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MAPS THAT TAKE GAUSSIAN MEASURES TO GAUSSIAN MEASURES

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For Don Burkholder, with apologies for what may be abstract nonsense

Abstract. Given a pair of separable, real Banach spaces E and F and a centered Gaussian measure μ on E, one can ask what sort of Borel measurable maps $\Phi : E \longrightarrow F$ map μ to a centered Gaussian measure on F. Obviously, a sufficient condition is that Φ be linear. On the other hand, linearity is far more than is really needed. Indeed, it suffices to know that Φ has the property that

$$\Phi\left(\frac{x_1 + x_2}{\sqrt{2}}\right) = \frac{\Phi(x_1) + \Phi(x_2)}{\sqrt{2}}$$

for $\mathcal{W}^2$-almost every $(x_1, x_2) \in E^2$. In this article, I will first prove a structure theorem which shows that any map Φ which satisfies this property arises from a linear map on the Cameron–Martin space associated with μ on E. I will then investigate which linear maps on the Cameron–Martin space determine a Φ, and finally I will discuss some of the properties of Φ which reflect properties of the linear map from which it is determined.

1. Abstract Wiener spaces

In this section, I will summarize a few facts about Gaussian measures on a Banach space. My treatment derives from L. Gross's theory of abstract Wiener space. For more details, I refer the reader to [2], [5], or Chapter 8 in [6]. In particular, it is important to know that any non-degenerate, centered Gaussian measure on a separable, real Banach space can be realized as the measure in an abstract Wiener space.

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I will call the pair (E, H) a *potential abstract Wiener space* if H is a real, separable Hilbert space, E is a real separable Banach space, and H is continuously embedded in E as a dense subspace. The following lemma summarizes some elementary facts about potential abstract Wiener spaces.

LEMMA 1.1. *Let (H, E) be a potential abstract Wiener space. For each $x^* \in E^*$ there exists a unique $h_{x^*} \in H$ with the property that $\langle h, x^* \rangle = (h, h_{x^*})_H$ for all $h \in H$. Moreover, $x^* \in E^* \longmapsto h_{x^*} \in H$ is a continuous, one-to-one, linear map whose range is dense, and, as a map from E^* into E, $x^* \rightsquigarrow h_{x^*}$ is continuous. Finally, $\{h_{x^*} : x^* \in E^*\}$ contains an orthonormal basis for H.*

Given a potential abstract Wiener space (H, E), the triple $(H, E, \mathcal{W})$ is called an *abstract Wiener space* if $\mathcal{W}$ is a Borel probability measure on E with the property that, for each $x^* \in E^*$, the random variable $x \rightsquigarrow \langle x, x^* \rangle$ under $\mathcal{W}$ is a centered Gaussian with variance $\|h_{x^*}\|_H^2$. Equivalently, the Fourier transform $\widehat{\mathcal{W}}$ of $\mathcal{W}$ is given by

$$\widehat{\mathcal{W}}(x^*) = \mathbb{E}^{\mathcal{W}}\big[e^{\sqrt{-1}\langle x, x^* \rangle}\big] = e^{-\frac{\|h_{x^*}\|_H^2}{2}} \quad \text{for } x^* \in E^*.$$

The Hilbert space H in $(H, E, \mathcal{W})$ is called the *Cameron–Martin space* associated with $\mathcal{W}$ on E.

Although the uniqueness of $\mathcal{W}$ is obvious, its existence is a highly nontrivial matter. Nonetheless, for each H there always exists an E on which there is a $\mathcal{W}$ for which it is the Cameron–Martin space (i.e., $(H, E, \mathcal{W})$ is an abstract Wiener space). When $N = \dim(\mathrm{H}) < \infty$ and one thinks of H as $\mathbb{R}^{\mathbb{N}}$ with some Hilbert norm, all choices of E can also be identified with $\mathbb{R}^N$, and $\mathcal{W}$ is the distribution of

$$(\xi_1, \dots, \xi_N) \in \mathbb{R}^{\mathbb{N}} \longrightarrow \sum_{k=1}^{N} \xi_k h_k \quad \text{under } \gamma_{0,1}^N,$$

where $\gamma_{0,1}$ is the standard Gauss measure on $\mathbb{R}$ and $\{h_k : 1 \le k \le N\}$ is an orthonormal basis for H. When $\dim(H) = \infty$, one has the following criterion for the existence of $\mathcal{W}$.

