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INSEPARABLE EXTENSIONS OF ALGEBRAS OVER THE STEENROD ALGEBRA WITH APPLICATIONS TO MODULAR INVARIANT THEORY OF FINITE GROUPS II

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Dedicated to Huyn Mùi on the occasion of his 60th birthday

Abstract. We continue our study of the homological properties of the purely inseparable extensions $\mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$ of integrally closed unstable Noetherian integral domains over the Steenrod algebra. It turns out that the projective dimension of H is a lower bound for the projective dimension of $\sqrt[\mathcal{P}^*]{\mathrm{H}}$. Furthermore, $\mathrm{depth}(\mathrm{H}) \geq \mathrm{depth}(\sqrt[\mathcal{P}^*]{\mathrm{H}})$, where depth denotes the depth. Moreover, both algebras have the same global dimension. We apply these results to extension $\mathbb{F}[V_\bullet]^G \hookrightarrow \mathbb{F}[V]^G$ of rings of invariants.

1. Introduction

Let H be a unstable reduced algebra over the Steenrod algebra of reduced powers $\mathcal{P}^*$. We denote the characteristic by p, and the order of the ground field $\mathbb{F}$ by q. Recall that the Steenrod algebra contains an infinite sequence of derivations iteratively defined as

$$\begin{aligned} P^{\Delta_1} &= P^1, \\ P^{\Delta_i} &= P^{\Delta_{i-1}} P^{q^{i-1}} - P^{q^{i-1}} P^{\Delta_{i-1}} \quad \text{for } i \geq 2. \end{aligned}$$

We set

$$P^{\Delta_0}(h) = \deg(h) h \quad \forall h \in \mathrm{H}.$$

Note that P^{Δ_0} is not an element of the Steenrod algebra.

The algebra H is called $\mathcal{P}^*$-*inseparably closed*, if whenever $h \in \mathrm{H}$ and

$$P^{\Delta_i}(h) = 0 \quad \forall i \geq 0,$$

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then there exists an element $h' \in \mathrm{H}$ such that

$$(h')^p = h.$$

The $\mathcal{P}^*$*-inseparable closure* of H is a $\mathcal{P}^*$-inseparably closed algebra $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ containing H such that the following universal property holds: Whenever we have a $\mathcal{P}^*$-inseparably closed algebra H' containing H there exists an embedding $\sqrt[\mathcal{P}^*]{\mathrm{H}} \hookrightarrow \mathrm{H}'$.

In Section 4.1 of [4], an explicit algorithm to construct the inseparable closure is given. We collect known results in the following proposition.

PROPOSITION 1.1. *Consider the natural inclusion*

$$\phi : \mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$$

of unstable reduced algebras over the Steenrod algebra. Then the following statements are valid:

(1) H *is an integral domain if and only if* $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ *is an integral domain.*
(2) $\mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$ *is an integral extension*, *and both algebras have the same Krull dimension.*
(3) *If* H *is integrally closed*, *then so is* $\sqrt[\mathcal{P}^*]{\mathrm{H}}$.
(4) H *is Noetherian if and only if* $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ *is Noetherian.*

If in addition H *is Noetherian*, *then*

(5) *The extension* ϕ *is finite.*
(6) $\overline{\mathrm{H}}$ *is Cohen–Macaulay if and only if* $\overline{\sqrt[\mathcal{P}^*]{\mathrm{H}}}$ *is Cohen–Macaulay*, *where* $\overline{-}$ *denotes the integral closure of* $-$.
(7) $\overline{\mathrm{H}}$ *is polynomial if and only if* $\overline{\sqrt[\mathcal{P}^*]{\mathrm{H}}}$ *is polynomial.*

Proof. For (1)–(3), see Proposition 4.2.1 in [4], for (4) see part (2) of Lemma 4.1.3, Lemma 4.2.2, Proposition 4.2.4, and Theorem 6.3.1 loc. cit., for (5) see Proposition 4.2.4 [4] and [9]. Statement (6) was proven in[1] [7]. Statement (7) was conjectured by C. W. Wilkerson around 1980, [9], and proven in [7]. □

In this paper, we proceed with the investigation of the similarities of an unstable integrally closed algebra over the Steenrod algebra and its inseparable closure. The proofs of statements (6) and (7) in the above proposition led to the conjecture that H and its inseparable closure $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ share all properties that have homological characterizations, like, e.g., the depth, the projective dimension, the global dimension, and the Gorenstein property. In this generality, this is not true. We illustrate this with the following two examples.

