Illinois Journal of Mathematics
Volume 59, Number 1, Spring 2015, Pages 85–98
S 0019-2082

INVARIANT BASIS NUMBER FOR C^*-ALGEBRAS

PHILIP M. GIPSON

ABSTRACT. We develop the ring-theoretic notion of Invariant Basis Number in the context of unital C^*-algebras and their Hilbert C^*-modules. Characterization of C^*-algebras with Invariant Basis Number is given in K-theoretic terms, closure properties of the class of C^*-algebras with Invariant Basis Number are given, and examples of C^*-algebras both with and without the property are explored. For C^*-algebras without Invariant Basis Number, we determine structure in terms of a "Basis Type" and describe a class of C^*-algebras which are universal in an appropriate sense. We conclude by investigating properties which are strictly stronger than Invariant Basis Number.

1. Introduction

Leavitt [8], [9] investigated unital rings R with the property that any free module X over R has a fixed basis size. Rings with this property are said to have Invariant Basis Number and examples of such include commutative and Noetherian rings. Leavitt characterizes [9, Corollary 1] rings with Invariant Basis Number in the following manner: a ring R has Invariant Basis Number if and only if there exists another ring R' with Invariant Basis Number and a unital homomorphism $\phi : R \to R'$. For rings without Invariant Basis Number, Leavitt assigns [9, Theorem 1] a pair of positive integers he terms the "module type" of the ring. Constructions [7], [8], [9] of rings, termed Leavitt algebras $L_K(m,n)$, with arbitrary module type are given.

The fundamental structure of the Leavitt algebras has appeared in some surprising contexts. The algebra $L_K(1,n)$ given by Leavitt [9, Section 3] is the purely algebraic analogue of the Cuntz C^*-algebra $\mathcal{O}_n$ and pre-dates Cuntz's investigations. Indeed, the close connection between Leavitt algebras and

Received February 5, 2015; received in final form September 13, 2015.
2010 *Mathematics Subject Classification.* Primary 46L05. Secondary 46L08, 46L80, 16D70.

Cuntz algebras inspired the formulation of Leavitt Path Algebras associated to graphs, which act as analogues to graph C^*-algebras. General Leavitt algebras $L_K(m,n)$ have been investigated by Ara and Goodearl [3] in the context of "separated" Leavitt Path Algebras. Several C^*-algebraic versions of the Leavitt algebras $L_K(m,n)$ have been recently used in the work of Ara and Exel [1], [2] related to dynamical systems.

In this paper, we will formulate the property of Invariant Basis Number in the context of C^*-algebras and their Hilbert C^*-modules. Using K-theoretic tools, we are able to formulate an improved characterization of C^*-algebras with Invariant Basis Number in Theorem 3.2. We reproduce in Theorem 4.1 Leavitt's type-classification for C^*-algebras without Invariant Basis Number and prove in Theorem 5.1 that each Basis Type is possible for some C^*-algebra. In Section 5, we determine that the C^*-algebras $U^{nc}_{m,n}$ studied by McCLanahan [10] are universal objects for C^*-algebras without Invariant Basis Number and, as such, are the correct analogue of the Leavitt algebras $L_K(m,n)$. Finally, we will investigate several stronger variations of Invariant Basis Number as proposed in the purely algebraic case by Cohn [4].

2. C^*-module preliminaries

We will always assume our C^*-algebras to be unital and denote the unit by 1 or 1_A. A *C^*-module* X over a C^*-algebra A (more briefly, an *A-module*) is a complex vector space which is a right A-module and is equipped with an A-valued inner-product $\langle\cdot,\cdot\rangle : X \times X \to A$ which is A-linear in the second coordinate and A-adjoint-linear in the first coordinate. If X is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|_A^{\frac{1}{2}}$ then it is termed a *Hilbert A-module*. We will use Wegge-Olsen [16, Chapter 15] as a reference for basic Hilbert C^*-module results.

The space of adjointable A-module homomorphisms between two A-modules X and Y will be denoted $L(X,Y)$. An adjointable homomorphism ϕ is *unitary* if it is bijective and isometric, that is, $\langle x,x'\rangle_X = \langle\phi(x),\phi(x')\rangle_Y$ for all $x,x' \in X$. We will say that X and Y are *unitarily equivalent*, and write $X \simeq Y$, if there exists a unitary in $L(X,Y)$.

An A-module X is *algebraically finitely generated* if there exist $x_1,\dots,x_n \in X$ such that $X = \mathrm{span}_A(x_1,\dots,x_n)$. We will never consider the weaker notion of topological finite generation, and so will omit the term "algebraically" in the remainder. An A-module X is *projective* if it is a direct summand of a free A-module. It is a known result ([16, Theorem 15.4.2] for example) that a finitely generated projective A-module is isomorphic (as an A-module) to a Hilbert A-module. Further, the finitely generated projective Hilbert A-modules are all of the form pA^n for some $n \geq 1$ and some matrix projection $p \in M_n(A)$.

We will denote the set of projections in $M_n(A)$ by $P_n(A)$. For $p \in P_n(A)$ and $q \in P_m(A)$ we will set $p \oplus q = \operatorname{diag}(p,q) \in P_{n+m}(A)$. We will say p and q are *stably equivalent* if there is a matrix projection r for which $p \oplus r \sim q \oplus r$, where "$\sim$" denotes (Murray–von Neumann) equivalence in $P_\infty(A) = \bigcup_{n=1}^\infty P_n(A)$. The stable equivalence class of p will be denoted $[p]_0$ and considered as an element of the group $K_0(A)$. The (additive) order of an element $[p]_0 \in K_0(A)$ will be denoted $|[p]_0|_{K_0(A)}$ or $|[p]_0|$ if the C^*-algebra A is clear from context.