LEMMA 1.2. *Let (H, E) be an infinite dimensional (i.e., $\dim(H) = \infty$) potential abstract Wiener space. Then there exists a $\mathcal{W}$ on E for which $(H, E, \mathcal{W})$ is an abstract Wiener space if there exists an orthonormal basis $\{h_k : k \ge 1\}$ in H for which the series*

(1.1) $\displaystyle\sum_{k=1}^{\infty} \xi_k h_k$ *converges in E for $\gamma_{0,1}^{\mathbb{Z}^+}$-almost every $(\xi_1, \dots, \xi_k, \dots) \in \mathbb{R}^{\mathbb{Z}^+}$.*

Conversely, if $(H, E, \mathcal{W})$ is an abstract Wiener space, then (1.1) holds for every choice of orthonormal basis, the convergence is in $L^p(\gamma_{0,1}^{\mathbb{Z}^+}; E)$ for every $p \in [1, \infty)$, and $\mathcal{W}$ is the distribution of the series under $\gamma_{0,1}^{\mathbb{Z}^+}$.

Given an abstract Wiener space $(H, E, \mathcal{W})$, there is a unique, linear isometric map $h \in H \longmapsto \mathcal{I}(h) \in L^2(\mathcal{W};\mathbb{R})$, known as the *Paley–Wiener map*, such that $\mathcal{I}(h_{x^*}) = \langle \cdot, x^* \rangle$ $\mathcal{W}$-almost surely for each $x^* \in E^*$. Indeed, the existence and uniqueness of $\mathcal{I}$ follow immediately from the facts that $\|h_{x^*}\|_H$ is the $L^2(\mathcal{W};\mathbb{R})$-norm of $\langle \cdot, x^* \rangle$ and that $\{h_{x^*} : x^* \in E^*\}$ is dense in H. Moreover, since $\langle \cdot, x^* \rangle$ is a centered Gaussian random variable under $\mathcal{W}$ for each $x^* \in E^*$, it follows that $\mathcal{I}(h)$ under $\mathcal{W}$ is a centered Gaussian random variable with variance $\|h\|_H^2$ for each $h \in H$. Hence, $\{\mathcal{I}(h) : h \in H\}$ is a closed, centered Gaussian family in $L^2(\mathcal{W};\mathbb{R})$.

Recall the (unnormalized) Hermite polynomials $H_n, n \geq 0$, given by

$$H_n(\xi) = (-1)^n e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n} e^{-\frac{\xi^2}{2}}, \quad \xi \in \mathbb{R}.$$

Familiar facts about these polynomials are that

$$(H_m, H_n)_{L^2(\gamma_{0,1};\mathbb{R})} = m!\delta_{m,n} \quad \text{and} \quad \frac{dH_m}{d\xi} = mH_{(m-1)^+}, \tag{1.2}$$

and the span of $\{H_n : n \geq 0\}$ is dense in $L^2(\gamma_{0,1};\mathbb{R})$. Now suppose that $(H, E, \mathcal{W})$ is an abstract Wiener space with $\dim(H) = \infty$, choose an orthonormal basis $\{h_k : k \geq 1\}$ in H, and, for $\alpha = (\alpha_1, \dots, \alpha_k, \dots) \in \mathbb{N}^{\mathbb{Z}^+}$ with $\|\alpha\| = \sum_{k=1}^{\infty} \alpha_k < \infty$, define

$$\mathcal{H}_\alpha = \prod_{k=1}^{\infty} H_{\alpha_k}\big(\mathcal{I}(h_k)\big).$$

