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NEW CHARACTERIZATIONS OF BESOV AND TRIEBEL–LIZORKIN SPACES VIA THE $T1$ THEOREM

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ABSTRACT. The main purpose of this paper is to provide new characterizations of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type with the "reverse" doubling property. To achieve our goal, the key idea is to prove a $T1$ theorem with only half the usual smoothness and cancellation conditions.

1. Introduction and statement of main results

In the earlier of 1970s, Coifman and Weiss [1] introduced spaces of homogeneous type in order to extend the Calderón–Zygmund singular operator theory to a more general setting. These spaces have no dilations, translations and analogues of the Fourier transform. Let us recall briefly spaces of homogeneous type in the sense of Coifman and Weiss. A quasi-metric ρ on a set X is a function $\rho : X \times X \to [0,\infty)$ satisfying (i) $\rho(x,y)=0$ if and only if $x=y$; (ii) $\rho(x,y)=\rho(y,x)$ for all $x,y\in X$; (iii) There exists a constant $A\in[1,\infty)$ such that for all x,y and $z\in X$,

$$\rho(x,y)\le A\big[\rho(x,z)+\rho(z,y)\big]. \tag{1}$$

Any quasi-metric defines a topology, for which the balls

$$B(x,r)=\big\{y\in X:\rho(y,x)<r\big\}$$

for all $x\in X$ and all $r>0$ form a basis. We say that (X,ρ,μ) is a space of homogeneous type in sense of Coifman and Weiss if ρ is a quasi-metric and μ is a nonnegative Borel regular measure on X satisfying the doubling condition, that is, for all $x\in X$, $r>0$, then $0<\mu(B(x,r))<\infty$ and

$$\mu\big(B(x,2r)\big)\le C\mu\big(B(x,r)\big), \tag{2}$$

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where μ is assumed to be defined on a σ-algebra which contains all Borel sets and all balls $B(x,r)$, and the constant $0<C<\infty$ is independent of $x \in X$ and $r>0$.

Macías and Segovia [8] showed that the quasi-metric ρ can be replaced by another quasi-metric d such that the topologies induced on X by ρ and d coincide. Moreover, d has the following regularity property: there exist constants $C>0$ and $0<\theta<1$ such that for all $x,x',y \in X$,

$$|d(x,y)-d(x',y)| \le Cd(x,x')^{\theta}\left[d(x,y)+d(x',y)\right]^{1-\theta}. \tag{3}$$

Moreover, if $B(x,r)$, the ball defined by the metric d, then

$$\mu\big(B(x,r)\big) \approx r. \tag{4}$$

Note that the condition (4) is much stronger than the doubling property (2).

In [9], Nagel and Stein developed the product theory on Carnot–Carathéodory spaces with a smooth quasi-metric d and a measure μ satisfying the condition (2) and the "reverse" doubling condition, that is, there exist constants $a_0, C \in (1,\infty)$ such that for all $x \in X$ and all $0<r<\sup_{x,y\in X} d(x,y)/a_0$,

$$C\mu\big(B(x,r)\big) < \mu\big(B(x,a_0 r)\big). \tag{5}$$

We point out that the doubling condition (2) and "reverse" doubling condition (5) implies that there exist positive constants ω (the *upper dimension* of μ), $\kappa \in (0,\omega]$ (the *lower dimension* of μ) and $c \in (0,1]$, $C>1$ such that for all $x \in X$, $0<r<\sup_{x,y\in X} d(x,y)/2$ and $1 \le \lambda < \sup_{x,y\in X} d(x,y)/2r$,

$$c\lambda^{\kappa}\mu\big(B(x,r)\big) \le \mu\big(B(x,\lambda r)\big) \le C\lambda^{\omega}\mu\big(B(x,r)\big). \tag{6}$$

Spaces of homogeneous type include many important examples, such as Euclidean space, Ahlfors regular metric measure spaces, Lie groups of polynomial growth and Carnot–Carathéodory spaces with doubling measures. All these examples fall under the scope of the study of RD-spaces introduced in [5]. To be more precise, a RD-space (X,d,μ) is a space of homogeneous type in sense of Coifman and Weiss where the quasi-metric d satisfies regularity (3) and the measure μ satisfies the "reverse" doubling property (6).

On the other hand, the seminal work on spaces of homogeneous type is the $T1$ theorem. More precisely, in order to study the L^2 boundedness of generalized Calderón–Zygmund singular integral operators, David and Journé [2] proved the remarkable $T1$ theorem on $\mathbb{R}^n$. David, Journé and Semmes [3] provided the $T1$ theorem on (X,d,μ) where d satisfy the regularity in (3) and the measure μ satisfy the property in (4). Han and Sawyer [6] established the $T1$ theorem for the Besov and Triebel–Lizorkin spaces on such spaces of homogeneous type. Moreover, they gave new characterizations of the Besov and Triebel–Lizorkin spaces with only half the usual smoothness and cancellation conditions on the approximate to the identity. See [6] for more details and

[4] and [7] for the related results. A theory of the Besov and Triebel–Lizorkin spaces on RD-spaces was established in [5], where these spaces were characterized by the family of operators whose kernels satisfy the usual smoothness and cancellation conditions and the $T1$ theorems with the usual smoothness and cancellation conditions were proved. See [5] for more details.

A natural question arises: Can one characterize the Besov and Triebel–Lizorkin spaces on spaces of homogeneous type of RD-spaces as established in [5] by only half the usual smoothness and cancellation conditions on the approximate to the identity?

The main purpose of this paper is to answer this question. The key tool to achieve our goal is to prove a new $T1$ theorem for the Besov and Triebel–Lizorkin spaces on spaces of homogeneous type of RD-spaces, where, however, the kernel of T satisfies only half the usual smoothness and cancellation conditions. To state the main results in this paper, throughout this paper, (X,d,μ) are spaces of homogeneous type of RD-spaces. We use C to denote positive constants, whose value may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. We denote by $f\sim g$ that there exists a constant $C>0$ independent of the main parameters such that $C^{-1}g<f<Cg$. M is the Hardy–Littlewood maximal operator. For any $1<q<\infty$, we denote by q' its conjugate index, that is, $1/q+1/q'=1$. We also denote $\min\{a,b\}$ by $a\wedge b$ for any $a,b\in\mathbb{R}$. For all $x\in X$ and all $r>0$, we use the abbreviations

$$V_r(x):=\mu\big(B(x,r)\big),\qquad V(x,y):=V\big(x,d(x,y)\big).$$

Now we begin with recalling the definition of an approximate to the identity, which plays the same role as the heat kernel $H(s,x,y)$ does in [9].

Definition 1.1 ([5]). Let θ be the regularity exponent of X. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of linear operators is said to be an approximation to the identity if there exists a constant $C>0$ such that for all $k\in\mathbb{Z}$ and all $x,x',y,y'\in X$, $S_k(x,y)$, the kernel of S_k, is a function from $X\times X$ into $\mathbb{C}$ satisfies

(i) $S_k(x,y)=0$ if $d(x,y)\ge C2^{-k}$ and $|S_k(x,y)|\le C\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;
(ii) $|S_k(x,y)-S_k(x',y)|\le C2^{k\theta}d(x,x')^\theta\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;
(iii) $|S_k(x,y)-S_k(x,y')|\le C2^{k\theta}d(y,y')^\theta\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;
(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\le C2^{2k\theta}d(x,x')^\theta d(y,y')^\theta\times\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;
(v) $\int S_k(x,y)\,d\mu(y)=1$;
(vi) $\int S_k(x,y)\,d\mu(x)=1$.

To define the Besov and Triebel–Lizokin spaces, we need the following spaces of test functions and distributions.

Definition 1.2 ([5]). Let θ be the regularity exponent of X and $0<\beta,\gamma\le\theta$. A function f defined on X is said to be a test function of type (β,γ) centered at $x_0\in X$ with width $r>0$ if f satisfies the following conditions:

(i) $|f(x)|\le C\frac{1}{V_r(x_0)+V(x_0,x)}\frac{r^\gamma}{(r+d(x,x_0))^\gamma}$;

(ii) $|f(x)-f(y)|\le C(\frac{d(x,y)}{r+d(x,x_0)})^\beta\frac{1}{V_r(x_0)+V(x_0,x)}\frac{r^\gamma}{(r+d(x,x_0))^\gamma}$ for $d(x,y)\le\frac{1}{2A}(r+d(x,x_0))$;

(iii) $\int f(x)\,d\mu(x)=0$.