3. Invariant Basis Number

Let A be a unital C^*-algebra. The *finitely generated free A-module of rank n* is $A^n := A \oplus \cdots \oplus A$ where there are n summands. The action of A on A^n is coordinate-wise multiplication on the right and the inner-product is given by $\langle (a_1,\dots,a_n),(b_1,\dots,b_n)\rangle = a_1^*b_1 + \cdots + a_n^*b_n$. Although we write them as tuples, that is, row vectors, it is often beneficial to view elements of A^n instead as column vectors. The coordinate projections $\pi_i : A^n \to A$ defined by $\pi_i(a_1,\dots,a_n) = a_i$ are bounded, contractive, adjointable A-module homomorphisms. Therefore, a Cauchy sequence in A^n is Cauchy in each coordinate and hence, as A itself is complete, converges in each coordinate. Thus A^n is a complete (i.e. Hilbert) A-module. In keeping with the literature, free Hilbert A-modules will henceforth be referred to as *standard A-modules*, where the completeness is understood.

The fundamental question we will consider is this: under what conditions are the standard modules distinct from one another? We will make this notion of distinctness precise with the next definition.

Definition 3.1. A C^*-algebra A has *Invariant Basis Number* (hereafter, has IBN) if

$$A^n \simeq A^m \quad \Leftrightarrow \quad n = m.$$

Unitary equivalence is, in general, a stronger condition than A-module isomorphism. In fact, unitaries are precisely the *isometric* A-module isomorphisms. However, in the case of standard modules every A-module homomorphism $\phi : A^n \to A^m$ may be represented as a $m \times n$ matrix with elements in A and so is automatically adjointable. Therefore, if $\phi : A^n \to A^m$ is an A-module isomorphism then the Polar Decomposition [16, Theorem 15.3.7] yields a unitary in $L(A^n, A^m)$. We have formulated the definition in terms of unitary equivalence, rather than module isomorphism, to emphasize the Hilbert structure of the standard modules.

A matrix $U \in M_{m,n}(A)$ will be termed a *unitary* if $UU^* = I_n$ and $U^*U = I_m$. As noted above, we may identify $L(A^n, A^m)$ with $M_{m,n}(A)$ and a unitary homomorphism in $L(A^n, A^m)$ corresponds to a unitary matrix in $M_{m,n}(A)$. The definition of Invariant Basis Number may thus be rephrased as follows: A has IBN if and only if every unitary matrix over A is square.

Example. It is not hard to verify that a matrix with entries in an commutative algebra is invertible if and only if it is square. Hence, commutative C^*-algebras have Invariant Basis Number.

The connection between matrices and Invariant Basis Number gives our first main result.

Theorem 3.2. *A C^*-algebra A has* IBN *if and only if the group element $[1_A]_0 \in K_0(A)$ has infinite order.*

Proof. If A does not have IBN, then $A^n \simeq A^m$ for some $n > m > 0$ and hence there is a unitary matrix in $M_{m,n}(A)$. This unitary implements the (Murray–von Neumann) matrix equivalence of the projections I_m and I_n and consequently we have

$$I_{n-m} \oplus I_m \sim I_n \sim I_m \sim 0 \oplus I_m.$$

Thus, I_{n-m} is stably equivalent to 0, that is, $(n-m)[1_A]_0 = [I_{n-m}]_0 = 0$, and so $[1_A]_0$ has finite order.

Conversely, if $[1_A]_0$ has finite order k then I_k is stably equivalent to 0, that is, there exists $N > 0$ and $p \in P_N(A)$ such that $p \oplus I_k \sim p \oplus 0 \sim p$. As $I_N \sim p \oplus (I_N - p)$ we have

$$I_N \oplus I_k \sim (I_N - p) \oplus p \oplus I_k \sim (I_N - p) \oplus p \sim I_N$$

and so $I_{N+k} \sim I_N$. The matrix implementing this equivalence is unitary and thus corresponds to a unitary homomorphism from A^N to A^{N+k}. Since $k > 0$ we must conclude that A does not have IBN. □

It is hinted in the above proof that when a C^*-algebra does not have IBN the order of $[1_A]_0$ yields information about equivalence of standard modules. We shall make this connection clear in Section 4 when we turn our attention fully to C^*-algebras without IBN.

The K-theoretic description of IBN immediately expands the class of C^*-algebras with that property beyond the commutative. In particular, it is well-known (see [14], for example) that stably-finite C^*-algebras, that is, those without any proper matrix isometries, have a totally ordered K_0 group. Further, in this case the element $[1_A]_0$ is an *order unit* for K_0 in the sense that for any $g \in K_0$ there is a positive integer k for which $-k[1_A]_0 < g < k[1_A]$. It follows that $[1_A]_0$ cannot have a finite order and, applying Theorem 3.2, we conclude that a stably-finite C^*-algebra must have IBN. We would like to remark that this could also be inferred from the matricial description of IBN, as any rectangular unitary could be "cut down" to a square proper isometry.

The functorial properties of K_0 also yield the following result which will be used extensively to demonstrate closure properties for the class of C^*-algebras with IBN.

Proposition 3.3. *A C^*-algebra A has IBN if and only if there exists a C^*-algebra B which has IBN and a unital $*$-homomorphism $\phi : A \to B$.*

Proof. Necessity is easily satisfied by letting $B = A$ and $\phi = id_A$.