Then

$$(\mathcal{H}_\alpha, \mathcal{H}_\beta)_{L^2(\mathcal{W};\mathbb{R})} = \alpha!\delta_{\alpha,\beta},$$

where $\alpha! = \prod_{k=1}^{\infty} \alpha_k!$. Moreover, because $\{H_n : n \geq 0\}$ is an orthogonal basis for $L^2(\gamma_{0,1};\mathbb{R})$, one can use the results in Lemma 1.2 to check that $\{\mathcal{H}_\alpha : \|\alpha\| < \infty\}$ is an orthogonal basis in $L^2(\mathcal{W};\mathbb{R})$. In particular, if

$$Z^{(n)}(\mathcal{W}) = \overline{\operatorname{span}\big(\{\mathcal{H}_\alpha : \|\alpha\| = n\}\big)}^{L^2(\mathcal{W};\mathbb{R})},$$

then $Z^{(m)}(\mathcal{W}) \perp Z^{(n)}(\mathcal{W})$ for $m \neq n$ and $L^2(\mathcal{W};\mathbb{R}) = \bigoplus_{n=0}^{\infty} Z^{(n)}(\mathcal{W})$. This is Wiener's decomposition of $L^2(\mathcal{W};\mathbb{R})$ into spaces of *homogeneous chaos*. It is important to recognize that $Z^{(n)}(\mathcal{W})$ does not depend on the particular choice of orthonormal basis $\{h_k : k \geq 1\}$ in terms of which it is defined. In particular, $Z^{(0)}(\mathcal{W})$ consists of the $\mathcal{W}$-almost surely constant elements of $L^2(\mathcal{W};\mathbb{R})$ and $Z^{(1)}(\mathcal{W}) = \{\mathcal{I}(h) : h \in H\}$.

Now choose $\{x_k^* : k \geq 1\} \subseteq E^*$ so that $\{h_k : k \geq 1\}$ is an orthonormal basis for H when $h_k = h_{x_k^*}$, and use this basis and the choice of $\langle \cdot, x_k^* \rangle$ to represent $\mathcal{I}(h_k)$ to define the $\mathcal{H}_\alpha$'s. Clearly, each φ from the span of the $\mathcal{H}_\alpha$'s is a polynomial in variable $\{\langle \cdot, x_k^* \rangle : \alpha_k \neq 0\}$, and therefore $\partial_k \varphi \equiv \frac{d}{dt}\varphi(x + th_k)|_{t=0}$

exists and is a polynomial function of the same variables as φ. Moreover, from (1.2) one sees that if $\varphi \in Z^{(n)}(\mathcal{W})$ for some $n \geq 1$, then $\partial_k \varphi \in Z^{(n-1)}(\mathcal{W})$ and

$$\sum_{k=1}^{\infty} \|\partial_k \varphi\|^2_{L^2(\mathcal{W};\mathbb{R})} = n\|\varphi\|^2_{L^2(\mathcal{W};\mathbb{R})}. \tag{1.3}$$

Hence, for each $n \geq 1$, ∂_k admits a unique continuous extension as a linear map from $Z^{(n)}(\mathcal{W})$ into $Z^{(n-1)}(\mathcal{W})$ for which (1.3) continues to hold. Similarly, for $h \in H$, there is a linear map ∂_h on the span of the $\mathcal{H}_\alpha$'s with the properties that $\partial_h \varphi = \sum_{k=1}^{\infty} (h, h_k)_H \partial_k \varphi$ when $h \in \operatorname{span}(\{h_k : k \geq 1\})$ and, for each $n \geq 1$, ∂_h takes $Z^{(n)}(\mathcal{W})$ into $Z^{(n-1)}(\mathcal{W})$ with

$$\|\partial_h \varphi\|^2_{L^2(\mathcal{W};\mathbb{R})} \leq n\|h\|^2_H \|\varphi\|^2_{L^2(\mathcal{W};\mathbb{R})} \quad \text{for } \varphi \in Z^{(n)}(\mathcal{W}). \tag{1.4}$$

Hence, for each $h \in H$, ∂_h has a unique extension to $\operatorname{span}(\bigcup_{n=0}^{\infty} Z^{(n)}(\mathcal{W}))$ as a linear operator with the property that (1.4) holds. Moreover, ∂_h maps $Z^{(0)}(\mathcal{W})$ to 0 and, when $n \geq 1$, $Z^{(n)}(\mathcal{W})$ to $Z^{(n-1)}(\mathcal{W})$.

2. Wiener maps

Given an abstract Wiener space $(H, E, \mathcal{W})$, it is clear that $(E^2, H^2, \mathcal{W}^2)$ is also an abstract Wiener space. Further, if $S : E^2 \longrightarrow E$ is given by

$$S(x_1, x_2) = \frac{x_1 + x_2}{\sqrt{2}},$$

then $S_* \mathcal{W}^2 = \mathcal{W}$.