If f is a test function of type (β,γ) centered at x_0 with width $r>0$, we write $f\in\mathcal{M}(x_0,r,\beta,\gamma)$, and the norm of f in $\mathcal{M}(x_0,r,\beta,\gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}=\inf\{C>0:\text{(i) and (ii) hold}\}.$$

We denote by $\mathcal{M}(\beta,\gamma)$ the class of all $f\in\mathcal{M}(x_0,1,\beta,\gamma)$. It is easy to see that $\mathcal{M}(x_1,r,\beta,\gamma)=\mathcal{M}(\beta,\gamma)$ with the equivalent norms for all $x_1\in X$ and $r>0$. Furthermore, it is also easy to check that $\mathcal{M}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta,\gamma)$.

Let $\widetilde{\mathcal{M}}(\beta,\gamma)$ be the completion of the space $\mathcal{M}(\theta,\theta)$ in $\mathcal{M}(\beta,\gamma)$ with $0<\beta,\gamma\le\theta$. If $f\in\widetilde{\mathcal{M}}(\beta,\gamma)$, we then define $\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}=\|f\|_{\mathcal{M}(\beta,\gamma)}$.

We define the distribution space $(\widetilde{\mathcal{M}}(\beta,\gamma))'$ by all linear functionals $\mathcal{L}$ from $\widetilde{\mathcal{M}}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a constant $C\ge 0$ such that for all $f\in\widetilde{\mathcal{M}}(\beta,\gamma)$,

$$\big|\mathcal{L}(f)\big|\le C\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}.$$

Now the Besov space $\dot{B}_p^{\alpha,q}$ and the Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}$ are defined as follows.

Definition 1.3 ([5]). Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity and $D_k=S_k-S_{k-1}$. Let $-\theta<\alpha<\theta$ and $1<p,q<\infty$. The Besov space $\dot{B}_p^{\alpha,q}$ is the collection of all $f\in(\widetilde{\mathcal{M}}(\beta,\gamma))'$ with $0<\beta,\gamma<\theta$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}}=\left\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|D_k(f)\big\|_{L^p}\big)^q\right\}^{\frac{1}{q}}<\infty.$$

The Triebel–Lizorkin space is the collection of all $f\in(\widetilde{\mathcal{M}}(\beta,\gamma))'$ with $0<\beta,\gamma<\theta$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}}=\left\|\left\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big|D_k(f)\big|\big)^q\right\}^{\frac{1}{q}}\right\|_{L^p}<\infty.$$

The authors [5] proved that Definition 1.3 is well defined, namely Definition 1.3 is independent of the choice of D_k. To state the main results in this paper, we also need the following definitions.

For $\eta \in (0,\theta]$, let $C_0^\eta(X)$ be the set of all continuous functions f on X with compact support such that

$$\|f\|_{C_0^\eta} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^\eta} < \infty.$$

Endow $C_0^\eta(X)$ with the natural topology and let $(C_0^\eta(X))'$ be its dual space.

DEFINITION 1.4. A continuous complex-valued function $K(x,y)$ defined on

$$\Omega = \big\{(x,y) \in X \times X : x \neq y\big\}$$

is called a standard kernel if there exist constants $\varepsilon \in (0,\theta]$ and $C_1 > 0$ such that

$$\big|K(x,y)\big| \le C_1 \frac{1}{V(x,y)}; \tag{7}$$

$$\big|K(x,y) - K\big(x',y\big)\big| \le C_1 \frac{d(x,x')^\varepsilon}{d(x,y)^\varepsilon} \frac{1}{V(x,y)} \quad \text{for } d\big(x,x'\big) \le d(x,y)/2A \tag{8}$$

and

$$\big|K(x,y) - K\big(x,y'\big)\big| \le C_1 \frac{d(y,y')^\varepsilon}{d(x,y)^\varepsilon} \frac{1}{V(x,y)} \quad \text{for } d\big(y,y'\big) \le d(x,y)/2A. \tag{9}$$

Calderón–Zygmund singular integral operators are given by the following.

DEFINITION 1.5. A continuous linear operator $T : C_0^\eta(X) \to (C_0^\eta(X))'$ is a Calderón–Zygmund singular integral operator if there exists a standard kernel K such that

$$\langle Tf, g\rangle = \int\int K(x,y) f(y) g(x)\, d\mu(x)\, d\mu(y)$$

for all $f, g \in C_0^\eta(X)$ with disjoint supports.

We also need the notion of the weak boundedness property.

DEFINITION 1.6 ([3]). A Calderón–Zygmund singular integral operator T is said to have the weak boundedness property, if there exist constants $C_2 > 0$ and $\eta \in (0,\theta]$ such that for all $x_0 \in X$ and $r > 0$

$$\big|\langle Tf, g\rangle\big| \le C_2 V_r(x_0) r^{2\eta} \|g\|_{C_0^\eta} \|f\|_{C_0^\eta},$$

where $f, g \in C_0^\eta(X)$ with supp $f, g \subset B(x_0, r)$, $\|f\|_\infty \le 1$, $\|g\|_\infty \le 1$, $\|f\|_{C_0^\eta} \le r^{-\eta}$ and $\|g\|_{C_0^\eta} \le r^{-\eta}$, and if T satisfies the weak boundedness property, we denote by $T \in WBP$.

The main results in this paper can be stated as follows.

THEOREM 1.1. *Suppose that T is a Calderón–Zygmund singular integral operator with the kernel satisfying (7), (8), $T(1) = 0$ and $T \in WBP$. Then T can be extended to a bounded linear operator on $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$ for $0 < \alpha < \varepsilon$ and $1 < p, q < \infty$.*

THEOREM 1.2. *Suppose that T is a Calderón–Zygmund singular integral operator with the kernel satisfying* (7), (9), $T^*(1)=0$ *and* $T\in WBP$. *Then T can be extended to a bounded linear operator on* $\dot{B}_p^{\alpha,q}$ *and* $\dot{F}_p^{\alpha,q}$ *for* $-\varepsilon<\alpha<0$ *and* $1<p,q<\infty$.

We would like to point out that, as mentioned, the $T1$ theorem where the kernel of T in [5] satisfies (7), (8), (9) and $T(1)=T^*(1)=0$. As a consequence of the above $T1$ theorems, we give new characterizations of the Besov and Triebel–Lizorkin spaces with only half the usual smoothness and cancellation conditions on the approximate to the identity.

THEOREM 1.3. *Let* $0<\alpha<\theta$ *and* $1<p,q<\infty$, $\{S_k\}_{k\in\mathbb{Z}}$ *be an approximation to the identity satisfies* (i), (ii) *and* (v) *of Definition* 1.1 *and* $E_k=S_k-S_{k-1}$.

(i) *For* $f\in\dot{B}_p^{\alpha,q}$, *then*

$$\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}}\sim\|f\|_{\dot{B}_p^{\alpha,q}}.$$

(ii) *For* $f\in\dot{F}_p^{\alpha,q}$, *then*

$$\Big\|\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big|E_k(f)\big|\big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^p}\sim\|f\|_{\dot{F}_p^{\alpha,q}}.$$

THEOREM 1.4. *Let* $-\theta<\alpha<0$ *and* $1<p,q<\infty$, $\{S_k\}_{k\in\mathbb{Z}}$ *be an approximation to the identity satisfies* (i), (iii) *and* (vi) *of Definition* 1.1 *and* $E_k=S_k-S_{k-1}$.

(i) *For* $f\in\dot{B}_p^{\alpha,q}$, *then*

$$\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}}\sim\|f\|_{\dot{B}_p^{\alpha,q}}.$$

(ii) *For* $f\in\dot{F}_p^{\alpha,q}$, *then*

$$\Big\|\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big|E_k(f)\big|\big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^p}\sim\|f\|_{\dot{F}_p^{\alpha,q}}.$$

We remark that Theorem 1.1 and Theorem 1.2 generalize the similar results proved in [6] on spaces of homogeneous type with the measure μ satisfies the condition (4).

2. Proof of Theorem 1.1 and Theorem 1.2

In this section, we show the $T1$ theorem of the Besov and Triebel–Lizorkin spaces on RD-spaces. We first give Calderón's reproducing formula which is a main tool in this paper.