[1] Note that the necessary assumption on H being integrally closed is missing in that reference.

EXAMPLE 1.2. Let $\mathbb{F}$ be a finite field of characteristic two. Consider the $\mathcal{P}^*$-purely inseparable extensions

$$\mathbb{F}\big[x^2,y^2\big] \hookrightarrow \mathbb{F}\big[x^2,y^2,y^3,x^2y\big] \hookrightarrow \mathbb{F}\big[x^2,y\big] \hookrightarrow \mathbb{F}[x,y].$$

All four algebras have Krull dimension two. Moreover, $\mathbb{F}[x,y]$ is the $\mathcal{P}^*$-inseparable closure of the other three. Their respective fields of fractions are

$$\mathbb{F}\big(x^2,y^2\big) \hookrightarrow \mathbb{F}\big(x^2,y\big) = \mathbb{F}\big(x^2,y\big) \hookrightarrow \mathbb{F}(x,y).$$

Thus, all of them are integrally closed except for $\mathbb{F}[x^2,y^2,y^3,x^2y]$. Observe that all of them have depth two, except $\mathbb{F}[x^2,y^2,y^3,x^2y]$ which has depth one. Thus, we see that the three integrally closed algebras are isomorphic as ungraded $\mathbb{F}$-algebras (even though not as algebras over the Steenrod algebra). However, we note that $\mathbb{F}[x^2,y^2,y^3,x^2y]$ is not only not isomorphic to $\mathbb{F}[x^2,y^2]$, nor is it isomorphic to $\mathbb{F}[x^2,y]$, but also they do not have the same depth either.

Here is another example illustrating that we cannot expect good results for algebras that are not integrally closed.

EXAMPLE 1.3. Consider the $\mathcal{P}^*$-purely inseparable extension

$$\mathrm{K} = \mathbb{F}\big[x^2,y,xy\big] \hookrightarrow \mathrm{H} = \mathbb{F}[x,y],$$

where $|\mathbb{F}| = 2$. Then $\overline{\mathrm{K}} = \mathrm{H}$, and its global dimension is

$$\text{gl-dim}(\mathrm{H}) = 2.$$

However,

$$\text{gl-dim}(\mathrm{K}) = \text{proj-dim}_{\mathrm{K}}(\mathbb{F}) = \infty,$$

where proj-dim denotes the projective dimension.

2. An unstable algebra and its inseparable closure

We assume from now on that H is an integral domain.

PROPOSITION 2.1. *Let* H *be an integrally closed unstable algebra over the Steenrod algebra. Then*

$$\text{gl-dim}(\mathrm{H}) = \text{gl-dim}\big(\sqrt[\mathcal{P}^*]{\mathrm{H}}\big).$$

Proof. The global dimension of H is finite if and only if H is a Noetherian polynomial algebra. By Theorem 7.4 in [7], this is equivalent to $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ being Noetherian and polynomial. Thus, the global dimensions of H and its inseparable closure are simultaneously finite and equal to their common Krull dimension by Theorem 6.3.1 in [4]. □

We denote by $\mathrm{H}^{[p]} \subseteq \mathrm{H}$ the subalgebra generated by the pth powers of elements in H. The classical Frobenius map

$$\mathrm{H} \longrightarrow \mathrm{H}^{[p]}, \qquad h \mapsto h^p$$

provides us with an (ungraded) isomorphism between the two $\mathbb{F}$-algebras.

PROPOSITION 2.2. *Let* H *be an integrally closed Noetherian integral domain. Then the extension* $\mathrm{H}^{[p]} \hookrightarrow \mathrm{H}$ *splits as a modules*[2] *over* $\mathrm{H}^{[p]} \odot \mathcal{P}^*$.