Now suppose that $(H, E, \mathcal{W})$ is an abstract Wiener space and that F is a second real, separable Banach space. A map $\Phi : E \longrightarrow F$ is a *Wiener map* if Φ is Borel measurable and

$$\Phi \circ S = \frac{\Phi \circ \pi_1 + \Phi \circ \pi_2}{\sqrt{2}} \quad \mathcal{W}^2\text{-almost surely,} \tag{2.1}$$

where $\pi_i : E^2 \longrightarrow E$ is the projection map $\pi_i(y_1, y_2) = y_i$ for $i \in \{1, 2\}$. Notice that if $\Phi : E \longrightarrow F$ is a Wiener map, then $\Phi_* \mathcal{W}$ is a centered Gaussian measure on F. Indeed, given any $y^* \in E^*$, the distribution μ of $\langle \Phi, y^* \rangle$ will satisfy the convolution equation $\mu = \mu_{2^{-\frac{1}{2}}} \star \mu_{2^{-\frac{1}{2}}}$, where, for $\alpha \in \mathbb{R}$, μ_α is the distribution of $x \rightsquigarrow \alpha x$ under μ, and (cf. Exercise 2.3.21 in Chapter 2 of [6]) the only solutions to this equation are centered Gaussians. Hence, by Fernique's theorem,[1]

$$\mathbb{E}^{\mathcal{W}}\left[e^{\lambda \|\Phi\|^2_F}\right] < \infty \quad \text{for some } \lambda \in (0, \infty). \tag{2.2}$$

In addition, if $\Psi : E \longrightarrow F$ is a second Borel measurable map which is $\mathcal{W}$-almost surely equal to Φ, then Ψ is also a Wiener map, and, more generally, a

[1] The statement of Fernique's theorem to which I am referring asserts that there is a $C < \infty$ such that for each $R > 0$ there is a $\lambda > 0$ for which $\mathbb{E}^\mu[e^{\lambda \|x\|^2_E}] \leq C$ whenever μ is a centered Gaussian measure on a separable, real Banach space E with $\mu(\|x\|_E \geq R) \leq \frac{1}{4}$.

Borel measurable Φ is a Wiener map if it is the $\mathcal{W}$-almost sure limit of Wiener maps.

In this section, I will investigate the structure of Wiener maps, and, since there is nothing more to say when $\dim(H) < \infty$, I will assume throughout that $\dim(H) = \infty$.

LEMMA 2.1. *If $\varphi \in Z^{(n)}(\mathcal{W})$ for some $n \geq 0$, then $\varphi \circ S \in Z^{(n)}(\mathcal{W}^2)$ and $\partial_{(h,-h)}\varphi \circ S = 0$ $\mathcal{W}$-almost surely for each $h \in H$.*

Proof. Obviously there is nothing to do when $n = 0$, and so I will assume that $n \geq 1$. In addition, since the set of $\varphi \in Z^{(n)}(\mathcal{W})$ for which these properties hold is a closed subspace of $L^2(\mathcal{W};\mathbb{R})$, it suffices to prove them when $\varphi = \mathcal{H}_\alpha$ for some α with $\|\alpha\| = n \geq 1$. Further, I will assume that the $\mathcal{H}_\alpha$'s are defined in terms of an orthonormal basis $\{h_k : k \geq 1\}$ where, for each $k \geq 1$, $h_k = h_{x_k^*}$ for some $x_k^* \in E^*$. Thus, $\mathcal{H}_\alpha$ can be taken to be a polynomial in the variables $\{\langle \cdot, x_k^* \rangle : \alpha_k \neq 0\}$.

