\substack{p\mid U_i\\ p\in\mathcal{S}}}(p-1),$$

we can rewrite (47) as

$$\sum_{i=1}^{k}\frac{b_i}{a_i}\prod_{\substack{p\mid U_i\\ p\notin\mathcal{S}}}\left(1-\frac{1}{p}\right)<1.$$

Note that $a_i\geq b_i$ for all i. Thus, by comparing this inequality with $H>1$ and examining the smallest prime factors of U outside $\mathcal{S}$, we can find a non-empty set $\mathcal{T}=\{p_1,\dots,p_w\}$ of prime factors of U with $p_1<\cdots<p_w$, which satisfies $\mathcal{S}\cap\mathcal{T}=\emptyset$ and two inequalities

$$\sum_{i=1}^{k}\frac{b_i}{a_i}\prod_{\substack{j=1\\ p\mid U_i}}^{w}\left(1-\frac{1}{p_j}\right)\leq 1,\qquad \sum_{i=1}^{k}\frac{b_i}{a_i}\prod_{\substack{j=1\\ p\mid U_i}}^{w-1}\left(1-\frac{1}{p_j}\right)>1. \tag{48}$$

By applying Lemma 6 to this pair of inequalities, we find that

$$a\Pi(\mathcal{T})=a\prod_{j=1}^{w}p_j\leq F_w(a+1),\qquad a=a_1\cdots a_k=\sigma(V)\Pi(\mathcal{S}),$$

i.e., (45) holds. For (44), we expand the definition of a_i and b_i in (48) to obtain

$$\sum_{i=1}^{k}\frac{V_i}{\sigma(V_i)}\prod_{\substack{p\mid U_i\\ p\in\mathcal{S}\cup\mathcal{T}}}\left(1-\frac{1}{p}\right)\leq 1.$$

Then this equality cannot hold by Lemma 8 so the condition (44) holds. Therefore, in any case, we succeeded to find a set $\mathcal{T}$ satisfying the desired conditions.

We next show that there is a non-empty subset $\mathcal{P}'$ of $\mathcal{S}\cup\mathcal{T}$ which satisfies

$$\sigma(V)\Pi(\mathcal{S})\Pi(\mathcal{T})\prod_{\substack{p^e\| U\\ p\in\mathcal{P}'}}\bigl(p^{e+1}-1\bigr)\le F_v\bigl(\sigma(V)\Pi(\mathcal{S})\Pi(\mathcal{T})\bigr),\quad v=\bigl|\mathcal{P}'\bigr|. \tag{49}$$

These $\mathcal{P}'$ and v will be the same objects as in the condition (i). Since we have $n/\sigma(n)\le 1$ for any positive integer n, (3) implies

$$\sum_{i=1}^{k}\frac{V_i}{\sigma(V_i)}\prod_{\substack{p^e\| U_i\\ p\in\mathcal{S}\cup\mathcal{T}}}\frac{1-1/p}{1-1/p^{e+1}}\ge 1. \tag{50}$$

By using notations

$$a_i'=\sigma(V_i)\prod_{\substack{p| U_i\\ p\in\mathcal{S}\cup\mathcal{T}}}p,\qquad b_i'=V_i\prod_{\substack{p| U_i\\ p\in\mathcal{S}\cup\mathcal{T}}}(p-1),$$

we can rewrite (50) as

$$\sum_{i=1}^{k}\frac{b_i'}{a_i'}\prod_{\substack{p^e\| U_i\\ p\in\mathcal{S}\cup\mathcal{T}}}\left(1-\frac{1}{p^{e+1}}\right)^{-1}\ge 1. \tag{51}$$

Similarly, (44) can be rewritten as

$$\sum_{i=1}^{k}\frac{b_i'}{a_i'}<1. \tag{52}$$

By comparing (51) and (52) and examining the smallest values of

$$p^{e+1}\quad\text{with}\quad p\in\mathcal{S}\cup\mathcal{T}\text{ and }p^e\parallel U,$$

we can find a non-empty subset $\mathcal{P}'=\{P_1,\dots,P_v\}$ of $\mathcal{S}\cup\mathcal{T}$ with

$$P_1^{e_1+1}<\cdots<P_v^{e_v+1},\qquad P_j^{e_j}\parallel U$$

satisfying two inequalities

$$\sum_{i=1}^{k}\frac{b_i'}{a_i'}\prod_{\substack{j=1\\ P_j| U_i}}^{v}\left(1-\frac{1}{P_j^{e_j+1}}\right)^{-1}\ge 1,\qquad \sum_{i=1}^{k}\frac{b_i'}{a_i'}\prod_{\substack{j=1\\ P_j| U_i}}^{v-1}\left(1-\frac{1}{P_j^{e_j+1}}\right)^{-1}<1. \tag{53}$$