To show sufficiency, we note that the functorial properties of K_0 induce a group homomorphism $K_0(\phi) : K_0(A) \to K_0(B)$. Since ϕ is unital we have $K_0(\phi)[1_A]_0 = [1_B]_0$. If B has IBN then $[1_B]_0$ has infinite order in $K_0(B)$ and so its preimage $[1_A]_0$ must have infinite order in $K_0(A)$. Thus A has IBN. □

The above statement mirrors the purely algebraic characterization of rings with IBN given by Leavitt [9, Corollary 1].

The proposition has immediate consequences for the closure properties of the class of C^*-algebras with Invariant Basis Number.

Corollary 3.4. IBN *is preserved under direct sums and unital extensions.*

Proof. Suppose that A is a C^*-algebra with IBN. If B is a unital C^*-algebra then the coordinate map $a \oplus b \mapsto a$ is a unital $*$-homomorphism and thus $A \oplus B$ has IBN.

If B is any unital extension of A, then there exists a C^*-algebra C and a short exact sequence

$$0 \to C \to B \xrightarrow{\phi} A \to 0.$$

Of course ϕ is a surjective $*$-homomorphism, hence is unital, and thus B has IBN. □

Note that a direct sum inherits IBN even if only one of the summands has that property. We conclude our discussion of C^*-algebras with IBN by leveraging the results to find non-commutative, non-stably-finite C^*-algebras which have IBN.

Example. Consider the Cuntz algebra $\mathcal{O}_\infty$, the universal C^*-algebra generated by a countable family of isometries with pairwise disjoint ranges. Since $\mathcal{O}_\infty$ contains proper isometries it is certainly neither commutative nor (stably) finite. However, it is a classical result of Cuntz [5, Corollary 3.11] that $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and is generated by $[1]_0$. Thus by, Theorem 3.2, $\mathcal{O}_\infty$ has IBN.

Example. On the opposite end of the spectrum, consider the Toeplitz algebra $\mathcal{T}$, the universal C^*-algebra generated by a single non-unitary isometry. Of course $\mathcal{T}$ is neither commutative nor (stably) finite but is well known to be an extension of the commutative C^*-algebra $C(\mathbb{T})$ by the compact operators $\mathcal{K}$. Thus by Corollary 3.4, $\mathcal{T}$ has IBN.

3.1. A remark on the non-unital case. It is a perfectly legitimate criticism that we are dealing solely with unital C^*-algebras. Let us briefly describe why we wish to avoid the nonunital case.

Suppose that A is a nonunital C^*-algebra. Unlike in the unital case, the adjointable A-module homomorphisms in $L(A^n, A^m)$ are not identified with

$M_{m,n}(A)$, but rather with $m \times n$ matrices over the *multiplier algebra* of A, which we'll denote by $\mathcal{M}(A)$. Of course $\mathcal{M}(A)$ is, practically by definition, unital. The unitary equivalence $A^n \simeq A^m$ thus implies the existence of a unitary matrix in $M_{m,n}(\mathcal{M}(A))$ and so $\mathcal{M}(A)^n \simeq \mathcal{M}(A)^m$. It is not hard to see that the logic is reversible and so $A^n \simeq A^m$ if and only if $\mathcal{M}(A)^n \simeq \mathcal{M}(A)^m$.

As a consequence of the above reasoning, we see that the statement "$A^n \simeq A^m$ if and only if $n = m$" is equivalent to "$\mathcal{M}(A)$ has IBN." This is what we believe should be the working definition of IBN for nonunital C^*-algebras. In fact, since $\mathcal{M}(A) = A$ when A is unital, it agrees with our unital definition.

Unfortunately, we do not feel this definition to be particularly useful. First, many nice properties of a C^*-algebra are not preserved in it's multiplier algebra. Seperability being a prime example. Second, we do not know of a method, outside a very few special cases, to detect information about $K_0(\mathcal{M}(A))$ based on information about A. Since our main tools are K-theoretic this is a major stumbling block.

4. C^*-algebras without Invariant Basis Number

We now turn our attention to those unital C^*-algebras which lack the Invariant Basis Number property. By Theorem 3.2, we may conclude that C^*-algebras A without IBN are characterized by having a finite order for the element $[1_A]_0 \in K_0(A)$. A particularly tractable case is when $[1_A]_0$ has order 1, i.e. is the zero element of $K_0(A)$.

Example. When H is an infinite dimensional Hilbert space $B(H)$ does not have IBN because $K_0(B(H)) = \{0\}$.

Example. The Cuntz algebra $\mathcal{O}_2$ is the universal C^*-algebra generated by two isometries v_1 and v_2 satisfying $v_1v_1^* + v_2v_2^* = 1$ and $v_1^*v_2 = v_2^*v_1 = 0$. A result of Cuntz [5, Theorem 3.7] is that $K_0(\mathcal{O}_2) = \{0\}$ and so $\mathcal{O}_2$ does not have IBN. In fact, we can concretely see the equivalence $\mathcal{O}_2 \simeq \mathcal{O}_2^2$ via the map $(a, b) \mapsto v_1a + v_2b$ which extends to a unitary homomorphism and corresponds to the 1×2 unitary matrix $[v_1v_2]$.

Example. For a slightly less trivial example, consider the Cuntz algebra $\mathcal{O}_3$. We have that $K_0(\mathcal{O}_3) = \mathbb{Z}/2\mathbb{Z}$ and is in fact generated by $[1]_0$. Thus $\mathcal{O}_3$ does not have IBN. Much like for $\mathcal{O}_2$ we can in fact write down a 1×3 unitary matrix $[v_1\ v_2\ v_3]$ which gives the unitary equivalence $\mathcal{O}_3 \simeq \mathcal{O}_3^3$. Of course in general we have $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$ and so no Cuntz algebra has IBN.