That $\partial_{(h,-h)}\mathcal{H}_\alpha \circ S = 0$ is essentially trivial. Indeed, for any $k \geq 1$, $\sqrt{2} \times \partial_{(h_k,-h_k)}\mathcal{H}_\alpha \circ S(x_1,x_2)$ equals

$$H'_{\alpha_k}\left(\frac{\langle x_1 + x_2, x_k^* \rangle}{\sqrt{2}}\right) \prod_{j \neq k} H_{\alpha_j}\left(\frac{\langle x_1 + x_2, x_j^* \rangle}{\sqrt{2}}\right) \\ - H'_{\alpha_k}\left(\frac{\langle x_1 + x_2, x_k^* \rangle}{\sqrt{2}}\right) \prod_{j \neq k} H_{\alpha_j}\left(\frac{\langle x_1 + x_2, x_j^* \rangle}{\sqrt{2}}\right) = 0.$$

To prove that $\mathcal{H}_\alpha \circ S \in Z^{(n)}(\mathcal{W}^2)$ when $\|\alpha\| = n$, use the generating function

$$e^{\lambda\xi - \frac{\lambda^2}{2}} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(\xi)$$

to see that

$$H_n\left(\frac{\xi_1 + \xi_2}{\sqrt{2}}\right) = 2^{-\frac{n}{2}} \sum_{m=0}^{n} \binom{n}{m} H_m(\xi_1) H_{n-m}(\xi_2),$$

and from this conclude that

$$\mathcal{H}_\alpha \circ S = 2^{-\frac{n}{2}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathcal{H}_\beta \circ \pi_1 \mathcal{H}_{\alpha-\beta} \circ \pi_2 \in Z^{(n)}\big(\mathcal{W}^2\big),$$

where $\beta \leq \alpha$ means that $\beta_k \leq \alpha_k$ for all $k \geq 1$ and $\binom{\alpha}{\beta} = \prod_{k=1}^{\infty} \binom{\alpha_k}{\beta_k}$. □

LEMMA 2.2. *A Borel measurable $\varphi : E \longrightarrow \mathbb{R}$ is a Wiener map if and only if $\varphi = \mathcal{I}(h)$ for some $h \in H$.*

Proof. First, suppose that $\varphi = \mathcal{I}(h)$. If $h = h_{x^*}$ for some $x^* \in E^*$, then $\varphi = \langle \cdot, x^* \rangle$ $\mathcal{W}$-almost surely and so, because $\langle \cdot, x^* \rangle$ is linear and therefore a Wiener map, it follows that φ is a Wiener map also. To extend the result to general $h \in H$, simply remember that the set of $\mathbb{R}$-valued Wiener maps is closed in $L^2(\mathcal{W};\mathbb{R})$.

Now suppose that φ is an $\mathbb{R}$-valued Wiener map. Since φ is a centered Gaussian under $\mathcal{W}$, $\varphi \in L^2(\mathcal{W};\mathbb{R})$. Now let φ_n denote the orthogonal projection of φ onto $Z^{(n)}(\mathcal{W})$. We will know that $\varphi = \mathcal{I}(h)$ for some $h \in H$ once we know that $\varphi_n = 0$ for $n \neq 1$. Since $\mathbb{E}^{\mathcal{W}}[\varphi] = 0$, $\varphi_0 = 0$. Thus, assume that $n \geq 2$. To show that $\varphi_n = 0$, I will first show that φ_n is a Wiener map. Indeed, from $\varphi = \sum_{m=0}^{\infty} \varphi_m$, we know that $\varphi \circ S = \sum_{m=0}^{\infty} \varphi_m \circ S$. Moreover, by Lemma 2.1, $\varphi_m \circ S \in Z^{(m)}(\mathcal{W}^2)$, and therefore $\varphi_n \circ S$ is the projection of $\varphi \circ S$ onto $Z^{(n)}(\mathcal{W}^2)$. At the same time, because $\varphi \circ S = \frac{\varphi \circ \pi_1 + \varphi \circ \pi_2}{\sqrt{2}}$ $\mathcal{W}^2$-almost surely, it is clear that $\frac{\varphi_n \circ \pi_1 + \varphi_n \circ \pi_2}{\sqrt{2}}$ is also the projection of $\varphi \circ S$ onto $Z^{(n)}(\mathcal{W}^2)$. Hence $\varphi_n \circ S = \frac{\varphi_n \circ \pi_1 + \varphi_n \circ \pi_2}{\sqrt{2}}$ $\mathcal{W}^2$-almost surely. From this and Lemma 2.1, it follows that $\partial_h(\varphi_n \circ \pi_1) = \partial_h(\varphi_n \circ \pi_2)$ $\mathcal{W}^2$-almost surely. But $\partial_h(\varphi_n \circ \pi_1)$ is independent of $\partial_h(\varphi_n \circ \pi_2)$ under $\mathcal{W}^2$, and therefore they can be $\mathcal{W}^2$-almost surely equal only if $\partial_h \varphi_n$ is $\mathcal{W}$-almost surely constant. Since this means that $\partial_h \varphi \in Z^{(0)}(\mathcal{W}) \cap Z^{(n-1)}(\mathcal{W})$ and $n \geq 2$, we now know that $\partial_h \varphi = 0$ $\mathcal{W}$-almost surely. In particular, if $\{h_k : k \geq 1\}$ is an orthonormal basis in H, then, by (1.3),