Proof. Since H is Noetherian the extension $\mathrm{H}^{[p]} \hookrightarrow \mathrm{H}$ is finite. Thus, we can pick a set of generators of H as a module over $\mathrm{H}^{[p]}$, say $\mathbf{t}_1, \ldots, \mathbf{t}_k$, and obtain

$$(\star) \qquad \mathrm{H} = \sum_{i=1}^{k} \mathrm{H}^{[p]} \mathbf{t}_i.$$

By Proposition 5.1 in [6], we can choose the $\mathbf{t}_i$'s to be Thom classes, i.e., for all $j = 1, \ldots, k$

$$\sum_{i=1}^{j} \mathrm{H}^{[p]} \mathbf{t}_i \Big/ \sum_{i=1}^{j-1} \mathrm{H}^{[p]} \mathbf{t}_i = \mathrm{H}^{[p]} \mathbf{t}_j \Big/ \left(\left(\sum_{i=1}^{j-1} \mathrm{H}^{[p]} \mathbf{t}_i \right) \cap \mathrm{H}^{[p]} \mathbf{t}_j \right)$$

is isomorphic to a suspension of an unstable cyclic module over $\mathrm{H}^{[p]}$. Without loss of generality, we can assume that $\mathbf{t}_1 = 1$. Consider the extension

$$\mathbb{FF}\big(\mathrm{H}^{[p]}\big) \hookrightarrow \sum_{i=1}^{k} \mathbb{FF}\big(\mathrm{H}^{[p]}\big) \mathbf{t}_i \hookrightarrow \mathbb{FF}(\mathrm{H}).$$

We claim that $\mathbb{FF}(\mathrm{H}) = \sum_{i=1}^{k} \mathbb{FF}(\mathrm{H}^{[p]}) \mathbf{t}_i$. To that end, take an element $\frac{h}{k} \in \mathbb{FF}(\mathrm{H})$ with $h, k \in \mathrm{H}$. Then

$$\frac{h}{k} = \frac{1}{k^p} h k^{p-1} = \frac{1}{k^p} \sum_{i=1}^{k} h_i \mathbf{t}_i = \sum_{i=1}^{k} \frac{h_i}{k^p} \mathbf{t}_i \in \sum_{i=1}^{k} \mathbb{FF}\big(\mathrm{H}^{[p]}\big) \mathbf{t}_i,$$

for suitable $h_i \in \mathrm{H}^{[p]}$. Since $\mathbb{FF}(\mathrm{H})$ is a finite dimensional vector space over $\mathbb{FF}(\mathrm{H}^{[p]})$ and $\{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ forms a spanning set, we find a basis among it and obtain

$$\mathbb{FF}\big(\mathrm{H}^{[p]}\big) \hookrightarrow \bigoplus_{i=1}^{l} \mathbb{FF}\big(\mathrm{H}^{[p]}\big) \mathbf{t}_i = \mathbb{FF}(\mathrm{H})$$

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[2] The notation $\mathrm{H}^{[p]} \odot \mathcal{P}^*$-module means that we are looking at modules over $\mathrm{H}^{[p]}$ that carry a Steenrod algebra action, that is compatible with the Steenrod algebra action of $\mathrm{H}^{[p]}$.

for some $l \leq k$. By choice of the $\mathbf{t}_i$'s, we can rewrite this and obtain a direct sum decomposition as $\mathbb{FF}(\mathrm{H}^{[p]}) \odot \mathcal{P}^*$-modules as follows

$$\mathbb{FF}(\mathrm{H}) = \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_1 \oplus \left(\bigoplus_{i=2}^{l} \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_i\right) \Big/ \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_1 \cap \bigoplus_{i=2}^{l} \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_i.$$

We take the unstable part of $\mathbb{FF}(\mathrm{H})$. By [3] we have that

$$\mathrm{H} = \mathsf{Un}\big(\mathbb{FF}(\mathrm{H})\big)$$

because H is assume to be integrally closed. Since Un commutes with direct sums (see [8]), we obtain

$$\mathrm{H} = \mathsf{Un}\big(\mathbb{FF}(\mathrm{H})\big) = \mathsf{Un}\big(\mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_1\big) \oplus \mathsf{Un}\left(\bigoplus_{i=2}^{l} \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_i\right) \Big/ \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_1.$$

Since $\mathbf{t}_1 = 1$ and $\mathrm{H}^{[p]}$ is integrally closed, we find $\mathsf{Un}(\mathbb{FF}(\mathrm{H}^{[p]})\mathbf{t}_1) = \mathrm{H}^{[p]}$. Thus,

$$\mathrm{H} = \mathrm{H}^{[p]}\mathbf{t}_1 \oplus \mathsf{Un}\left(\bigoplus_{i=2}^{l} \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_i\right) \Big/ \mathbb{FF}\big(\mathrm{H}^{[p]}\big)\mathbf{t}_1$$

as desired. □

In Chapter 4 of [4], an explicit algorithm to construct the inseparable closure is given. We recollect the few steps we need in what follows:

Denote by $\mathcal{C}(\mathrm{H}) \subseteq \mathrm{H}$ the subalgebra consisting of the so-called $\mathcal{P}^*$-constants:

$$\mathrm{H}^{[p]} \subseteq \mathcal{C}(\mathrm{H}) = \big\{h \in \mathrm{H} | P^{\Delta_i}(h) = 0\, \forall i \geq 0\big\} \subseteq \mathrm{H}.$$

Let $\{s_i, i \in I\}$ be a set of generators of $\mathcal{C}(\mathrm{H})$ as a module over $\mathrm{H}^{[p]}$. We adjoin the pth roots of the s_i's and obtain

$$\mathrm{H}' = \mathrm{H}[\gamma_1, \gamma_2, \ldots]/\sqrt{\big(\gamma_i^p - s_i, i = 1, 2, \ldots\big)}.$$

Set $\mathrm{H} = \mathrm{H}_0$ and $\mathrm{H}' = \mathrm{H}_1$. Then we define $\mathrm{H}_i = (\mathrm{H}_{i-1})'$ and we obtain an ascending chain of unstable algebras

$$\mathrm{H} = \mathrm{H}_0 \hookrightarrow \mathrm{H}_1 \hookrightarrow \mathrm{H}_2 \hookrightarrow \cdots.$$

The $\mathcal{P}^*$-inseparable closure is then the colimit

$$\sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathrm{colim}_i\{\mathrm{H}_i\};$$

see Proposition 4.1.5 in [4]. Furthermore, for the corresponding fields of fractions we have the following:

$$\mathbb{FF}(\mathrm{H}_{i+1}) = \mathbb{FF}(\mathrm{H}_i)[\gamma_1, \gamma_2, \ldots]/\sqrt{\big(\gamma_i^p - s_i, i = 1, 2, \ldots\big)};$$

see Proposition 2.4 in [7].

We note that $\mathcal{C}(\mathrm{H})$ is itself an unstable Noetherian integral domain over the Steenrod algebra, see Lemmata 4.1.1 and 4.1.2 in [4] if H is. We need another property of $\mathcal{C}(\mathrm{H})$.

LEMMA 2.3. *If* H *is an integrally closed integral domain, then so is* $\mathcal{C}(\mathrm{H})$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{FF}\big(\mathcal{C}(\mathrm{H})\big) & \hookrightarrow & \mathbb{FF}(\mathrm{H}) \\ \cup & & \cup \\ \mathcal{C}(\mathrm{H}) & \hookrightarrow & \mathrm{H}. \end{array}$$

Let $\frac{c}{d} \in \mathbb{FF}(\mathcal{C}(\mathrm{H}))$, with $c, d \in \mathcal{C}(\mathrm{H})$, be integral over $\mathcal{C}(\mathrm{H})$. Thus $\frac{c}{d} \in \mathbb{FF}(\mathrm{H})$ is integral over H. Since H is integrally closed, we find that $\frac{c}{d} = h \in \mathrm{H}$. Thus, $c = dh$ and we have for all i that

$$0 = P^{\Delta_i}(c) = P^{\Delta_i}(dh) = P^{\Delta_i}(d)h + dP^{\Delta_i}(h) = dP^{\Delta_i}(h).$$

Since H is an integral domain, we have $P^{\Delta_i}(h) = 0$ and thus $\frac{c}{d} = h \in \mathcal{C}(\mathrm{H})$ as desired. □

LEMMA 2.4. *Let* H *be an unstable algebra over the Steenrod algebra. Then*

$$\mathcal{C}(\mathrm{H}) = (\mathrm{H}_1)^{[p]}.$$

Proof. By construction, the extension $\mathrm{H} \hookrightarrow \mathrm{H}_1$ is purely inseparable of exponent one. Thus $(\mathrm{H}_1)^{[p]} \subseteq \mathrm{H}$, and since this algebra consists of $\mathcal{P}^*$-constants we have

$$(\mathrm{H}_1)^{[p]} \subseteq \mathcal{C}(\mathrm{H}).$$

To prove the reverse inclusion note that every element in $\mathcal{C}(\mathrm{H})$ has a pth root in H_1, thus is contained in $(\mathrm{H}_1)^{[p]}$. □