LEMMA 2.1 ([5]). *Let* $\{S_k\}_{k\in\mathbb{Z}}$ *be an approximation to the identity defined in Definition* 1.1 *and* $D_k = S_k - S_{k-1}$. *Then there exist families of linear operators* $\widetilde{D}_k$ *and* $\widetilde{\widetilde{D}}_k$ *for* $k \in \mathbb{Z}$ *such that for all* $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$ *with* $0<\beta,\gamma<\theta$,

$$f = \sum_{k\in\mathbb{Z}} \widetilde{D}_k D_k(f) = \sum_{k\in\mathbb{Z}} D_k \widetilde{\widetilde{D}}_k(f), \tag{10}$$

where the series converges in both the norms of $\widetilde{\mathcal{M}}(\beta',\gamma')$ *with* $0<\beta'<\beta$ *and* $0<\gamma'<\gamma$, *and* L^p *with* $1<p<\infty$. *When* $f \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$, *the series converges in the norm of* $(\widetilde{\mathcal{M}}(\beta',\gamma'))'$ *with* $\beta<\beta'<\theta$, $\gamma<\gamma'<\theta$. *Moreover, for any* $\theta' \in (0,\theta)$, $\widetilde{D}_k(x,y)$ *and* $\widetilde{\widetilde{D}}_k(x,y)$, *the kernels of* $\widetilde{D}_k$ *and* $\widetilde{\widetilde{D}}_k$, *satisfy the similar estimates but with* x *and* y *interchange in* (ii):

(i)

$$\bigl|\widetilde{D}_k(x,y)\bigr| \le C \frac{1}{V_{2^{-k}}(x)+V(x,y)} \frac{2^{-k\theta'}}{(2^{-k}+d(x,y))^{\theta'}}; \tag{11}$$

(ii)

$$\begin{aligned} &\bigl|\widetilde{D}_k(x,y)-\widetilde{D}_k\bigl(x',y\bigr)\bigr| \\ &\quad \le C\biggl(\frac{d(x,x')}{2^{-k}+d(x,y)}\biggr)^{\theta'} \frac{1}{V_{2^{-k}}(x)+V(x,y)} \frac{2^{-k\theta'}}{(2^{-k}+d(x,y))^{\theta'}} \end{aligned} \tag{12}$$

for $d(x,x') \le \frac{1}{2A}(2^{-k}+d(x,y))$;

(iii)

$$\int \widetilde{D}_k(x,y)\,d\mu(y) = \int \widetilde{D}_k(x,y)\,d\mu(x) = 0. \tag{13}$$

In order to prove Theorem 1.1, we need the following estimates.

PROPOSITION 2.1. *Suppose that* T *satisfies the hypotheses of Theorem* 1.1 *and* $\{D_k\}_{k\in\mathbb{Z}}$ *be given in Definitions* 1.3. *Then there exists a constant* $C>0$ *such that*

$$\begin{aligned} &\bigl|D_l T D_k(x,y)\bigr| \\ &\quad \le C\bigl(2^{(k-l)\varepsilon}\wedge 1\bigr) \frac{1+(l-l\wedge k)}{V_{2^{-(k\wedge l)}}(x)+V(x,y)} \frac{2^{-(k\wedge l)\varepsilon}}{(2^{-(k\wedge l)}+d(x,y))^{\varepsilon}}. \end{aligned} \tag{14}$$

Proof. We need to consider four cases.

Case 1: $l\le k$, $d(x,y) \ge 4A^3C2^{-l}$.

We first make the following observations: note that if $d(x,u) \le C2^{-l}$ and $d(v,y)\le C2^{-k}$, then

$$\begin{aligned} d(x,y) &\le Ad(x,u)+Ad(u,y) \le Ad(x,u)+A^2d(u,v)+A^2d(v,y) \\ &\le 2A^2C2^{-l}+A^2d(u,v), \end{aligned}$$

so $d(u,v)\geq\frac{d(x,y)}{2A^2}$ and $\frac{1}{2A}d(u,v)\geq d(x,u)$; since $d(x,y)\geq 4A^3C2^{-l}$, we have

$$V_{2^{-l}}(x)+V_{2^{-l}}(y)\leq CV(x,y).$$

Using the above facts, the smoothness condition on the kernel $K(u,v)$ and the moment property of $D_l(x,u)$, we obtain

$$\begin{aligned}
&\left|\int\int D_l(x,u)K(u,v)D_k(v,y)\,d\mu(u)\,d\mu(v)\right|\\
&\quad=\left|\int\int D_l(x,u)\big[K(u,v)-K(x,v)\big]D_k(v,y)\,d\mu(u)\,d\mu(v)\right|\\
&\quad\leq C\int\int\big|D_l(x,u)\big|\frac{d(x,u)^\varepsilon}{d(u,v)^\varepsilon}\frac{1}{V(u,v)}\big|D_k(v,y)\big|\,d\mu(u)\,d\mu(v)\\
&\quad\leq C\frac{2^{-l\varepsilon}}{d(x,y)^\varepsilon}\frac{1}{V(x,y)}\\
&\quad\leq C\frac{2^{-l\varepsilon}}{(2^{-l}+d(x,y))^\varepsilon}\frac{1}{V_{2^{-l}}(x)+V(x,y)}.
\end{aligned}$$

Case 2: $l\leq k$, $d(x,y)<4A^3C2^{-l}$.

Suppose that a smooth cut-off function $\phi_0\in C^\infty(\mathbb{R})$ satisfies that $\phi_0(x)=1$ when $|x|\leq 1$ and $\phi_0(x)=0$ when $|x|\geq 2$. Set $\phi_1=1-\phi_0$ and write

$$\begin{aligned}
&\left|\int\int D_l(x,u)K(u,v)D_k(v,y)\,d\mu(u)\,d\mu(v)\right|\\
&\quad\leq\left|\int\int D_l(x,u)K(u,v)D_k(v,y)\phi_0\left(\frac{d(x,v)}{C_3 2^{-l}}\right)d\mu(u)\,d\mu(v)\right|\\
&\qquad+\left|\int\int D_l(x,u)K(u,v)D_k(v,y)\phi_1\left(\frac{d(x,v)}{C_3 2^{-l}}\right)d\mu(u)\,d\mu(v)\right|\\
&\quad:=I_1+I_2.
\end{aligned}$$

For I_1, set $\psi_k(v)=D_k(v,y)\phi_0(\frac{d(x,v)}{C_3 2^{-l}})$. Since $T\in WBP$, then

$$I_1=\big|\big\langle D_l(x,\cdot),T\psi_k(\cdot)\big\rangle\big|\leq CV_{2^{-l}}(x)2^{-2l\eta}\|\psi_k\|_{C_0^\eta}\big\|D_l(x,\cdot)\big\|_{C_0^\eta}.$$

It is easy to verify that $\|D_l(x,\cdot)\|_{C_0^\eta}\leq C2^{l\eta}[V_{2^{-l}}(x)]^{-1}$ and

$$\|\psi_k\|_{C_0^\eta}\leq C2^{l\eta}\big[V_{2^{-l}}(y)\big]^{-1}.$$

From the above estimates, we get

$$I_1\leq C\frac{1}{V_{2^{-l}}(x)}\leq C\frac{2^{-l\varepsilon}}{(2^{-l}+d(x,y))^\varepsilon}\frac{1}{V_{2^{-l}}(x)+V(x,y)}.$$

We now deal with term I_2. Note that $d(x,u)\leq C2^{-l}$ and that by the support of ϕ_1, $d(x,v)\geq C_3 2^{-l}$, where C_3 is large enough such that $d(x,u)\leq Cd(u,v)$, and then

$$V(u,v) \geq CV_{2^{-l}}(u) \sim CV_{2^{-l}}(x).$$

Thus, we have

$$\begin{aligned}
I_2 \leq & C \int\int |D_l(x,u)||D_k(v,y)| \frac{1}{V(u,v)}\, d\mu(u)\, d\mu(v) \\
\leq & C \frac{1}{V_{2^{-l}}(x)} \leq C \frac{2^{-l\varepsilon}}{(2^{-l}+d(x,y))^{\varepsilon}} \frac{1}{V_{2^{-l}}(x)+V(x,y)}.
\end{aligned}$$

Case 3: $l>k$, $d(x,y) \geq 4A^3C2^{-k}$.

In this case, using the fact that $D_l(1)=0$, we get

$$\begin{aligned}
&\left| \int\int D_l(x,u)K(u,v)D_k(v,y)\, d\mu(u)\, d\mu(v) \right| \\
&\quad = \left| \int\int D_l(x,u)\big[K(u,v)-K(x,v)\big] D_k(v,y)\, d\mu(u)\, d\mu(v) \right| \\
&\quad \leq C \int\int |D_l(x,u)| \frac{d(x,u)^{\varepsilon}}{d(u,v)^{\varepsilon}} \frac{1}{V(u,v)} |D_k(v,y)|\, d\mu(u)\, d\mu(v) \\
&\quad \leq C \frac{2^{-l\varepsilon}}{d(x,y)^{\varepsilon}} \frac{1}{V(x,y)},
\end{aligned}$$

which implies (14) for the case whenever $l>k$, $d(x,y) \geq 4A^3C2^{-k}$.

Case 4: $l>k$, $d(x,y) < 4A^3C2^{-k}$.