Then by applying Lemma 7 to (53), we find that

$$a'\prod_{j=1}^{v}\bigl(P_j^{e_j+1}-1\bigr)\le F_v\bigl(a'\bigr),\qquad a'=a_1'\cdots a_k'=\sigma(V)\Pi(\mathcal{S})\Pi(\mathcal{T}),$$

i.e., (49) holds. Thus our $\mathcal{P}'$ satisfies the desired condition.

Finally, we choose $\mathcal{S}'$, V_i', and U_i' by

$$\mathcal{S}' = (\mathcal{S} \cup \mathcal{T}) \setminus \mathcal{P}', \qquad V_i' = V_i \prod_{\substack{p^e \| U_i \\ p \in \mathcal{P}'}} p^e, \qquad M_i = U_i' V_i'.$$

Then it is clear that $(U_i', V_i') = 1$, $V \,\|\, V'$. The set $\mathcal{S}'$ consists of some prime factors of U' since the prime factors of U' are those of U outside the set $\mathcal{P}'$.

The remaining task is to check the conditions (i), (ii), and (iii). Note that the notations on $\mathcal{P}'$ and v keep its consistency. Since $\mathcal{P}'$ is non-empty, the condition (i) is satisfied. For the consistency on w, it suffices to see

$$v + \left|\mathcal{S}'\right| - |\mathcal{S}| = \left|\mathcal{P}'\right| + |\mathcal{S} \cup \mathcal{T}| - \left|\mathcal{P}'\right| - |\mathcal{S}| = |\mathcal{S} \cup \mathcal{T}| - |\mathcal{S}| = |\mathcal{T}|.$$

We prove the inequality in (ii) and (iii). By our choice of V_i' and $\mathcal{S}'$,

$$\begin{aligned} \sigma\big(V'\big)\Pi\big(\mathcal{S}'\big)\Pi\big(\mathcal{P}'\big)\Phi\big(\mathcal{P}'\big) &= \sigma(V)\sigma\Bigg(\prod_{\substack{p^e \| U \\ p \in \mathcal{P}'}} p^e\Bigg)\Pi(\mathcal{S})\Pi(\mathcal{T})\Phi\big(\mathcal{P}'\big) \\ &= \sigma(V)\Pi(\mathcal{S})\Pi(\mathcal{T}) \prod_{\substack{p^e \| U \\ p \in \mathcal{P}'}} \big(p^{e+1} - 1\big). \end{aligned}$$

By (49) and the definition of Ψ, this implies

$$\sigma\big(V'\big)\Pi\big(\mathcal{S}'\big)\Psi\big(\mathcal{P}'\big) \le F_v\big(\sigma(V)\Pi(\mathcal{S})\Pi(\mathcal{T})\big).$$

If $w = 0$, then $\mathcal{T} = \emptyset$ so that this inequality already gives the inequality in (iii). We substitute (45) here by using Lemma 1. Then, by Lemma 3,

$$\sigma\big(V'\big)\Pi\big(\mathcal{S}'\big)\Psi\big(\mathcal{P}'\big) \le F_v\big(F_w\big(\sigma(V)\Pi(\mathcal{S}) + 1\big)\big) \le F_{v+w}\big(\sigma(V)\Pi(\mathcal{S}) + 1\big)$$

since the assumption $x \ge 2$ of Lemma 3 is satisfied by

$$\sigma(V)\Pi(\mathcal{S}) + 1 \ge 2.$$

Thus the inequality in (ii) also holds. This completes the proof. $\square$

6. Completion of the proof of Theorem 1

We start with carrying out the induction given in Section 5.

LEMMA 10. *For any anarchic harmonious tuple* $(M_i)_{i=1}^k$,

$$\sigma(M_1 \cdots M_k)\frac{\Phi(\mathcal{P})}{\Pi(\mathcal{P})} \le F_{2K}(2)\Pi(\mathcal{P})^{-2},$$

where $\mathcal{P}$ *is the set of all prime factors of* $M_1 \cdots M_k$ *and* $K = \omega(M_1 \cdots M_k)$.