THEOREM 2.5. *Let* H *be an integrally closed Noetherian integral domain. Then*

$$\text{proj-dim}(\mathrm{H}_{i-1}) \leq \text{proj-dim}(\mathrm{H}_i) \quad \forall i,$$

where proj-dim *denotes the projective dimension.*

Proof. Since H_i is an integrally closed integral domain whenever H is (see Lemma 2.2 in [7]), it is enough to show the statement for $i = 1$. We note that the projective dimension of H can be calculated by finding the projective dimension as a module over a system of parameters, say S. Since $\mathrm{H} \hookrightarrow \mathrm{H}_1$ is finite $\mathrm{S} \subseteq \mathrm{H}_1$ is a Noether normalization as well. Consider the following commutative diagram of S-module homomorphisms and exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ & & 0 & \hookrightarrow & \mathrm{H}/\mathcal{C}(\mathrm{H}) & \longrightarrow & \mathrm{H}/\mathcal{C}(\mathrm{H}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{H}^{[p]} & \hookrightarrow & \mathrm{H} & \longrightarrow & \mathrm{H}/\mathrm{H}^{[p]} & \longrightarrow & 0 \\ & & || & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{H}^{[p]} & \hookrightarrow & \mathcal{C}(\mathrm{H}) & \longrightarrow & \mathcal{C}(\mathrm{H})/\mathrm{H}^{[p]} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

By Lemma 6.3 in [7], the algebras H and $\mathrm{H}^{[p]}$ have the same projective dimension. Set $\text{proj-dim}(\mathrm{H}) = \text{proj-dim}(\mathrm{H}^{[p]}) = d$ and $\text{proj-dim}(\mathcal{C}(\mathrm{H})) = t$. Since $\mathrm{H}^{[p]} \hookrightarrow \mathrm{H}$ splits by Proposition 2.2, we read off the second exact row that

$$\text{proj-dim}(\mathrm{H}/\mathrm{H}^{[p]}) \leq d.$$

We want to show that $d \leq t$. Assume to the contrary that $d > t$. We proceed by depth chasing: The last row tells us that $\text{proj-dim}(\mathcal{C}(\mathrm{H})/\mathrm{H}^{[p]}) = d+1$. Thus, the last column gives that $\text{proj-dim}(\mathrm{H}/\mathcal{C}(\mathrm{H})) = d+2$. However, the middle column says $\text{proj-dim}(\mathrm{H}/\mathcal{C}(\mathrm{H})) = d$. This is the desired contradiction. Thus, we have

$$\text{proj-dim}(\mathrm{H}^{[p]}) = \text{proj-dim}(\mathrm{H}) \leq \text{proj-dim}\big(\mathcal{C}(\mathrm{H})\big).$$

To conclude the proof, note that $\mathcal{C}(\mathrm{H}) = (\mathrm{H}_1)^{[p]}$ by Lemma 2.4 and thus

$$\text{proj-dim}(\mathrm{H}) \leq \text{proj-dim}\big((\mathrm{H}_1)^{[p]}\big) = \text{proj-dim}(\mathrm{H}_1)$$

as claimed. □

Corollary 2.6. *Let* H *be a Noetherian integrally closed integral domain. Then*

$$\text{proj-dim}(\mathrm{H}) \leq \text{proj-dim}(\sqrt[\mathcal{P}^*]{\mathrm{H}}).$$

Proof. Since H is Noetherian the chain of algebras

$$\mathrm{H} = \mathrm{H}_0 \hookrightarrow \mathrm{H}_1 \hookrightarrow \mathrm{H}_2 \hookrightarrow \cdots \hookrightarrow \mathrm{H}_r = \sqrt[\mathcal{P}^*]{\mathrm{H}}$$

stabilizes at some $r \in \mathbb{N}$; see Theorem 6.3.1 in [4]. Furthermore, if H is integrally closed, then so is H_i for all i; see Proposition 4.2.1 (5) in [4]. Thus, the result follows from the preceding by induction on r. □

We have the following immediate corollary.