Using the facts that $D_l(1)=T(1)=0$, we have

$$\begin{aligned}
&\left| \int\int D_l(x,u)K(u,v)D_k(v,y)\, d\mu(u)\, d\mu(v) \right| \\
&\quad = \left| \int\int D_l(x,u)K(u,v)\big[D_k(v,y)-D_k(x,y)\big]\phi_0\left(\frac{d(x,v)}{4A^3C2^{-l}}\right) d\mu(u)\, d\mu(v) \right| \\
&\qquad + \left| \int\int D_l(x,u)\big[K(u,v)-K(x,v\big]\big[D_k(v,y)-D_k(x,y)\big] \right. \\
&\qquad \left. \times \phi_1\left(\frac{d(x,v)}{4A^3C2^{-l}}\right) d\mu(u)\, d\mu(v) \right| \\
&\quad := I_3+I_4.
\end{aligned}$$

Since $\|[D_k(\cdot,y)-D_k(y,x)]\phi_0(\frac{d(x,\cdot)}{4A^3C2^{-l}})\|_{C_0^\eta} \leq C[V_{2^{-k}}(y)]^{-1}2^{k\varepsilon}2^{-l\varepsilon}2^{l\eta}$, then, by the fact that $T \in WBP$,

$$\begin{aligned}
I_3 &\leq CV_{2^{-l}}(x)2^{-2l\eta}\big[V_{2^{-k}}(y)\big]^{-1}2^{k\varepsilon}2^{-l\varepsilon}2^{l\eta}2^{l\eta}\big[V_{2^{-l}}(x)\big]^{-1} \\
&\leq 2^{(k-l)\varepsilon}\big[V_{2^{-k}}(x)\big]^{-1},
\end{aligned}$$

which is dominated by the right-hand side of (14) whenever $l > k$, $d(x,y) < 4A^3C2^{-k}$.

Using the smoothness of $K(x,y)$ in x, we have

$$\begin{aligned}
I_4 \leq C2^{-l\varepsilon} & \int_{4A^3C2^{-l}<d(x,v)<\frac{1}{2A}2^{-k}} d(x,v)^{-\varepsilon}\big[V(x,v)\big]^{-1} \\
& \times \big|D_k(v,y)-D_k(x,y)\big|\,d\mu(v) \\
& + C2^{-l\varepsilon}\int_{d(x,v)\geq\frac{1}{2A}2^{-k}} d(x,v)^{-\varepsilon}\big[V(x,v)\big]^{-1}\big|D_k(v,y)-D_k(x,y)\big|\,d\mu(v) \\
\leq{} & C2^{(k-l)\varepsilon}\big[V_{2^{-k}}(x)\big]^{-1}\int_{4A^3C2^{-l}<d(x,v)<\frac{1}{2A}2^{-k}}\big[V(x,v)\big]^{-1}\,d\mu(v) \\
& + C2^{-l\varepsilon}\big[V_{2^{-k}}(x)\big]^{-1}\int_{d(x,v)\geq\frac{1}{2A}2^{-k}} d(x,v)^{-\varepsilon}\big[V(x,v)\big]^{-1}\,d\mu(v) \\
\leq{} & C\big(1+(l-k)\big)2^{(k-l)\varepsilon}\big[V_{2^{-k}}(x)\big]^{-1},
\end{aligned}$$

which completes the proof of Proposition 2.1. $\square$

We now are ready to give the

Proof of Theorem 1.1. Lemma 2.1, Proposition 2.1 and Hölder's inequality are enable us to get

$$\begin{aligned}
&\|Tf\|_{\dot{B}_p^{\alpha,q}} \\
&\quad= \Big\{\sum_{l\in\mathbb{Z}}\big(2^{l\alpha}\big\|D_l(Tf)\big\|_{L^p}\big)^q\Big\}^{1/q} \\
&\quad\leq \Big\{\sum_{l\in\mathbb{Z}}\Big(2^{l\alpha}\sum_{k\in\mathbb{Z}}\big\|D_lTD_k\widetilde{\widetilde{D}}_k(f)\big\|_{L^p}\Big)^q\Big\}^{1/q} \\
&\quad\leq C\Big\{\sum_{l\in\mathbb{Z}}\Big(2^{l\alpha}\sum_{k\in\mathbb{Z}}\big(1+(l-l\wedge k)\big)\big(2^{(k-l)\varepsilon}\wedge 1\big)\big\|M\big(\widetilde{\widetilde{D}}_k(f)\big)\big\|_{L^p}\Big)^q\Big\}^{1/q} \\
&\quad\leq C\Big\{\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\big(1+(l-l\wedge k)\big)\big(2^{(k-l)\varepsilon}\wedge 1\big)2^{(l-k)\alpha}\big(2^{k\alpha}\big\|\widetilde{\widetilde{D}}_k(f)\big\|_{L^p}\big)^q\Big\}^{1/q} \\
&\quad\leq C\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|\widetilde{\widetilde{D}}_k(f)\big\|_{L^p}\big)^q\Big\}^{1/q} \\
&\quad\leq C\|f\|_{\dot{B}_p^{\alpha,q}},
\end{aligned}$$

where $0 < \alpha < \theta$.

Applying Lemma 2.1, Proposition 2.1, Hölder's inequality and Fefferman–Stein's vector-valued maximal inequality, we also have

$$\begin{aligned}
&\|Tf\|_{\dot F_p^{\alpha,q}}\\
&\quad= \Big\|\Big\{\sum_{l\in\mathbb{Z}}\big(2^{l\alpha}\big|D_l(Tf)\big|\big)^q\Big\}^{1/q}\Big\|_{L^p}\\
&\quad\le \Big\|\Big\{\sum_{l\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}2^{l\alpha}\big|D_lTD_k\widetilde{\widetilde{D}}_k(f)\big|\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\
&\quad\le \Big\|\Big\{\sum_{l\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}2^{l\alpha}\big(1+(l-l\wedge k)\big)\big(2^{(k-l)\varepsilon}\wedge 1\big)M\big(\widetilde{\widetilde{D}}_k(f)\big)\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\
&\quad\le \Big\|\Big\{\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\big(1+(l-l\wedge k)\big)\big(2^{(k-l)\varepsilon}\wedge 1\big)2^{(l-k)\alpha}\big(2^{k\alpha}M\big(\widetilde{\widetilde{D}}_k(f)\big)\big)^q\Big\}^{1/q}\Big\|_{L^p}\\
&\quad\le \Big\|\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}M\big(\widetilde{\widetilde{D}}_k(f)\big)\big)^q\Big\}^{1/q}\Big\|_{L^p}\\
&\quad\le \Big\|\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big|\widetilde{\widetilde{D}}_k(f)\big|\big)^q\Big\}^{1/q}\Big\|_{L^p}\\
&\quad\le C\|f\|_{\dot F_p^{\alpha,q}},
\end{aligned}$$

where $0<\alpha<\theta$, and which concludes the proof of Theorem 1.1. □

Similar to the proof of Proposition 1.1, we can get

PROPOSITION 2.2. *Suppose that T satisfies the hypotheses of Theorem 1.2 and $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Definition 1.3. Then there exists a constant $C>0$ such that*

$$\big|D_lTD_k(x,y)\big|\le C\big(2^{(l-k)\varepsilon}\wedge 1\big)\frac{(1+(k-l\wedge k))}{V_{2^{-(k\wedge l)}}(x)+V(x,y)}\frac{2^{-(k\wedge l)\varepsilon}}{(2^{-(k\wedge l)}+d(x,y))^{\varepsilon}}.$$

Combining Lemma 2.1 with Proposition 2.2, by an analogous argument to Theorem 1.1, it is immediate to obtain Theorem 1.2. Here we leave the details to the interested reader.

3. Proof of Theorem 1.3 and Theorem 1.4

Before we verify Theorem 1.3 and Theorem 1.4, we state the duality properties of the Besov and Triebel–Lizorkin spaces as follows.

LEMMA 3.1 ([5]). *Let $-\theta<\alpha<\theta$ and $1<p,q<\infty$, then*

(i) $(\dot B_p^{\alpha,q}(X))'=\dot B_{p'}^{-\alpha,q'}(X)$;

(ii) $(\dot F_p^{\alpha,q}(X))'=\dot F_{p'}^{-\alpha,q'}(X)$.

We also need Calderón's reproducing formula where the series converges in the norms of the Besov and Triebel–Lizorkin spaces.

Proposition 3.1 ([5]). *Let all notation be the same as in Lemma 2.1, then Calderón's reproducing formula* (10) *converges in both norms of* $\dot{B}_p^{\alpha,q}$ *and* $\dot{F}_p^{\alpha,q}$ *with* $|\alpha|<\theta$ *and* $1<p,q<\infty$.