Proof. Let $(M_i)_{i=1}^k$ be an anarchic harmonious tuple and $K = \omega(M_1 \cdots M_k)$. We apply Lemma 9 inductively to construct tuples of decompositions

$$M_i(\nu) = U_i(\nu)V_i(\nu), \qquad \big(U_i(\nu), V_i(\nu)\big) = 1 \quad (\nu = 0, 1, 2, \ldots)$$

and sets of primes

$$\mathcal{S}(\nu), \mathcal{P}(\nu) \quad (\nu = 0, 1, 2, \ldots),$$

where $\mathcal{S}(\nu)$ is a set of prime factors of $U(\nu) := U_1(\nu) \cdots U_k(\nu)$. We start with

$$U_i(0) = M_i, \qquad V_i(0) = 1, \qquad \mathcal{S}(0) = \emptyset, \qquad \mathcal{P}(0) = \emptyset.$$

In general steps, we apply Lemma 9 to the $(\nu-1)$th term with the choice

$$U_i = U_i(\nu-1), \qquad V_i = V_i(\nu-1), \qquad \mathcal{S} = \mathcal{S}(\nu-1)$$

and define the νth term by

$$U_i(\nu) = U_i', \qquad V_i(\nu) = V_i', \qquad \mathcal{S}(\nu) = \mathcal{S}', \qquad \mathcal{P}(\nu) = \mathcal{P}'.$$

Then we can continue this induction step as long as $U(\nu-1) > 1$. Let

$$v(\nu) = \big|\mathcal{P}(\nu)\big|, \qquad w(\nu) = v(\nu) + \big|\mathcal{S}(\nu)\big| - \big|\mathcal{S}(\nu-1)\big|$$

for $\nu \geq 1$ as long as the induction step is available. By the definition of $\mathcal{P}(\nu)$,

$$\mathcal{P}(\nu) = \big\{p \text{ prime}: \ p \mid V(\nu)\big\} \setminus \big\{p \text{ prime}: \ p \mid V(\nu-1)\big\} \tag{54}$$

for $\nu \geq 1$, where $V(\nu) := V_1(\nu) \cdots V_k(\nu)$, so

$$v(\nu) = \omega\big(V(\nu)\big) - \omega\big(V(\nu-1)\big)$$

since $V(\nu-1) \parallel V(\nu)$. Thus, by (i) of Lemma 9,

$$s \leq v(1) + \cdots + v(s) = \omega\big(V(s)\big) \leq \omega(M_1 \cdots M_k) = K \tag{55}$$

for any $s \geq 1$ if the induction step is available until the sth step. Thus, the induction step stops in finitely many steps. Suppose that the induction step stops at the nth step to produce

$$U_i(n) = 1, \qquad V_i(n) = M_i, \qquad \mathcal{S}(n) = \emptyset.$$

By (iii) of Lemma 9, for $2 \leq s \leq n$ satisfying $w(s) = 0$, we have

$$\sigma\big(V(s)\big)\Pi\big(\mathcal{S}(s)\big)\Psi\big(\mathcal{P}(s)\big) \leq F_{v(s)+w(s)}\big(\sigma\big(V(s-1)\big)\Pi\big(\mathcal{S}(s-1)\big)\big).$$

Since $v(s) + w(s) = v(s) \geq 1$, Lemma 2 gives

$$\begin{aligned} &\sigma\big(V(s)\big)\Pi\big(\mathcal{S}(s)\big)\Psi\big(\mathcal{P}(s)\big) \\ &\quad \leq \Psi\big(\mathcal{P}(s-1)\big)^{-1} F_{v(s)+w(s)}\big(\sigma\big(V(s-1)\big)\Pi\big(\mathcal{S}(s-1)\big)\Psi\big(\mathcal{P}(s-1)\big)\big). \end{aligned} \tag{56}$$

For remaining $2 \leq s \leq n$ with $w(s) \geq 1$, we use (ii) of Lemma 9 to obtain

$$\begin{aligned} \sigma\big(V(s)\big)\Pi\big(\mathcal{S}(s)\big)\Psi\big(\mathcal{P}(s)\big) &\leq F_{v(s)+w(s)}\big(\sigma\big(V(s-1)\big)\Pi\big(\mathcal{S}(s-1)\big) + 1\big) \\ &\leq F_{v(s)+w(s)}\left(\frac{4}{3}\sigma\big(V(s-1)\big)\Pi\big(\mathcal{S}(s-1)\big)\right) \end{aligned}$$

since $\sigma(V(s-1)) \geq 3$ for $s \geq 2$. Also, since $\mathcal{P}(s-1) \neq \emptyset$ for $s \geq 2$,

$$\Psi\big(\mathcal{P}(s-1)\big) = \Pi\big(\mathcal{P}(s-1)\big)\Phi\big(\mathcal{P}(s-1)\big) \geq 2 \geq \left(\frac{4}{3}\right)^2.$$