Recalling the definition of Invariant Basis Number, a C^*-algebra lacks IBN precisely when two or more standard modules with differing ranks are equivalent. The restrictions on when such equivalence may occur give some structural information for C^*-algebras without IBN. The precise nature of that information is contained in our next main result.

THEOREM 4.1. *If A is a C^*-algebra without* IBN, *then there exists a unique largest positive integer N and a unique smallest positive integer K satisfying*:

(1) *if $n, m \geq 1$, $n < N$, and $A^n \simeq A^m$ then $n = m$, and*
(2) *if $n, m \geq 1$ and $A^n \simeq A^m$ then $(n - m) \equiv 0 \mod K$.*

This result is comparable to [9, Theorem 1]. The first condition characterizes N as the least rank for which distinctness of the standard A-modules fails: all standard A-modules of rank less than N are distinct. The second condition characterizes K as the minimum "jump" in rank possible between equivalent standard A-modules.

DEFINITION 4.2. If A is a C^*-algebra without IBN, then the pair (N, K) given by Theorem 4.1 is the *Basis Type* of A. For notational purposes we may write $\mathrm{type}(A) = (N, K)$ or (N_A, K_A).

Proof of Theorem 4.1. Since A does not have IBN there are at least two distinct ranks n, m for which $A^n \simeq A^m$. In particular, the set $E := \{j \geq 0 : \exists k \neq j \text{ s.t. } A^j \simeq A^k\}$ is nonempty and so $N := \min\{n : n \in E\}$ is well defined. If $n < N$ then $n \notin E$ and so $A^n \simeq A^m$ only if $m = n$. So our choice of N satisfies the first condition. That our N is the largest possible is immediate, since if $N' > N$ then there is at least one rank (N itself) less than N' for which the first condition does not hold.

Let N be as above and define $K = \min\{k > 0 : A^N \simeq A^{N+k}\}$, which exists by our choice of N. Note that for any $n \geq N + K$ we have

$$A^n = A^{n-N-K+N+K} \simeq A^{n-N-K} \oplus A^{N+K} \simeq A^{n-N-K} \oplus A^N \simeq A^{n-K}.$$

Through iteration of this process, we obtain an integer n' satisfying $N \leq n' < N + K$, $n' \equiv n \mod K$, and $A^{n'} \simeq A^n$. Because of this, it is enough to check a simpler version of the second condition: if $A^n \simeq A^m$ for $N \leq n, m < N + K$ then $n = m$. (Note this will guarantee the minimality of K.) Suppose that n, m are two ranks satisfying the simplified hypothesis but with $m > n$. Then

$$A^N \simeq A^{N+K} \simeq A^{N+K-m} \oplus A^m \simeq A^{N+K-m} \oplus A^n \simeq A^{N+K-(m-n)}$$

and, as $K - (m - n) < K$, we have contradicted the minimality of K. $\square$

The Basis Type of a C^*-algebra determines the equivalences of standard modules. In particular, if $\mathrm{type}(A) = (N, K)$ then there are precisely $N + K$ unitary equivalence classes of standard modules: the distinct ones of rank less than N and the K classes for ranks $N, N + 1, \ldots, N + K - 1$.

EXAMPLE. Revisiting the examples from the beginning of the section, we find that $B(H)$ and $\mathcal{O}_2$ both have Basis Type $(1, 1)$. The Cuntz algebra $\mathcal{O}_3$ is of Basis Type $(1, 2)$ since (as may be checked) $\mathcal{O}_3 \not\simeq \mathcal{O}_3^2$ but $\mathcal{O}_3 \simeq \mathcal{O}_3^3$.

Recalling that $K_0(\mathcal{O}_2) = K_0(B(H)) = 0$ while $K_0(\mathcal{O}_3) = \mathbb{Z}/2\mathbb{Z}$ the following proposition is perhaps unsurprising.

Proposition 4.3. *If A is a C^*-algebra with Basis Type (N,K), then the order of $[1_A]_0$ in $K_0(A)$ is equal to K.*

Proof. Since A does not have IBN the element $[1_A]_0$ must have some finite order J. Since $A^N \simeq A^{N+K}$ by definition of the Basis Type we conclude that I_N and I_{N+K} are (Murray–von Neumann) equivalent matrix projections; consequently we have $K[1_A]_0 = [I_K]_0 = 0$ in $K_0(A)$ and thus $K \equiv 0 \mod J$. Re-examination of the proof for Theorem 3.2 yields that as $J[1_A]_0 = 0$ there exists some M such that $I_{M+J} \sim I_M$, i.e. $A^M \simeq A^{M+J}$. Thus, by definition of K, we have $J \equiv 0 \mod K$. We must then conclude that $J = K$, as desired. □

Following Leavitt [9, Section 2], we will give the Basis Types a lattice structure as follows:

$$(N_1,K_1) \le (N_2,K_2) \quad \Leftrightarrow \quad N_1 \le N_2 \text{ and } K_2 \equiv 0 \mod K_1,$$
$$(N_1,K_1) \vee (N_2,K_2) = \big(\max(N_1,N_2), \operatorname{lcm}(K_1,K_2)\big),$$
$$(N_1,K_1) \wedge (N_2,K_2) = (\min(N_1,N_2), \gcd(K_1,K_2).$$

We are able to relate this lattice structure to various algebraic operations primarily through the following proposition.

Proposition 4.4. *Let A and B be C^*-algebras, A without* IBN, *and $\phi: A \to B$ a unital $*$-homomorphism. Then B is without* IBN *and* $\operatorname{type}(B) \le \operatorname{type}(A)$.