$$0 = \sum_{k=1}^{\infty} \|\partial_{h_k} \varphi_n\|^2_{L^2(\mathcal{W};\mathbb{R})} = n \|\varphi_n\|^2_{L^2(\mathcal{W};\mathbb{R})},$$

and so $\varphi_n = 0$ $\mathcal{W}$-almost surely. $\square$

THEOREM 2.3. *If $\Phi : E \longrightarrow F$ is Borel measurable, then Φ is a Wiener map if and only if there is a bounded, linear map $A : H \longrightarrow F$ such that $\langle \Phi, y^* \rangle = \mathcal{I}(A^\top y^*)$ $\mathcal{W}$-almost surely for each $y^* \in F^*$, where $A^\top : F^* \longrightarrow H$ is the adjoint of A. Moreover, if A exists, then it is unique, it is continuous from the weak* topology on H into the strong topology on F, and, for any orthonormal basis $\{h_k : k \geq 1\}$,*

$$\Phi = \sum_{m=1}^{\infty} \mathcal{I}(h_k) A h_k \quad \mathcal{W}\text{-almost surely},$$

where the convergence is $\mathcal{W}$-almost sure as well as in $L^p(\mathcal{W};\mathbb{R})$ for each $p \in [1, \infty)$. In particular, if F_A is the closure in F of the range AH of A, then $\Phi \in F_A$ $\mathcal{W}$-almost surely.

Proof. First, suppose that A exists. Then, by Lemma 2.2, for each $y^* \in E^*$, $\langle \Phi, y^* \rangle$ is an $\mathbb{R}$-valued Wiener map and therefore

$$\langle \Phi \circ S, y^* \rangle = \frac{\langle \Phi \circ \pi_1 + \Phi \circ \pi_2, y^* \rangle}{\sqrt{2}} \quad \mathcal{W}^2\text{-almost surely.}$$

Hence, since F^* is separable in the weak* topology, Φ is a Wiener map. Furthermore, because

$$\big\langle Ah, y^*\big\rangle = \big(h, A^\top y^*\big)_H = \mathbb{E}^{\mathcal{W}}\big[\mathcal{I}(h)\mathcal{I}\big(A^\top y^*\big)\big] = \mathbb{E}^{\mathcal{W}}\big[\mathcal{I}(h)\big\langle \Phi, y^*\big\rangle\big],$$

it is clear that there is at most one choice of A.

Now suppose that Φ is a Wiener map. I begin by constructing the map $A^\top : F^* \longrightarrow H$ which will be the adjoint of the A for which we are looking. Namely, given $y^* \in F^*$, set $\varphi = \langle \Phi, y^*\rangle$. Then, because Φ is a Wiener map, φ is an $\mathbb{R}$-valued Wiener map. Hence, by Lemma 2.2, there exists a necessarily unique $A^\top y^* \in H$ such that $\langle \Phi, y^*\rangle = \mathcal{I}(A^\top y^*)$ $\mathcal{W}$-almost surely. By uniqueness, $A^\top$ is linear. Furthermore, because, by (2.2), $\Phi \in L^2(\mathcal{W}; F)$ and

$$\big\|A^\top y^*\big\|_H = \big\|\mathcal{I}\big(A^\top y^*\big)\big\|_{L^2(\mathcal{W};\mathbb{R})} \le \big\|y^*\big\|_{F^*} \|\Phi\|_{L^2(\mathcal{W};F)},$$

it is clear that $A^\top$ is bounded from F^* into H. In fact, if $y_n^* \longrightarrow 0$ in the weak* topology on F^*, then, because, by the uniform boundedness principle, $C = \sup\{\|y_n^*\|_{F^*} : n \ge 1\} < \infty$ and therefore $|\langle \Phi, y_n^*\rangle| \le C\|\Phi\|_F \in L^2(\mathcal{W};\mathbb{R})$, it follows from Lebesgue's Dominated Convergence theorem that

$$\big\|A^\top y_n^*\big\|_H^2 = \int \big\langle \Phi, y_n^*\big\rangle^2 \, d\mathcal{W} \longrightarrow 0.$$