We also need the following almost orthogonal estimates.

Proposition 3.2. (i) *Suppose that* S_{k_i} *satisfies* (i), (ii) *and* (v) *of Definition* 1.1 *and* $E_{k_i}=S_{k_i}-S_{k_i-1}$ *for* $k_i\in\mathbb{Z}$, $i=1,2$. *Then there exists a constant* $C>0$ *such that*

$$\tag{15} \begin{aligned}&\big|E_{k_1}E_{k_2}(x,y)\big|\\ &\quad\le C\big(2^{(k_2-k_1)\theta}\wedge 1\big)\frac{1}{V_{2^{-(k_1\wedge k_2)}}(x)+V(x,y)}\left(\frac{2^{-(k_1\wedge k_2)}}{2^{-(k_1\wedge k_2)}+d(x,y)}\right)^{\theta}.\end{aligned}$$

(ii) *Suppose that* S_{k_i} *satisfies* (i), (iii) *and* (vi) *of Definition* 1.1 *and* $E_{k_i}=S_{k_i}-S_{k_i-1}$ *for* $k_i\in\mathbb{Z}$, $i=1,2$. *Then there exists a constant* $C>0$ *such that*

$$\tag{16} \begin{aligned}&\big|E_{k_1}E_{k_2}(x,y)\big|\\ &\quad\le C\big(2^{(k_1-k_2)\theta}\wedge 1\big)\frac{1}{V_{2^{-(k_1\wedge k_2)}}(x)+V(x,y)}\left(\frac{2^{-(k_1\wedge k_2)}}{2^{-(k_1\wedge k_2)}+d(x,y)}\right)^{\theta}.\end{aligned}$$

Proof. We only verify (15) and the proof of (16) is similar to (15). When $k_1\ge k_2$, if $d(x,z)<C2^{-k_1}$, $d(z,y)<C2^{-k_2}$ or $d(x,y)<C2^{-k_2}$ implies that $d(x,y)<C2^{-k_2}$. Thus, by the cancellation condition of $E_{k_1}(x,z)$, we have

$$\begin{aligned}&\big|E_{k_1}E_{k_2}(x,y)\big|\\ &\quad=\left|\int E_{k_1}(x,z)\big[E_{k_2}(z,y)-E_{k_2}(x,y)\big]\,d\mu(z)\right|\\ &\quad\le C\frac{2^{-k_2\theta}}{(2^{-k_2}+d(x,y))^{\theta}}\\ &\qquad\times\int_{\{z|d(x,z)<C2^{-k_1}\}}\frac{2^{k_2\theta}}{V_{2^{-k_1}}(x)+V(x,z)}\frac{d(x,z)^{\theta}}{V_{2^{-k_2}}(x)+V(x,y)}\,d\mu(z)\\ &\quad\le C2^{(k_2-k_1)\theta}\frac{1}{V_{2^{-k_2}}(x)+V(x,y)}\frac{2^{-k_2\theta}}{(2^{-k_2}+d(x,y))^{\theta}},\end{aligned}$$

which implies (15) for the case whenever $k_1\ge k_2$. When $k_1<k_2$, we only use the size condition, we also have

$$\begin{aligned}&\big|E_{k_1}E_{k_2}(x,y)\big|\\ &\quad=\left|\int E_{k_1}(x,z)E_{k_2}(z,y)\,d\mu(z)\right|\end{aligned}$$

$$\le C\int_{\{z|d(x,z)<C2^{-k_1},d(y,z)<C2^{-k_2}\}}\frac{1}{V_{2^{-k_1}}(x)}\frac{1}{V_{2^{-k_2}}(x)+V(z,y)}\,d\mu(z)$$
$$\le C\frac{1}{V_{2^{-k_1}}(x)+V(x,y)}\frac{2^{-k_1\theta}}{(2^{-k_1}+d(x,y))^{\theta}},$$

which finishes the proof of (15). □

PROPOSITION 3.3. (i) *Let E_k be the same as in Theorem 1.3 for $k\in\mathbb{Z}$ and $E_k(x,y)$ is the kernel of E_k. Then $E_k(x,y)\in\dot{B}_p^{\alpha,q}$ and $E_k(x,y)\in\dot{F}_p^{\alpha,q}$ for any fixed $x\in X$, $-\theta<\alpha<0$ and $1<p,q<\infty$.*

(ii) *Let E_k be the same as in Theorem 1.4 for $k\in\mathbb{Z}$ and $E_k(x,y)$ is the kernel of E_k. Then $E_k(x,y)\in\dot{B}_p^{\alpha,q}$ and $E_k(x,y)\in\dot{F}_p^{\alpha,q}$ for any fixed $y\in X$, $0<\alpha<\theta$ and $1<p,q<\infty$.*

Proof. We begin with the following estimate: for given $x\in X$ and D_l is the same as in Definition 1.3 for $l\in\mathbb{Z}$, then

$$\begin{aligned}&\left|\int\big[D_l(z,y)-D_l(z,x)\big]E_k(x,y)\,d\mu(y)\right|\\&\qquad\le C2^{(l-k)\theta}\frac{1}{V_{2^{-l}}(z)+V(z,x)}\frac{2^{-l\theta}}{(2^{-l}+d(z,x))^{\theta}},\end{aligned}$$

where $l<k$ and E_k be given in Proposition 3.3 (i). The above inequality follows from (16).

We now return to prove (i). By $\int E_k(x,y)\,d\mu(y)=0$ and the above fact, we have

$$\begin{aligned}&\Big\{\sum_{l<k}\big(2^{l\alpha}\big\|D_l\big(E_k(x,\cdot)\big)\big\|_{L^p}\big)^q\Big\}^{1/q}\\&\quad=C\Big\{\sum_{l<k}\Big(2^{l\alpha}\Big(\int\Big|\int\big[D_l(z,y)-D_l(z,x)\big]E_k(x,y)\,d\mu(y)\Big|^p\,d\mu(z)\Big)^{1/p}\Big)^q\Big\}^{1/q}\\&\quad\le C\Big\{\sum_{l<k}\Big(2^{l\alpha}2^{(l-k)\theta}\\&\qquad\times\Big(\int\Big(\frac{1}{V_{2^{-l}}(z)+V(z,x)}\frac{2^{-l\theta}}{(2^{-l}+d(z,x))^{\theta}}\Big)^p\,d\mu(z)\Big)^{1/p}\Big)^q\Big\}^{1/q}\\&\le C\sum_{l<k}2^{l\alpha}2^{(l-k)\theta}\frac{1}{V_{2^{-l}}(x)^{1-\frac{1}{p}}}\\&\le C\sum_{l<k}2^{(l-k)(\theta+\alpha)}2^{k\alpha}\frac{1}{V_{2^{-k}}(x)^{1-\frac{1}{p}}}\le C_k<\infty,\end{aligned}$$

where $-\theta<\alpha<0$. When $l\ge k$, using the size conditions of $D_l(z,y)$ and $E_k(x,z)$, then

$$\begin{aligned}
&\Big\{\sum_{l\ge k}\big(2^{l\alpha}\big\|D_l\big(E_k(x,\cdot)\big)\big\|_{L^p}\big)^q\Big\}^{1/q}\\
&\quad=C\bigg\{\sum_{l\ge k}\bigg(2^{l\alpha}\bigg(\int\bigg|\int D_l(z,y)E_k(x,y)\,d\mu(y)\bigg|^p d\mu(z)\bigg)^{1/p}\bigg)^q\bigg\}^{1/q}\\
&\quad\le C\bigg\{\sum_{l\ge k}\bigg(2^{l\alpha}\bigg(\int\bigg(\int\frac{1}{V_{2^{-l}}(z)+V(z,y)}\frac{2^{-l\theta}}{(2^{-l}+d(z,y))^\theta}\\
&\qquad\times\frac{1}{V_{2^{-k}}(y)+V(y,x)}\frac{2^{-k\theta}}{(2^{-k}+d(y,x))^\theta}\,d\mu(y)\bigg)^p d\mu(z)\bigg)^{1/p}\bigg)^q\bigg\}^{1/q}\\
&\quad\le C\bigg\{\sum_{l\ge k}\bigg(2^{l\alpha}\bigg(\int\bigg(\frac{1}{V_{2^{-k}}(x)+V(z,x)}\frac{2^{-k\theta}}{(2^{-k}+d(z,x))^\theta}\bigg)^p d\mu(z)\bigg)^{1/p}\bigg)^q\bigg\}^{1/q}\\
&\quad\le C\sum_{l\ge k}2^{l\alpha}\frac{1}{V_{2^{-k}}(x)^{1-\frac1p}}\\
&\quad\le C\sum_{l\ge k}2^{(l-k)\alpha}2^{k\alpha}\frac{1}{V_{2^{-k}}(x)^{1-\frac1p}}\le C_k<\infty,
\end{aligned}$$

where $-\theta<\alpha<0$ and which concludes the proof of $E_k(x,y)\in\dot B^{\alpha,q}_p$ for any fixed $x\in X$. We also can deal with that $E_k(x,y)\in\dot F^{\alpha,q}_p$ for any fixed $x\in X$, $-\theta<\alpha<0$ and $1<p,q<\infty$. Here we omit the details.