Proof. Note that by Proposition 3.3 B cannot have IBN. Let $\operatorname{type}(A) = (N_A, K_A)$ and $\operatorname{type}(B) = (N_B, K_B)$. The functoriality of K_0 induces a group homomorphism $K_0(\phi): K_0(A) \to K_0(B)$ which takes $[1_A]_0$ to $[1_B]_0$. Being a group homomorphism, it follows that the order of $K_0(\phi)[1_A]_0 \in K_0(B)$ must divide the order of $[1_A]_0 \in K_0(A)$. We thus have

$$\big|[1_A]_0\big|_{K_0(A)} \equiv 0 \mod \big|[1_B]_0\big|_{K_0(B)}$$

which combines with Proposition 4.3 to give us $K_A \equiv 0 \mod K_B$.

We may ampliate ϕ to $\phi^{(m,n)}: M_{m,n}(A) \to M_{m,n}(B)$ by applying ϕ entry-wise. Since ϕ is unital any unitary matrix in $M_{m,n}(A)$ is sent, via $\phi^{(m,n)}$, to a unitary matrix in $M_{m,n}(B)$. Thus if $A^n \simeq A^m$ then so too $B^n \simeq B^m$; in particular we have $B^{N_A} \simeq B^{N_A+K_A}$. By construction (see Theorem 4.1) $N_B = \min\{n : \exists j \ne n \text{ s.t. } B^n \simeq B^j\}$ and so we conclude that $N_B \le N_A$. □

The primary utility of the previous proposition is to prove various closure properties of the class of C^*-algebras without IBN.

Corollary 4.5. *If A does not have* IBN *and B is a quotient of A, then B does not have* IBN.

This is Proposition 4.4 applied to the quotient map.

Corollary 4.6. *If A and B are C^*-algebras without* IBN *then* $\mathrm{type}(A \oplus B) = \mathrm{type}(A) \vee \mathrm{type}(B)$.

Proof. Proposition 4.4 applied to the coordinate projections $(a, b) \mapsto a$ and $(a, b) \mapsto b$ has us conclude that $\mathrm{type}(A) \leq \mathrm{type}(A \oplus B)$ and $\mathrm{type}(B) \leq \mathrm{type}(A \oplus B)$ and so $\mathrm{type}(A) \vee \mathrm{type}(B) \leq \mathrm{type}(A \oplus B)$.

As $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$ we use Proposition 4.3 to conclude that $K_{A\oplus B} = \mathrm{lcm}(K_A, K_B)$.

Suppose, without loss of generality, that $\max(N_A, N_B) = N_A$. With $K_{A\oplus B} = \mathrm{lcm}(K_A, K_B)$, we have

$$A^{N_A} \simeq A^{N_A+K_A} \simeq A^{N_A+2K_A} \simeq \cdots \simeq A^{N_A+K_{A\oplus B}}$$

and, as $B^{N_A} \simeq B^{N_A-N_B} \oplus B^{N_B} \simeq B^{N_A-N_B} \oplus B^{N_B+K_B} \simeq B^{N_A+K_B}$, we have also

$$B^{N_A} \simeq B^{N_A+K_B} \simeq B^{N_A+2K_B} \simeq \cdots \simeq B^{N_A+K_{A\oplus B}}.$$

Consequently

$$(A \oplus B)^{N_A} = A^{N_A} \oplus B^{N_A} \simeq A^{N_A+K_{A\oplus B}} \oplus B^{N_A+K_{A\oplus B}} \simeq (A \oplus B)^{N_A+K_{A\oplus B}}.$$

We conclude that $N_{A\oplus B} \leq N_A = \max(N_A, N_B)$. As $\mathrm{type}(A) \wedge \mathrm{type}(B) \leq \mathrm{type}(A \oplus B)$, that is, $\max(N_A, N_B) \leq N_{A\oplus B}$, we have equality.

In conclusion, $N_{A\oplus B} = \max(N_A, N_B)$ and $K_{A\oplus B} = \mathrm{lcm}(K_A, K_B)$ and so $\mathrm{type}(A \oplus B) = \mathrm{type}(A) \vee \mathrm{type}(B)$. □

In contrast to Corollary 3.4, it is quite necessary that neither summand of $A \oplus B$ has IBN. It is natural to suspect that the remaining lattice operation will correspond to tensor products.

Corollary 4.7. *If A and B are C^*-algebras without* IBN, *then* $\mathrm{type}(A \otimes B) \leq \mathrm{type}(A) \wedge \mathrm{type}(B)$.

The proof of this corollary is nothing but Proposition 4.4 applied to the embeddings $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$. Two remarks are in order: first, that the result holds for any norm structure we may place on $A \otimes B$; second, that it is unknown (even, to our knowledge, in the purely algebraic case) whether inequality ever occurs.

Corollary 4.8. *If $\{A_i, \phi_i\}$ is an inductive system of C^*-algebras, each without* IBN, *and each ϕ_i is unital, then the direct limit C^*-algebra A of the system does not have IBN.*

The proof of this corollary is Proposition 4.4 applied to the universal maps $\psi_i : A_i \to A$, which are unital.

Finally, we will demonstrate that the class of C^*-algebras without Invariant Basis Number is unfortunately *not* closed under Morita equivalence. A good reference for the theory of Morita equivalence is [11]. Our motivating example is the algebra $\mathcal{O}_\infty$ and the fact that the identity of a corner C^*-algebra $p\mathcal{O}_\infty p$ is the projection p.