Therefore, by Lemma 2 with the choice $\alpha = \frac{3}{4}\Psi(\mathcal{P}(s-1)) \geq \Psi(\mathcal{P}(s-1))^{\frac{1}{2}}$,

$$\begin{aligned}&\sigma\big(V(s)\big)\Pi\big(\mathcal{S}(s)\big)\Psi\big(\mathcal{P}(s)\big)\\&\quad\leq \Psi\big(\mathcal{P}(s-1)\big)^{-2^{v(s)+w(s)-2}}\\&\qquad\times F_{v(s)+w(s)}\big(\sigma\big(V(s-1)\big)\Pi\big(\mathcal{S}(s-1)\big)\Psi\big(\mathcal{P}(s-1)\big)\big)\end{aligned}$$

so by using $v(s)+w(s) \geq v(s)+1 \geq 2$, we again arrive at (56). Thus the estimate (56) holds for every $2 \leq s \leq n$. Then by using (56) inductively with Lemmas 1, 2, and 3,

$$\begin{aligned}&\sigma\big(V(n)\big)\Pi\big(\mathcal{S}(n)\big)\Psi\big(\mathcal{P}(n)\big)\\&\quad\leq \Psi\big(\mathcal{P}(n-1)\big)^{-1}F_{v(n)+w(n)}\big(\sigma\big(V(n-1)\big)\Pi\big(\mathcal{S}(n-1)\big)\Psi\big(\mathcal{P}(n-1)\big)\big)\\&\quad\leq \Psi\big(\mathcal{P}(n-1)\big)^{-1}\Psi\big(\mathcal{P}(n-2)\big)^{-1}\\&\qquad\times F_{v(n)+w(n)}\big(F_{v(n-1)+w(n-1)}\big(\sigma\big(V(n-2)\big)\Pi\big(\mathcal{S}(n-2)\big)\Psi\big(\mathcal{P}(n-2)\big)\big)\big)\\&\quad\leq \Psi\big(\mathcal{P}(n-1)\big)^{-1}\Psi\big(\mathcal{P}(n-2)\big)^{-1}\\&\qquad\times F_{v(n)+w(n)+v(n-1)+w(n-1)}\big(\sigma\big(V(n-2)\big)\Pi\big(\mathcal{S}(n-2)\big)\Psi\big(\mathcal{P}(n-2)\big)\big)\\&\quad\leq\cdots\\&\quad\leq \Psi\left(\bigsqcup_{\nu=1}^{n-1}\mathcal{P}(\nu)\right)^{-1}F_{v(n)+w(n)+\cdots+v(2)+w(2)}\big(\sigma\big(V(1)\big)\Pi\big(\mathcal{S}(1)\big)\Psi\big(\mathcal{P}(1)\big)\big),\end{aligned}$$

where, in order to apply Lemma 3, we used

$$\sigma\big(V(s)\big)\Pi\big(\mathcal{S}(s)\big)\Psi\big(\mathcal{P}(s)\big) \geq \sigma\big(V(s)\big) \geq \sigma\big(V(1)\big) \geq 3 \quad (1 \leq s \leq n).$$

By using (ii) of Lemma 9 once more and recalling (54),

$$\begin{aligned}\sigma(M_1\cdots M_k) &= \Psi\big(\mathcal{P}(n)\big)^{-1}\sigma\big(V(n)\big)\Pi\big(\mathcal{S}(n)\big)\Psi\big(\mathcal{P}(n)\big)\\&\leq \Psi(\mathcal{P})^{-1}F_{\sum_{\nu=1}^{n}(v(\nu)+w(\nu))}\big(\sigma\big(V(0)\big)\Pi\big(\mathcal{S}(0)\big)+1\big)\\&= \Psi(\mathcal{P})^{-1}F_{\sum_{\nu=1}^{n}(v(\nu)+w(\nu))}(2).\end{aligned}$$

By definition of $w(\nu)$ and $\mathcal{S}(0)=\mathcal{S}(n)=\emptyset$, we have

$$\begin{aligned}\sum_{\nu=1}^{n}\big(v(\nu)+w(\nu)\big) &= 2\sum_{\nu=1}^{n}v(\nu)+\big|\mathcal{S}(n)\big|-\big|\mathcal{S}(0)\big|\\&= 2\sum_{\nu=1}^{n}v(\nu) = 2\omega\big(V(n)\big) = 2\omega(M_1\cdots M_k) = 2K.\end{aligned}$$

Thus the lemma follows. □

We next prove the following lemma, which is an amicable number analogue of the lemma used in the clever trick of Chen and Tang [4, Lemma 2.3] or of Kobayashi (see the remark before Corollary 1.7 of [13]).