To verify (ii), for any fixed $y\in X$, $0<\alpha<\theta$ and $1<p,q<\infty$, we only consider $E_k(x,y)\in\dot F^{\alpha,q}_p$, and $E_k(x,y)\in\dot B^{\alpha,q}_p$ can be handled similarly. When $l\le k$, we get

$$\begin{aligned}
&\bigg\|\Big\{\sum_{l\le k}\big(2^{l\alpha}\big|D_l\big(E_k(\cdot,y)\big)\big|\big)^q\Big\}^{1/q}\bigg\|_{L^p}\\
&\quad=\bigg(\int\bigg\{\sum_{l\le k}\bigg(2^{l\alpha}\bigg|\int D_l(x,z)E_k(z,y)\,d\mu(z)\bigg|\bigg)^q\bigg\}^{p/q}d\mu(x)\bigg)^{1/p}\\
&\quad\le C\sum_{l\le k}2^{l\alpha}\bigg(\int\bigg(\frac{1}{V_{2^{-l}}(x)+V(x,y)}\frac{2^{-l\theta}}{(2^{-l}+d(x,y))^\theta}\bigg)^p d\mu(x)\bigg)^{1/p}\\
&\quad\le C\sum_{l\le k}2^{l\alpha}\frac{1}{V_{2^{-l}}(y)^{1-\frac1p}}\\
&\quad\le C\sum_{l\le k}2^{(l-k)\alpha}2^{k\alpha}\frac{1}{V_{2^{-k}}(y)^{1-\frac1p}}\le C_k<\infty,
\end{aligned}$$

where $0 < \alpha < \theta$. When $l > k$, by an analogous argument to (15), then

$$
\begin{aligned}
(17)\qquad & \left| \int D_l(x,z)\big[E_k(z,y) - E_k(x,y)\big]\, d\mu(z) \right| \\
& \leq C 2^{(k-l)\theta} \frac{1}{V_{2^{-k}}(x) + V(x,y)} \frac{2^{-k\theta}}{(2^{-k} + d(x,y))^{\theta}}.
\end{aligned}
$$

Using the estimate (17), we obtain

$$
\begin{aligned}
& \left\| \left\{ \sum_{l>k} \big(2^{l\alpha} \big| D_l\big(E_k(\cdot,y)\big)\big|\big)^q \right\}^{1/q} \right\|_{L^p} \\
& = \sum_{l>k} 2^{l\alpha} \left(\int \left| \int D_l(x,z)\big[E_k(z,y) - E_k(x,y)\big]\, d\mu(z) \right|^p d\mu(x) \right)^{1/p} \\
& \leq \sum_{l>k} 2^{l\alpha} 2^{(k-l)\theta} \left(\int \left| \frac{1}{V_{2^{-k}}(x) + V(x,y)} \frac{2^{-k\theta}}{(2^{-k} + d(x,y))^{\theta}} \right|^p d\mu(x) \right)^{1/p} \\
& \leq \sum_{l>k} 2^{l\alpha} 2^{(k-l)\theta} \frac{1}{V_{2^{-k}}(y)^{1-\frac{1}{p}}} \\
& \leq C \sum_{l>k} 2^{(k-l)(\theta-\alpha)} 2^{k\alpha} \frac{1}{V_{2^{-k}}(y)^{1-\frac{1}{p}}} \leq C_k < \infty,
\end{aligned}
$$

where $0 < \alpha < \theta$ and the proof of (ii) is finished. □

We now prove the following proposition.

PROPOSITION 3.4. *Let $0 < \alpha < \theta$ and $1 < p, q < \infty$. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity satisfying* (i), (ii) *and* (v) *of Definition* 1.1 *and $E_k = S_k - S_{k-1}$. Then there exists a constant $C > 0$ such that*

$$
(18)\qquad \left\{ \sum_{k\in\mathbb{Z}} \big(2^{k\alpha} \big\| E_k(f) \big\|_{L^p}\big)^q \right\}^{\frac{1}{q}} \leq C \|f\|_{\dot{B}_p^{\alpha,q}},
$$

and

$$
(19)\qquad \left\| \left\{ \sum_{k\in\mathbb{Z}} \big(2^{k\alpha} E_k(f)\big)^q \right\}^{\frac{1}{q}} \right\|_{L^p} \leq C \|f\|_{\dot{F}_p^{\alpha,q}}.
$$

Proof. Suppose that $f \in \dot{B}_p^{\alpha,q}$ with $0 < \alpha < \theta$ and $1 < p, q < \infty$. By Lemma 3.1 and Proposition 3.1, we can write

$$
(20)\qquad E_k(f) = E_k\left(\sum_{l\in\mathbb{Z}} \widetilde{D}_l D_l(f) \right) = \sum_{l\in\mathbb{Z}} E_k \widetilde{D}_l D_l(f).
$$

Applying (20), (15), Minkowski's inequality, Young's inequality and Hölder's inequality, we have

$$
\begin{aligned}
(21)\qquad & \Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}} \\
&\le \Big\{\sum_{k\in\mathbb{Z}}\Big(2^{k\alpha}\sum_{l\in\mathbb{Z}}\big\|E_k\widetilde{D}_lD_l(f)\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big\{\sum_{k\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}}2^{k\alpha}\|E_k\widetilde{D}_l\|_{L^1\to L^1}\big\|D_l(f)\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big\{\sum_{k\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}}\big(2^{(l-k)\theta}\wedge 1\big)2^{(k-l)\alpha}2^{l\alpha}\big\|D_l(f)\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big\{\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\big(2^{(l-k)\theta}\wedge 1\big)2^{(k-l)\alpha}\big(2^{l\alpha}\big\|D_l(f)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big\{\sum_{l\in\mathbb{Z}}\big(2^{l\alpha}\big\|D_l(f)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}} \\
&= C\|f\|_{\dot{B}^{\alpha,q}_p}
\end{aligned}
$$

with $0<\alpha<\theta$. When $f\in\dot{F}^{\alpha,q}_p$ with $0<\alpha<\theta$ and $1<p,q<\infty$, (19) can be dealt with similarly, and which finishes the proof of Proposition 3.4. □

To finish the proof of Theorem 1.3, we need to show the reverse inequalities (18) and (19). For this purpose, we will use Theorem 1.1 and Coifman's idea. To be precisely, let I be the identity operator and E_k be the same as in Theorem 1.3 for $k\in\mathbb{Z}$, then $I=\sum_{k\in\mathbb{Z}}E_k$ in L^2. Applying Coifman's idea, we rewrite

$$
\begin{aligned}
(22)\qquad I &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}E_lE_k = \sum_{k\in\mathbb{Z}}\sum_{|l|\le N}E_{l+k}E_k + \sum_{k\in\mathbb{Z}}\sum_{|l|>N}E_{l+k}E_k \\
&= \sum_{k\in\mathbb{Z}}E_k^NE_k + \sum_{k\in\mathbb{Z}}\sum_{|l|>N}E_{l+k}E_k := T_N + R_N,
\end{aligned}
$$

where $E_k^N=\sum_{|l|\le N}E_{l+k}$.

Similar to the proof of (17), we observe that for given integer N, let E_k and E_l^N be as in (22) for $k,l\in\mathbb{Z}$, then

$$
(23)\qquad \big|E_kE_l^N(f)\big| \le C_N\big(2^{(l-k)\theta}\wedge 1\big)M(f)(x),
$$

where constant C_N depends only on N.