PROPOSITION 4.9. *Let A be a infinite simple unital C^*-algebra, then there is a C^*-algebra B Morita equivalent to A which does not have* IBN.

Proof. If A is infinite, then there exists a proper isometry $v \in A$. As $vv^* \sim v^*v = 1_A$ we have

$$[1_A]_0 = \left[1_A - vv^*\right]_0 + \left[vv^*\right]_0 = \left[1_A - vv^*\right]_0 + [1_A]_0$$

and so $[1_A - vv^*]_0 = 0$ in $K_0(A)$. Now consider the full corner $B = (1_A - vv^*)A(1_A - vv^*)$, which is Morita-equivalent to A [11, Example 3.6], and note that $1_B = 1_A - vv^*$. Thus, $[1_B]_0 = 0$ in $K_0(B)$ and so B does not have IBN. □

Returning to the concrete example, $\mathcal{O}_\infty$ is a unital simple infinite C^*-algebra. We have seen before that $\mathcal{O}_\infty$ has IBN but now, by the above proposition, it contains many full corners which does not have IBN.

5. Universal algebras for Basis Types

A natural question stemming from the discussion of Basis Type is this: are all pairs (N, K) of positive integers realized as the Basis Types of C^*-algebras? We shall answer this in the affirmative and further we will exhibit C^*-algebras which are "universal" for their Basis Type.

Our investigation will be motivated by the situation for the Basis Types $(1, K)$. If $\text{type}(A) = (1, K)$ then necessarily $A \simeq A^{K+1}$ and so there is a unitary $1 \times (K + 1)$ matrix, that is, a row unitary. The elements of such a matrix are isometries satisfying the Cuntz relations and so there is an induced unital $*$-homomorphism (in fact, an embedding) of $\mathcal{O}_{K+1}$ into A. Now as $\mathcal{O}_{K+1} \simeq \mathcal{O}_{K+1}^{K+1}$ and $K_0(\mathcal{O}_{K+1}) = \mathbb{Z}/K\mathbb{Z}$ we conclude via Proposition 4.3 that $\text{type}(\mathcal{O}_{K+1}) = (1, K)$. We consider the Cuntz algebra $\mathcal{O}_{K+1}$ "universal" for Basis Type $(1, K)$ in this sense: whenever $\text{type}(A) = (1, K)$ there is an induced unital $*$-homomorphism $\phi : \mathcal{O}_{K+1} \to A$. We use the term universal loosely because this homomorphism is not necessarily unique. For example, when A is itself a Cuntz algebra then ϕ can be given by any permutation of the generating isometries.

In [10], McClanahan investigated C^*-algebras $U^{nc}_{m,n}$ defined as follows:

$$U^{nc}_{m,n} := C^*\big(u_{ij} : U = [u_{ij}] \in M_{m,n} \text{ satisfies } UU^* = I_m, U^*U = I_n\big).$$

The C^*-algebra $U^{nc}_{m,n}$ has the universal property that whenever A is a C^*-algebra with elements $\{a_{ij}\}$ such that $[a_{ij}] \in M_{m,n}(A)$ is unitary then there is a unital $*$-homomorphism $\phi : U^{nc}_{m,n} \to A$ with $\phi(u_{ij}) = a_{ij}$. Since there is a natural identification of $U^{nc}_{m,n}$ with $U^{nc}_{n,m}$ (taking u_{ij} to u^*_{ji}) we shall only consider the cases where $n > m$.

Suppose that A is a C^*-algebra with $\text{type}(A) = (N, K)$. Then by definition $A^N \simeq A^{N+K}$ and so there is an $N \times (N + K)$ unitary matrix over A. By the universal property we have a unital $*$-homomorphism $\phi : U^{nc}_{N,N+K} \to A$.

Thus we may recast the universal property enjoyed by the $U^{nc}_{m,n}$ as follows: if A is a C^*-algebra of Basis Type $(m, n-m)$ then there is a unital $*$-homomorphism $\phi : U^{nc}_{m,n} \to A$. McClanahan proved that $U^{nc}_{1,n} = \mathcal{O}_n$ and so there is no conflict with our previous discussion. He further demonstrated that $U^{nc}_{m,n}$ is not simple whenever $m > 0$ (there is always a unital $*$-homomorphism $\phi : U^{nc}_{m,n} \to \mathcal{O}_{n-m+1}$) and so, unlike for the Cuntz algebras, the universal property does not guarantee an embedding of $U^{nc}_{m,n}$ into a C^*-algebra when $m > 1$.

Since $U^{nc}_{m,n}$, by definition, has a unitary $m \times n$ matrix we conclude that its standard modules of ranks n and m are equivalent, and so $U^{nc}_{m,n}$ does not have IBN. Ara and Goodearl have recently shown in [3] that $K_0(U^{nc}_{m,n}) = \mathbb{Z}/(n-m)\mathbb{Z}$ (and is generated by $[1]_0$) and so by Proposition 4.3 we have that $\mathrm{type}(U^{nc}_{m,n}) = (N, n-m)$ for some $N \leq m$. To prove that we have $N = m$, we shall exploit the universal property of $U^{nc}_{m,n}$ together with our next main result.

Theorem 5.1. *For each pair (N, K) of positive integers there is a C^*-algebra A with* $\mathrm{type}(A) = (N, K)$.

Proof. We have already seen that for $K > 0$, $\mathrm{type}(\mathcal{O}_{K+1}) = (1, K)$. As $(1, K) \vee (N, 1) = (N, K)$ we conclude by Corollary 4.6 that it is enough, given $N > 0$, to exhibit a C^*-algebra of Basis Type $(N, 1)$.