Corollary 2.7. *Let* H *be a Noetherian integrally closed unstable integral domain over the Steenrod algebra and let* $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ *be its* $\mathcal{P}^*$*-inseparable closure. Then*

$$\operatorname{depth}(\mathrm{H}) \geq \operatorname{depth}(\mathrm{H}_i) \geq \operatorname{depth}(\sqrt[\mathcal{P}^*]{\mathrm{H}})$$

for all i.

Proof. Since H is Noetherian, the extensions $\mathrm{H} \hookrightarrow \mathrm{H}_i \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$ is finite. Thus, a Noether normalization $\mathrm{S} \subseteq \mathrm{H}$ of H is a Noether normalization for H_i and $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ as well. Thus, the statement follows from the Auslander–Buchsbaum formula. □

Remark 2.8. Note that the above result contains as a special case Corollary 2.2 in [5], where the above inequality was proven for Cohen–Macaulay $\sqrt[\mathcal{P}^*]{\mathrm{H}}$.

3. Applications to modular invariant theory

Let $\rho\colon G \hookrightarrow \mathsf{GL}(n,\mathbb{F})$ be a faithful representation of a finite group over a finite field $\mathbb{F}$ of characteristic p and order q. Denote by $V = \mathbb{F}^n$ the n-dimensional vector space over $\mathbb{F}$, and by $\mathbb{F}[V]$ the symmetric algebra on the dual V^*. The representation ρ induces a linear action of G on $\mathbb{F}[V]$. Denote by $\mathbb{F}[V]^G \subseteq \mathbb{F}[V]$ the subring of G-invariant polynomials. By the Galois Embedding theorem, an integrally closed $\mathcal{P}^*$-inseparably closed Noetherian unstable integral domain over the Steenrod algebra is such a ring of invariants $\mathbb{F}[V]^G$ for a suitable representation ρ of some group G, see [1] and Theorem 7.1.1 in [4].

Let $V = W_0 \oplus \cdots \oplus W_e$ be a vector space decomposition. Set

$$\mathbb{F}[V_\bullet] = \mathbb{F}[W_0] \otimes_{\mathbb{F}} \mathbb{F}[W_1]^{[p]} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[W_e]^{[p^e]}.$$

The generalized Galois Embedding theorem states that H is isomorphic to $\mathbb{F}[V_\bullet]^G$ as an algebra over the Steenrod algebra for some suitable flag $V_\bullet$, group G, and representation ρ if and only if H is an integrally closed Noetherian unstable integral domain over the Steenrod algebra, see [9] and Theorem 5.2 in [7]. Furthermore, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H} = \mathbb{F}[V_\bullet]^G & \hookrightarrow & \mathbb{F}[V_\bullet] \\ \downarrow & & \downarrow \\ \sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V]^G & \hookrightarrow & \mathbb{F}[V], \end{array}$$

where the horizontal inclusions are Galois and the vertical inclusions are purely $\mathcal{P}^*$-inseparable.

In Proposition 6.4 and Theorem 7.4 in [7], we saw that $\mathbb{F}[V_\bullet]^G$ and $\mathbb{F}[V]^G$ are simultaneously Cohen–Macaulay, or polynomial. Based on the results of the preceding section, we can add to that list the following properties.

Theorem 3.1. *Let $\mathbb{F}[V_\bullet]^G \hookrightarrow \mathbb{F}[V]^G$ be an extension of rings of invariants. Then*

(1) $\mathbb{F}[V_\bullet]^G$ *and* $\mathbb{F}[V]^G$ *have the same global dimension*,
(2) $\operatorname{proj-dim}(\mathbb{F}[V_\bullet]^G) \leq \operatorname{proj-dim}(\mathbb{F}[V]^G)$, *and*
(3) $\operatorname{depth}(\mathbb{F}[V_\bullet]^G) \geq \operatorname{depth}(\mathbb{F}[V]^G)$.