Using (23), by an analogous argument to (21), for $f \in L^2 \cap \dot{B}_p^{\alpha,q}$, we have

$$\begin{aligned}
\|T_N(f)\|_{\dot{B}_p^{\alpha,q}} &= \Big\{\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\big\|D_k\big(T_N(f)\big)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}} \\
&\le \Big\{\sum_{k\in\mathbb{Z}} \Big(2^{k\alpha}\sum_{l\in\mathbb{Z}}\big\|D_kE_l^NE_l(f)\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big\{\sum_{k\in\mathbb{Z}} \Big(\sum_{l\in\mathbb{Z}}\big(2^{(l-k)\theta}\wedge 1\big)2^{k\alpha}\big\|M\big(E_l(f)\big)\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big\{\sum_{l\in\mathbb{Z}}\big(2^{l\alpha}\big\|E_l(f)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}}.
\end{aligned} \tag{24}$$

We first assume that T_N^{-1} exists and is bonded on $\dot{B}_p^{\alpha,q}$ for large integer N. For $f \in L^2 \cap \dot{B}_p^{\alpha,q}$, then

$$\begin{aligned}
\|f\|_{\dot{B}_p^{\alpha,q}} &= \big\|T_N^{-1}T_N(f)\big\|_{\dot{B}_p^{\alpha,q}} \le C\big\|T_N(f)\big\|_{\dot{B}_p^{\alpha,q}} \\
&\le C\Big\{\sum_{l\in\mathbb{Z}}\big(2^{l\alpha}\big\|E_l(f)\big\|_{L^p}\big)^q\Big\}^{\frac{1}{q}}.
\end{aligned} \tag{25}$$

For $f \in \dot{B}_p^{\alpha,q}$ and $\{\sum_{l\in\mathbb{Z}}(2^{l\alpha}\|E_l(f)\|_{L^p})^q\}^{\frac{1}{q}} < \infty$, we can choose a sequence $\{f_n\}_{n=1}^\infty$ with $f_n \in L^2 \cap \dot{B}_p^{\alpha,q}$ such that

$$\lim_{n\to\infty}\|f_n - f\|_{\dot{B}_p^{\alpha,q}} = 0,$$

since $L^2 \cap \dot{B}_p^{\alpha,q}$ is dense in $\dot{B}_p^{\alpha,q}$. Thus, using the above fact and (18), then

$$\begin{aligned}
\|f\|_{\dot{B}_p^{\alpha,q}} = \lim_{n\to\infty}\|f_n\|_{\dot{B}_p^{\alpha,q}} &\le C\lim_{n\to\infty}\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f_n)\big\|_{L^p}\big)^q\Big\}^{1/q} \\
&\le C\lim_{n\to\infty}\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f_n-f)\big\|_{L^p}\big)^q\Big\}^{1/q} \\
&\quad + C\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f)\big\|_{L^p}\big)^q\Big\}^{1/q} \\
&\le C\lim_{n\to\infty}\|f_n - f\|_{\dot{B}_p^{\alpha,q}} + C\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f)\big\|_{L^p}\big)^q\Big\}^{1/q} \\
&= C\Big\{\sum_{k\in\mathbb{Z}}\big(2^{k\alpha}\big\|E_k(f)\big\|_{L^p}\big)^q\Big\}^{1/q},
\end{aligned}$$

This finishes proof of the reverse inequality of (18). We can similarly get the desired result for $f \in \dot{F}_p^{\alpha,q}$.

We now need to verify that T_N^{-1} exists and is bonded on $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$ for a fixed large integer N. By the fact that $T_N^{-1} = (I - R_N)^{-1} = \sum_{m=0}^{\infty} R_N^m$, it suffices to prove that R_N is bounded on $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$ with an operator norm less than 1. We write $R_N = \sum_{|l-k|>N} E_l E_k$, and consider the sums for $k - l > N$ and $l - k > N$, respectively.

Proposition 3.5. *Let E_l be the same as in Theorem 1.3 for $l \in \mathbb{Z}$. Suppose that $0 < \alpha < \theta$ and $1 < p, q < \infty$. Then there exists a constant $C > 0$ such that*

$$\Big\| \sum_{k-l>N} E_l E_k(f) \Big\|_{\dot{B}_p^{\alpha,q}} \le C 2^{-N\alpha} \|f\|_{\dot{B}_p^{\alpha,q}} \tag{26}$$

and

$$\Big\| \sum_{k-l>N} E_l E_k(f) \Big\|_{\dot{F}_p^{\alpha,q}} \le C 2^{-N\alpha} \|f\|_{\dot{F}_p^{\alpha,q}}. \tag{27}$$

Proof. We only prove (26) and the proof of (27) is similar. By the definition of $\dot{B}_p^{\alpha,q}$, Proposition 3.3, Hölder's inequality and the fact that $0 < \alpha < \theta$, then we get

$$\begin{aligned}
&\Big\| \sum_{k-l>N} E_l E_k(f) \Big\|_{\dot{B}_p^{\alpha,q}} \\
&= \Big\{ \sum_{j\in\mathbb{Z}} \Big(2^{j\alpha} \Big\| D_j \Big(\sum_{k-l>N} E_l E_k(f) \Big) \Big\|_{L^p} \Big)^q \Big\}^{\frac{1}{q}} \\
&\le C \Big\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{k-l>N} 2^{j\alpha} \big(2^{(l-j)\theta} \wedge 1 \big) \big\| E_k(f) \big\|_{L^p} \Big)^q \Big\}^{\frac{1}{q}} \\
&= C \Big\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{k-l>N} \big(2^{(l-j)\theta} \wedge 1 \big) 2^{(j-l)\alpha} 2^{(l-k)\alpha} 2^{k\alpha} \big\| E_k(f) \big\|_{L^p} \Big)^q \Big\}^{\frac{1}{q}} \\
&\le C \Big\{ \sum_{j\in\mathbb{Z}} \sum_{k-l>N} \big(2^{(l-j)\theta} \wedge 1 \big) 2^{(j-l)\alpha} 2^{(l-k)\alpha} \big(2^{k\alpha} \big\| E_k(f) \big\|_{L^p} \big)^q \Big\}^{\frac{1}{q}} \\
&\le C 2^{-N\alpha} \Big\{ \sum_{k\in\mathbb{Z}} \big(2^{k\alpha} \big\| E_k(f) \big\|_{L^p} \big)^q \Big\}^{\frac{1}{q}} \\
&\le C 2^{-N\alpha} \|f\|_{\dot{B}_p^{\alpha,q}},
\end{aligned} \tag{28}$$

where the last inequality follows from Proposition 3.4 and this is a desired result. □

Now we prove the following proposition.

PROPOSITION 3.6. *Let E_l be the same as in Theorem 1.3 for $l \in \mathbb{Z}$. Suppose that $0 < \alpha < \theta$ and $1 < p, q < \infty$. Then there exist constants $C, \delta > 0$ such that*

$$\Big\| \sum_{l-k>N} E_l E_k(f) \Big\|_{\dot{B}_p^{\alpha,q}} \le C 2^{-N\delta} \|f\|_{\dot{B}_p^{\alpha,q}} \tag{29}$$

and

$$\Big\| \sum_{l-k>N} E_l E_k(f) \Big\|_{\dot{F}_p^{\alpha,q}} \le C 2^{-N\delta} \|f\|_{\dot{F}_p^{\alpha,q}}. \tag{30}$$

Proof. Let $\widetilde{R}_N = \sum_{l-k>N} E_l E_k$ and $\widetilde{R}_N(x,y)$ be the kernel of $\widetilde{R}_N$. We claim that for $0 < \theta' < \theta$, there exist constants $C, \delta > 0$ such that $\widetilde{R}_N$ satisfies $\widetilde{R}_N(1) = 0$, $\widetilde{R}_N \in WBP$ and the kernel $\widetilde{R}_N(x,y)$ satisfies

$$\big|\widetilde{R}_N(x,y)\big| \le C 2^{-N\delta} \frac{1}{V(x,y)}; \tag{31}$$

$$\big|\widetilde{R}_N(x,y) - \widetilde{R}_N(x',y)\big| \le C 2^{-N\delta} \frac{d(x,x')^{\theta'}}{d(x,y)^{\theta'}} \frac{1}{V(x,y)} \tag{32}$$

for $d(x,x') \le \frac{1}{2A} d(x,y)$.