By combining [13, Theorem 3.5] and [12, Theorem 5.3] we may, for fixed $N > 0$, obtain a unital C^*-algebra A with the following properties:

(1) for $n < N$ the C^*-algebras $M_n(A)$ are finite,
(2) for $m \geq N$ the C^*-algebras $M_m(A)$ are properly infinite, and
(3) $K_0(A) = 0$.

Since $K_0(A) = 0$ it follows that from Theorem 3.2 and Proposition 4.3 that A does not have IBN and has basis type $(N', 1)$ for some $N' > 0$. Since $K_0(M_N(A)) = K_0(A) = 0$ and $M_N(A)$ is properly infinite there is an embedding (see [15, Proposition 4.2.3]) of $\mathcal{O}_2$ into $M_N(A)$. Thus there is a 1×2 unitary matrix (with entries consisting of the images of the Cuntz isometries) over $M_N(A)$ which, viewed in a different light, is an $N \times 2N$ unitary matrix over A itself. Thus $A^N \simeq A^{2N}$ and we conclude that $N' \leq N$. Suppose that $N' < N$. As $\mathrm{type}(A) = (N', 1)$ we have $A^{N'} \simeq A^{N'+1}$ and so there is a unitary $N' \times (N'+1)$ matrix. Deleting any one column from this matrix yields a $N' \times N'$ proper isometry, contradicting the fact that $M_{N'}(A)$ is finite. Hence, $N' = N$ and $\mathrm{type}(A) = (N, 1)$. □

We emphasize that the C^*-algebras in Theorem 5.1 (obtained from [12] and [13]) are not simple. Since the C^*-algebras $U^{nc}_{m,n}$ are also not simple in general, it is a question of some interest to us if Basis Types beyond $(1, K)$ are possible for simple C^*-algebras.

Corollary 5.2. $\mathrm{type}(U^{nc}_{m,n}) = (m, n-m)$.

This is obtained from Theorem 5.1, Proposition 4.4, and the universal property of $U^{nc}_{m,n}$.

Corollary 5.3. *$U^{nc}_{m,n} = U^{nc}_{m',n'}$ if and only if $n = n'$ and $m = m'$.*

Note that the Basis Types are able to distinguish the C^*-algebras $U^{nc}_{m,n}$ and $U^{nc}_{m+1,n+1}$ while the K-theory cannot: they share the same K_0 group, $\mathbb{Z}/(n-m)\mathbb{Z}$, and both have trivial K_1 (see [3, Section 5]).

Finally, we are able to use the C^*-algebras $U^{nc}_{m,n}$ to prove that IBN is preserved under inductive limits. In [10, Remark, p. 1066] McClanahan notes that $U^{nc}_{m,n}$ is *semiprojective* in the sense of [6, Section 3]: that whenever $\{B_i\}$ is an inductive system of C^*-algebras with limit B and $\phi : U^{nc}_{m,n} \to B$ is a unital $*$-homomorphism then there exists a unital $*$-homomorphism $\phi_k : U^{nc}_{m,n} \to B_k$ for some k.

Proposition 5.4. *If $\{A_i, \phi_i\}$ is an inductive family of C^*-algebras, each with IBN and each ϕ_i unital, then the C^*-algebraic direct limit A of the system has IBN.*

Proof. If the limit A did not have IBN, then it must have some Basis Type (N, K). By the universal property there is a unital $*$-homomorphism $\psi : U^{nc}_{N,N+K} \to A$ and hence also, because of the semiprojectivity, a unital $*$-homomorphism $\psi_n : U^{nc}_{N,N+K} \to A_n$ for some n. But, as A_n has IBN, we would then conclude by Proposition 3.3 that $U^{nc}_{N,N+K}$ has IBN, a clear contradiction. ☐

6. Stronger notions

In [4], Cohn considered two ring-theoretic properties strictly stronger than Invariant Basis Number. The C^*-algebraic analogues are formulated below.

Definition 6.1. A C^*-algebra has IBN_1 if, whenever n, m are integers and X an A-module, $A^n \simeq A^m \oplus X$ implies $n \geq m$.

Definition 6.2. A C^*-algebra A has IBN_2 if for all $n > 0$, $A^n \simeq A^n \oplus X$ for some A-module X implies $X = 0$.

The next proposition is nearly immediate.

Proposition 6.3. $\mathrm{IBN}_2 \Rightarrow \mathrm{IBN}_1 \Rightarrow \mathrm{IBN}$.

Proof. Suppose A has IBN_2. If $n < m$ and $A^n \simeq A^m \oplus X$ for some A-module X then $A^n \simeq A^n \oplus A^{m-n} \oplus X$ and we conclude by IBN_2 that $A^{m-n} \oplus X = 0$, i.e. $m - n = 0$ a contradiction. Suppose that A has IBN_1. If $A^n \simeq A^m$ for $n > m$ then $A^m \simeq A^n \oplus 0$ and so $n \leq m$, a contradiction. ☐

Our main goal for this section is twofold: first, to demonstrate that these properties are distinct; and second, to better characterize C^*-algebras satisfying the properties IBN_1 and IBN_2. This goal is easily accomplished for the property IBN_2.

THEOREM 6.4. *A C^*-algebra A has* IBN_2 *if and only if A is stably finite.*

Proof. Suppose that A is not stably finite, that is, there is a proper isometry $V \in M_n(A)$ for some $n \geq 1$. Note that $I_n \sim VV^*$ and $I_n \sim I_n - VV^* \oplus VV^* \sim I_n - VV^* \oplus I_n$. Thus, $A^n \simeq A^n \oplus (I - VV^*)A^n$ where $(I_n - VV^*)A^n \neq 0$ as V is proper. Thus, A does not have IBN_2.