Proof. The first statement follows from Proposition 2.1. The second statement follows from Corollary 2.6 and the last from Corollary 2.7. □

Remark 3.2. We note that the preceding result can be refined for the extension

$$\mathbb{F}[V_\star]^G \hookrightarrow \mathbb{F}[V_\bullet]^G,$$

where $V_\star \subseteq V_\bullet$ denotes a G-subflag. We find that $\mathbb{F}[V_\bullet]^G$, $\mathbb{F}[V_\star]^G$, and $\mathbb{F}[V]^G$ have the same global dimension by Proposition 2.1. If in addition $\mathbb{F}[V_\bullet]^G = (\mathbb{F}[V_\star]^G)_i$ for some $i \geq 0$, then it follows that

(2′) $\operatorname{proj-dim}(\mathbb{F}[V_\star]^G) \leq \operatorname{proj-dim}(\mathbb{F}[V_\bullet]^G)$ by Theorem 2.5, and similarly
(3′) $\operatorname{depth}(\mathbb{F}[V_\star]^G) \geq \operatorname{depth}(\mathbb{F}[V_\bullet]^G)$ by Corollary 2.7.

We note that in the case of the preceding result the image of G under ρ necessarily consists of matrices of the form

$$\begin{bmatrix} A_0 & 0 & & \cdots & 0 \\ * & A_1 & 0 & \cdots & 0 \\ & * & \ddots & & \vdots \\ \cdots & & \ddots & & 0 \\ * & & \cdots & * & A_e \end{bmatrix},$$

where A_i is an invertible $n_i \times n_i$-matrix with $n_i = \dim(W_i)$.

PROPOSITION 3.3. *Let $\mathbb{F}$ be a field of characteristic p and order q. Assume that $\rho(G)$ consists of matrices of block diagonal form*

$$\begin{bmatrix} A_0 & 0 & & \cdots & 0 \\ 0 & A_1 & 0 & \cdots & 0 \\ & 0 & \ddots & & \vdots \\ \cdots & & \ddots & & 0 \\ 0 & & \cdots & 0 & A_e \end{bmatrix}.$$

Furthermore, let

$$\mathbb{F}[V_\bullet] = \mathbb{F}[W_0] \otimes_{\mathbb{F}} \mathbb{F}[W_1]^{[q]} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[W_e]^{[q^e]}$$

then $\mathbb{F}[V_\bullet]^G$ and $\mathbb{F}[V]^G$ are ungraded isomorphic.

Proof. Consider the (ungraded) isomorphism

$$\phi : \mathbb{F}[V] \longrightarrow \mathbb{F}[V_\bullet], \qquad x_i \mapsto x_i^{q^j}$$

for $x_i \in \mathbb{F}[W_j]$ as basis element. Since G acts on $\mathbb{F}[W_j]$ for all $j = 0, \ldots, e$, the map ϕ commutes with the group action. Thus, the result follows. $\square$

We want to illustrate these results with an example taken from [7]; see Example 7.6 loc.cit.

EXAMPLE 3.4. Let p be odd, and $\mathbb{F}$ a field of characteristic p. Consider the four dimensional modular representation $\mathbb{Z}/p \hookrightarrow \mathsf{GL}(4, \mathbb{F})$ afforded by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Its ring of invariants turns out to be a hypersurface

$$\mathbb{F}[x_1, y_1, x_2, y_2]^{\mathbb{Z}/p} = \mathbb{F}[c_1, y_1, c_2, y_2, q]/(r),$$

where $c_i = x_i^p - x_i y_i^{p-1}$ are the top orbit Chern classes of x_i, $i = 1, 2$, and $q = x_1 y_2 - x_2 y_1$ is an invariant quadratic form. The relation is given by

$$r = q^p - c_1 y_2^p + c_2 y_1^p + q y_1^{p-1} y_2^{p-1},$$

see Theorem 2.1 in [2]. Certainly, $\mathbb{Z}/p$ acts also on $\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p]$ and we find that

$$\left(\mathbb{F}[x_1, y_1] \otimes \mathbb{F}\left[x_2^p, y_2^p\right]\right)^{\mathbb{Z}/p} = \mathbb{F}\left[c_1, y_1, c_2^p, y_2^p, q'\right] / \left(r'\right),$$

where $q' = x_1 y_2^p - x_2^p y_1$ and $r' = (q')^p - c_1 y_2^{p^2} + c_2^p y_1^p - q' y_1^{p-1} y_2^{p(p-1)}$. We note that the two rings are isomorphic, but not graded isomorphic, nor (in the case of a finite ground field $\mathbb{F}$) isomorphic as algebras over the Steenrod algebra.

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