To show the claim, we rewrite

$$\widetilde{R}_N = \sum_{l-k>N} E_l E_k = \sum_{l>N} \sum_{k\in\mathbb{Z}} E_{k+l} E_k.$$

Let $\widetilde{\widetilde{R}}_N = \sum_{k\in\mathbb{Z}} E_{k+l} E_k$ and $\widetilde{\widetilde{R}}_N(x,y)$ be the kernel of $\widetilde{\widetilde{R}}_N$. Proposition 3.3 and [5, Lemma 3.5] imply

$$\begin{aligned} \big|\widetilde{\widetilde{R}}_N(x,y)\big| &= \Big| \sum_{k\in\mathbb{Z}} \int E_{k+l}(x,z) E_k(z,y)\, d\mu(z) \Big| \\ &\le C \sum_{k\in\mathbb{Z}} 2^{-l\theta} \frac{1}{V_{2^{-k}}(x) + V(x,y)} \frac{2^{-k\theta}}{(2^{-k} + d(x,y))^{\theta}} \\ &\le C 2^{-l\theta} \frac{1}{V(x,y)}. \end{aligned} \tag{33}$$

Thus, we have

$$\big|\widetilde{R}_N(x,y)\big| \le C \sum_{l>N} 2^{-l\theta} \frac{1}{V(x,y)} \le C 2^{-N\theta} \frac{1}{V(x,y)}, \tag{34}$$

which implies (31). When $2^{-l} d(x,y) < d(x,x') \le \frac{1}{2A} d(x,y)$, by (34), we have

$$\big|\widetilde{\widetilde{R}}_N(x,y) - \widetilde{\widetilde{R}}_N(x',y)\big| \le C 2^{-l\theta} \bigg(\frac{1}{V(x,y)} + \frac{1}{V(x',y)} \bigg) \le C 2^{-l\theta} \frac{1}{V(x,y)}. \tag{35}$$

When $d(x,x') \le 2^{-l} d(x,y)$, using the kernel of E_{k+l}, for large N, we get

$$d(x,z) \le C 2^{-(k+l)} \le \frac{1}{2A} 2^{-k}. \tag{36}$$

Thus, we obtain

$$\begin{aligned}
&\big|\widetilde{\widetilde{R}}_N(x,y) - \widetilde{\widetilde{R}}_N(x',y)\big| \\
&\le \sum_{k\in\mathbb{Z}} \left| \int \big[E_{k+l}(x,z) - E_{k+l}(x',z)\big] E_k(z,y)\, d\mu(z) \right| \\
&= \sum_{k\in\mathbb{Z}} \left| \int \big[E_{k+l}(x,z) - E_{k+l}(x',z)\big] \big[E_k(z,y) - E_k(x,y)\big]\, d\mu(z) \right| \\
&\le C \sum_{k\in\mathbb{Z}} \frac{1}{V_{2^{-k}}(x) + V(x,y)} \frac{2^{-k\theta}}{(2^{-k} + d(x,y))^\theta} \int \frac{d(x,x')^\theta}{(2^{-(k+l)} + d(x,z))^\theta} \\
&\quad \times \frac{1}{V_{2^{-(k+l)}}(x) + V(x,z)} \frac{2^{-(k+l)\theta}}{(2^{-(k+l)}) + d(x,z)^\theta} \left(\frac{d(x,z)}{2^{-k} + d(x,y)} \right)^\theta d\mu(z) \\
&\le C \sum_{k\in\mathbb{Z}} \frac{1}{V_{2^{-k}}(x) + V(x,y)} \frac{2^{-k\theta}}{(2^{-k} + d(x,y))^\theta} d(x,x')^\theta \frac{1}{d(x,y)^\theta} \\
&\le C \frac{d(x,x')^\theta}{d(x,y)^\theta} \frac{1}{V(x,y)},
\end{aligned} \tag{37}$$

where we use the fact $E_{k+l}(1) = 0$. By the geometric mean of (35) and (37), we can get

$$\big|\widetilde{\widetilde{R}}_N(x,y) - \widetilde{\widetilde{R}}_N(x',y)\big| \le C 2^{-l\delta} \frac{d(x,x')^{\theta'}}{d(x,y)^{\theta'}} \frac{1}{V(x,y)}$$

with $0 < \theta' < \theta$ and $\delta = \theta - \theta'$. Thus when $d(x,x') \le \frac{1}{2A} d(x,y)$, we have

$$\begin{aligned}
\big|\widetilde{R}_N(x,y) - \widetilde{R}_N(x',y)\big| &\le C \sum_{l>N} 2^{-l\delta} \frac{d(x,x')^{\theta'}}{d(x,y)^{\theta'}} \frac{1}{V(x,y)} \\
&\le C 2^{-N\delta} \frac{d(x,x')^{\theta'}}{d(x,y)^{\theta'}} \frac{1}{V(x,y)}.
\end{aligned}$$

Obviously, $\widetilde{R}_N(1) = 0$. Fix two functions ϕ and ψ with supp ϕ, supp $\psi \in B(x_0, r)$ for $x_0 \in X$ and $r > 0$, $\|\phi\|_\infty \le 1$, $\|\psi\|_\infty \le 1$, $\|\phi\|_{\mathcal{C}_0^\eta} \le r^{-\eta}$ and $\|\psi\|_{\mathcal{C}_0^\eta} \le r^{-\eta}$. We have

$$\begin{aligned}
&\big|\langle \widetilde{R}_N \phi, \psi \rangle\big| \\
&\le \sum_{l>N} \sum_{k\in\mathbb{Z}} \left| \int\int\int E_{k+l}(x,z) E_k(z,y) \phi(y) \psi(x)\, d\mu(z)\, d\mu(y)\, d\mu(x) \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{l>N}\sum_{2^{-k}>r}\Big|\int\int\int E_{k+l}(x,z)\big[E_k(z,y)-E_k(x,y)\big] \\
&\quad \times \phi(y)\psi(x)\,d\mu(z)\,d\mu(y)\,d\mu(x)\Big| \\
&\quad + \sum_{l>N}\sum_{2^{-k}\le r}\Big|\int\int\int E_{k+l}(x,z)\big[E_k(z,y)-E_k(x,y)\big]\big[\phi(y)-\phi(x)\big] \\
&\quad \times \psi(x)\,d\mu(z)\,d\mu(y)\,d\mu(x)\Big| \\
&:= I_1 + I_2.
\end{aligned}
$$

In order to estimate I_1, since supp $\psi \subset B(x_0,r)$, we have $d(x,x_0)\le r<2^{-k}$, and then

$$V_{2^{-k}}(x) \sim V_{2^{-k}}(x_0).$$

Thus, we get

$$
\begin{aligned}
(38)\quad I_1 &\le C\sum_{l>N}\sum_{2^{-k}>r}\int\int\int \big|E_{k+l}(x,z)\big|\frac{d(x,z)^\theta}{(2^{-k}+d(x,y))^\theta} \\
&\quad \times \frac{1}{V_{2^{-k}}(x)+V(x,y)}\frac{2^{-k\theta}}{(2^{-k}+d(x,y))^\theta}\big|\phi(y)\big|\big|\psi(x)\big|\,d\mu(z)\,d\mu(y)\,d\mu(x) \\
&\le C\sum_{l>N}\sum_{2^{-k}>r} 2^{-(k+l)\theta}2^{k\theta}\frac{1}{V_{2^{-k}}(x_0)}\big[V_r(x_0)\big]^2 \\
&\le C2^{-N\theta}V_r(x_0).
\end{aligned}
$$

For I_2, by the fact $d(x,y)\le C2^{-k}$ and (36), we obtain

$$
\begin{aligned}
(39)\quad I_2 &\le C\sum_{l>N}\sum_{2^{-k}\le r}\int\int\int \big|E_{k+l}(x,z)\big|\frac{d(x,z)^\theta}{(2^{-k}+d(x,y))^\theta} \\
&\quad \times \frac{1}{V_{2^{-k}}(x)+V(x,y)}\frac{2^{-k\varepsilon}}{(2^{-k}+d(x,y))^\theta} \\
&\quad \times d(x,y)^\eta\|\phi\|_{C_0^\eta}\big|\psi(x)\big|\,d\mu(z)\,d\mu(y)\,d\mu(x) \\
&\le C\sum_{l>N}\sum_{2^{-k}\le r} 2^{-(k+l)\theta}2^{k\theta}2^{-k\eta}r^{-\eta}V_r(x_0) \\
&\le C2^{-N\theta}V_r(x_0).
\end{aligned}
$$

Combining (38) with (39), then

$$\big|\langle \widetilde{R}_N\phi,\psi\rangle\big| \le C2^{-N\theta}V_r(x_0),$$

which shows $\widetilde{R}_N \in WBP$. By Theorem 1.1, for all $f \in \dot{B}_p^{\alpha,q}$, then

$$\Big\| \sum_{l-k>N} E_l E_k(f) \Big\|_{\dot{B}_p^{\alpha,q}} \le C 2^{-N\delta} \|f\|_{\dot{B}_p^{\alpha,q}}.$$

For all $f \in \dot{F}_p^{\alpha,q}$, we also have

$$\Big\| \sum_{l-k>N} E_l E_k(f) \Big\|_{\dot{F}_p^{\alpha,q}} \le C 2^{-N\delta} \|f\|_{\dot{F}_p^{\alpha,q}}.$$

We finish the proof of Proposition 3.6. □

From Lemma 3.1, Propositions 3.1, 3.2 and 3.3, the proof of Theorem 1.4 is similar to Theorem 1.3 with necessary modification. We leave the details to the interested reader.

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