Suppose that A does not have IBN_2. Then $A^n \simeq A^n \oplus X$ for some $n \geq 1$ and nontrivial A-module X. Note that the embedding $\iota : A^n \to A^n \oplus X$ is an adjointable A-module homomorphism which is isometric in the sense that $\iota^*\iota = I_n$. Let $U \in L(A^n \oplus X, A^n)$ be a unitary, then $V = U \circ \iota : A^n \to A^n$ is an adjointable A-module homomorphism with $V^*V = I_n$ and $VV^* = U(I_n \oplus 0)U^* \neq I_n$. Thus, V corresponds to a $n \times n$ proper matrix isometry and $M_n(A)$ is not finite. □

Since there are C^*-algebras with IBN which are not stably finite (for example, the Toeplitz algebra) we conclude that IBN_2 is strictly stronger than IBN.

Although we do not yet know of a better characterization for C^*-algebras with IBN_1, we are nevertheless able to conclude that it is a distinct property from IBN.

EXAMPLE. Consider the C^*-algebra $\mathcal{T}_2$ which is the universal algebra for two isometries v_1 and v_2 satisfying $v_1^*v_2 = v_2^*v_1 = 0$ and $v_1v_1^* + v_2v_2^* < 1$. Note that $V = [v_1v_2] \in M_{1,2}(\mathcal{T}_2)$ is a proper matrix isometry in the sense that $V^*V = I_2$ and $VV^* < 1$. Since V is adjointable the submodule $V\mathcal{T}_2^2 \subset \mathcal{T}_2$ is complementable (with complement $\ker V^*$) and so

$$\mathcal{T}_2 = V\mathcal{T}_2^2 \oplus \ker V^* \simeq \mathcal{T}_2^2 \oplus \ker V^*.$$

Thus, $\mathcal{T}_2$ does not have IBN_1 but Cuntz [5, Proposition 3.9] has shown $K_0(\mathcal{T}_2) = \mathbb{Z}$ and is generated by $[1]_0$, hence $\mathcal{T}_2$ does have IBN.

Indeed, the relationship $A \simeq A^2 \oplus X$ guarantees a unital $*$-homomorphism $\phi : \mathcal{T}_2^2 \to A$ in much the same way the relationship $A \simeq A^2$ guarantees an embedding $\psi : \mathcal{O}_2 \to A$.

Acknowledgments. I wish to thank my Ph.D. advisor Dr. David Pitts for his constant support and insightful questions; my colleague Dr. Adam Fuller for our constructive conversations; Dr. N. Christopher Phillips at the University of Oregon for directing my attention to Rørdam's work, suggesting Proposition 4.9, and remarking that isomorphic standard modules are necessarily unitarily equivalent; and lastly the referee for his or her helpful report. The results present in this paper formed part of my doctoral dissertation while at the University of Nebraska.

References

[1] P. Ara and R. Exel, *Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions*, Adv. Math. **252** (2014), 748–804. MR 3144248

[2] P. Ara, R. Exel and T. Katsura, *Dynamical systems of type* (m, n) *and their* C*-*algebras*, Systems **33** (2013), no. 5, 1291–1325. MR 3103084

[3] P. Ara and K. R. Goodearl, *C^*-algebras of separated graphs*, J. Funct. Anal. **261** (2011), no. 9, 2540–2568. MR 2826405

[4] P. M. Cohn, *Some remarks on the invariant basis property*, Topology **5** (1966), 215–228. MR 0197511

[5] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. (2) **113** (1981), no. 1, 181–197. MR 0604046

[6] E. G. Effros and J. Kaminker, *Homotopy continuity and shape theory for C^*-algebras*, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., vol. 123, Longman Sci. Tech, Harlow, 1986, pp. 152–180. MR 0866493

[7] W. G. Leavitt, *Modules over rings of words*, Proc. Amer. Math. Soc. **7** (1956), 188–193. MR 0077520

[8] W. G. Leavitt, *Modules without invariant basis number*, Proc. Amer. Math. Soc. **8** (1957), 322–328. MR 0083986

[9] W. G. Leavitt, *The module type of a ring*, Trans. Amer. Math. Soc. **103** (1962), 113–130. MR 0132764

[10] K. McClanahan, *K-theory and* Ext-*theory for rectangular unitary C^*-algebras*, Rocky Mountain J. Math. **23** (1993), no. 3, 1063–1080. MR 1245464

[11] I. Raeburn and D. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, Amer. Math. Soc., Providence, RI, 1998. MR 1634408

[12] M. Rørdam, *Stability of C^*-algebras is not a stable property*, Doc. Math. **2** (1997), 375–386. (electronic). MR 1490456

[13] M. Rørdam, *On sums of finite projections*, Operator algebras and operator theory (Shanghai, 1997), Contemp. Math., vol. 228, Amer. Math. Soc., Providence, RI, 1998, pp. 327–340. MR 1667668

[14] M. Rørdam, F. Larsen and N. Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, Cambridge, 2000. MR 1783408

[15] M. Rørdam and E. Størmer, *Classification of nuclear C^*-algebras. Entropy in operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 126, Springer-Verlag, Berlin, 2002. MR 1878881

[16] N. E. Wegge-Olsen, *A friendly approach, K-theory and C^*-algebras*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993. MR 1222415

Philip M. Gipson, Department of Mathematics, State University of New York College at Cortland, Cortland, NY 13045-0900, USA

E-mail address: philip.gipson@cortland.edu