Hence, $A^\top$ is continuous from the weak* topology on F^* into the strong topology on H. In particular, this means that if $h \in H$ and $y^{**} = (A^\top)^\top h \in F^{**}$, then

$$\big\langle y_n^*, y^{**}\big\rangle = \big(h, A^\top y_n^*\big)_H \longrightarrow 0$$

when $\{y_n^* : n \ge 1\} \subseteq F^*$ tends to 0 in the weak* topology. Thus (cf. Theorem 9 on p. 421 of [1]), $y^{**} \in F$, and so $(A^\top)^\top$ determines a bounded linear map, which I will call A, from H into F, and clearly $A^\top$ is the adjoint of A.

The continuity of A with respect to the weak* topology on H into to the strong topology on F is an immediate consequence of the corresponding continuity property of $A^\top$ and the fact that the closed unit ball in F^* is weak* compact. To prove the concluding assertion, let $\{h_k : k \ge 1\}$ be an orthonormal basis for H and take $\mathcal{F}_n$ to be the σ-algebra generated by $\{\mathcal{I}(h_k) : 1 \le k \le n\}$. Then, because $\Phi \in L^p(\mathcal{W}; F)$ for every $p \in [1,\infty)$ and the $\mathcal{W}$-completion of $\bigvee_{n=1}^\infty \mathcal{F}_n$ contains the Borel field over E, we know that $\mathbb{E}^{\mathcal{W}}[\Phi|\mathcal{F}_n] \longrightarrow \Phi$ both $\mathcal{W}$-almost surely as well as in $L^p(\mathcal{W}; F)$ for every $p \in [1,\infty)$. On the other hand,

$$\begin{aligned}
&\big\langle \mathbb{E}^{\mathcal{W}}[\Phi|\mathcal{F}_n], y^*\big\rangle \\
&\quad = \mathbb{E}^{\mathcal{W}}\big[\big\langle \Phi, y^*\big\rangle|\mathcal{F}_n\big] = \mathbb{E}^{\mathcal{W}}\big[\mathcal{I}\big(A^\top y^*\big)|\mathcal{F}_n\big] \\
&\quad = \sum_{k=1}^n \big\langle Ah_k, y^*\big\rangle \mathcal{I}(h_k) = \left\langle \sum_{k=1}^n \mathcal{I}(h_k) Ah_k, y^*\right\rangle \quad \mathcal{W}\text{-almost surely}
\end{aligned}$$

for each $y^* \in F^*$. Hence, since the weak* topology on F^* is separable,

$$\mathbb{E}^{\mathcal{W}}[\Phi|\mathcal{F}_n] = \sum_{k=1}^{n} \mathcal{I}(h_k)Ah_k. \qquad \square$$

3. A's which determine Wiener maps

Given a bounded, linear map $A : H \longrightarrow F$ and a Wiener map $\Phi : E \longrightarrow F$, I will say that Φ *comes from* A if $\langle \Phi, y^* \rangle = \mathcal{I}(A^\top y^*)$ $\mathcal{W}$-almost surely for each $y^* \in F^*$, in which case I will say that A *determines* Φ. Again because F^* is weak* separable, it is obvious that, up to a set of $\mathcal{W}$-measure 0, A can determine at most one Φ. On the other hand, the problem of deciding whether a given A determines any Φ is much more difficult. Indeed, it turns out to be tantamount to finding out whether a certain potential abstract Wiener space can be made into an abstract Wiener space. To explain this, let $H_A = AH$ be the range of A, turn H_A into a Hilbert space with norm $\|\cdot\|_{H_A}$ determined by $\|Ah\|_{H_A} = \|h\|_H$ when $h \perp \mathrm{Null}(A)$. Next, take F_A to be the closure of H_A in F, and turn F_A into a Banach space by restricting $\|\cdot\|_F$ to F_A. Obviously, (H_A, F_A) is a potential Wiener space. Moreover, F_A^* can be identified as the quotient space $F^*/\sim$, where $z^* \sim y^*$ means that $\langle y, z^* - y^* \rangle = 0$ for all $y \in F_A$. Finally, if $y^* \in F^*$ and $[y^*]_A$ is the $\sim$-equivalence class containing y^*, then $AA^\top y^*$ is the unique $g_A \in H_A$ with the property that $\langle h_A, [y^*]_A \rangle = (g_A, h_A)_{H_A}$ for all $h_A \in H_A$. Thus, $(h_A)_{[y^*]_A} = AA^\top y^*$, and therefore $\|(h_A)_{[y^*]_A}\|_{H_A} = \|A^\top y^*\|_H$.