LEMMA 11. *For any anarchic harmonious tuple* $(M_i)_{i=1}^k$,

$$\sigma(M_1 \cdots M_k)\frac{\Phi(\mathcal{P})}{\Pi(\mathcal{P})} \le F_K\big(\Pi(\mathcal{P})\big)\Pi(\mathcal{P})^{-2},$$

where $\mathcal{P}$ *is the set of all prime factors of* $M_1 \cdots M_k$ *and* $K = \omega(M_1 \cdots M_k)$.

Proof. Since $(M_i)_{i=1}^k$ is a harmonious tuple, (3) holds. Then by using

$$a_i = \prod_{p \mid M_i} p, \qquad b_i = \prod_{p \mid M_i} (p-1), \qquad a = a_1 \cdots a_k,$$

we can rewrite the identity (3) as

$$\sum_{i=1}^k \frac{b_i}{a_i} \prod_{p^e \| M_i} \left(1 - \frac{1}{p^{e+1}}\right)^{-1} = 1.$$

Also, if we remove some prime factor of $M_1 \cdots M_k$ from this identity, then the left-hand side becomes smaller. Thus, we can apply Lemma 7 to obtain

$$a \prod_{p^e \| M_1 \cdots M_k} \big(p^{e+1} - 1\big) \le F_K(a).$$

Since $a = \Pi(\mathcal{P})$, this gives

$$\sigma(M_1 \cdots M_k)\Pi(\mathcal{P})\Phi(\mathcal{P}) \le F_K\big(\Pi(\mathcal{P})\big).$$

This completes the proof. □

Proof of Theorem 1. If $\Pi(\mathcal{P}) > 2^{2^K}$, then we use Lemma 10 to obtain

$$\sigma(M_1 \cdots M_k)\frac{\Phi(\mathcal{P})}{\Pi(\mathcal{P})} \le F_{2K}(2)\Pi(\mathcal{P})^{-2} < F_{2K}(2)2^{-2\cdot 2^K}.$$

On the other hand, if $\Pi(\mathcal{P}) \le 2^{2^K}$, then we use Lemma 11 to obtain

$$\begin{aligned} \sigma(M_1 \cdots M_k)\frac{\Phi(\mathcal{P})}{\Pi(\mathcal{P})} &\le F_K\big(\Pi(\mathcal{P})\big)\Pi(\mathcal{P})^{-2} \\ &= \Pi(\mathcal{P})^{2^{K-1}-2}\big(\Pi(\mathcal{P})^{2^{K-1}} - 1\big) \le F_{2K}(2)2^{-2\cdot 2^K}. \end{aligned}$$

Thus in any case we have

$$\sigma(M_1 \cdots M_k)\frac{\Phi(\mathcal{P})}{\Pi(\mathcal{P})} \le F_{2K}(2)2^{-2\cdot 2^K}. \tag{57}$$

Note that

$$\sigma(M_1 \cdots M_k)\frac{\Phi(\mathcal{P})}{\Pi(\mathcal{P})} = M_1 \cdots M_k \prod_{p^e \| M_1 \cdots M_k} \left(1 - \frac{1}{p^{e+1}}\right).$$

Combining this identity with (57), we obtain

$$M_1 \cdots M_k \le \prod_{p^e \| M_1 \cdots M_k} \left(1 - \frac{1}{p^{e+1}}\right)^{-1} F_{2K}(2)2^{-2\cdot 2^K}.$$

By using

$$\prod_{p^e \,\|\, M_1 \cdots M_k} \left(1 - \frac{1}{p^{e+1}}\right)^{-1} < \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \frac{\pi^2}{6}, \quad F_{2K}(2) < 2^{4^K},$$

we finally arrive at

$$M_1 \cdots M_k < \frac{\pi^2}{6} F_{2K}(2) 2^{-2\cdot 2^K} < \frac{\pi^2}{6} 2^{4^K - 2\cdot 2^K}.$$

This completes the proof. □

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