Theorem 3.1. *Refer to the preceding discussion. Then the following are equivalent.*

(1) *A determines a Wiener map $\Phi : E \longrightarrow F$.*
(2) *There is an orthonormal basis $\{h_k : k \geq 1\}$ in H such that the series*

$$\sum_{k=1}^{\infty} \xi_k A h_k \text{ converges in } F \text{ for } \gamma_{0,1}^{\mathbb{Z}^+}\text{-almost every } (\xi_1, \dots, \xi_k, \dots) \in \mathbb{R}^{\mathbb{Z}^+}.$$

(3) *There is a $\mathcal{W}_A$ on F_A for which $(H_A, F_A, \mathcal{W}_A)$ is an abstract Wiener space.*

Moreover, if (1) *holds and $\{h_k : k \geq 1\}$ is an orthonormal basis in H, then*

$$\Phi = \sum_{k=1}^{\infty} \mathcal{I}(h_k)Ah_k \quad \mathcal{W}\text{-almost surely},$$

where the series converges in $L^p(\mathcal{W}; F_A)$ for every $p \in [1, \infty)$ as well as $\mathcal{W}$-almost surely. In addition, the $\mathcal{W}_A$ in (3) *equals the restriction to F_A of $\Phi_*\mathcal{W}$.*

Proof. First suppose that (1) holds, and let $\{h_k : k \geq 1\}$ be an orthonormal basis in H. Then, by the last part of Theorem 2.3, $\sum_{k=1}^{\infty} \mathcal{I}(h_k) A h_k$ converges in F to Φ $\mathcal{W}$-almost surely as well as in $L^p(\mathcal{W};F)$ for every $p \in [1,\infty)$. Hence, we now know that the concluding assertion is true. Also, because $\gamma_{0,1}^{\mathbb{Z}^+}$ is the $\mathcal{W}$-distribution of $(\mathcal{I}(h_1),\dots,\mathcal{I}(h_k),\dots)$, we know that (1) implies (2).

Next suppose that (2) holds, and denote by Φ the sum of the series. Then, because, by Lemma 2.2, each of the summands is a Wiener map, it follows that Φ is also a Wiener map. In addition, for any $y^* \in F^*$, $\mathcal{W}$-almost surely

$$\langle \Phi, y^* \rangle = \sum_{k=1}^{\infty} \mathcal{I}(h_k) \langle A h_k, y^* \rangle = \sum_{k=1}^{\infty} \mathcal{I}(h_k) \big(h_k, A^\top y^*\big)_H = \mathcal{I}\big(A^\top y^*\big).$$

Hence, (2) implies (1).

Next, suppose that (1), and therefore (2), holds. By (2), $\Phi \in F_A$ $\mathcal{W}$-almost surely, and so $\Phi_* \mathcal{W}(F_A) = 1$. In addition,

$$\mathbb{E}^{\mathcal{W}}\big[e^{\sqrt{-1}\langle \Phi, y^* \rangle}\big] = \mathbb{E}^{\mathcal{W}}\big[e^{\sqrt{-1}\mathcal{I}(A^\top y^*)}\big] = e^{-\frac{1}{2}\|A^\top y^*\|_H^2}.$$

Hence, since $\|(h_A)_{[y^*]_A}\|_{H_A} = \|A^\top y^*\|_H$, $(H_A, F_A, \Phi_*\mathcal{W} \upharpoonright F_A)$ is an abstract Wiener space, and so (1) implies (3) and the $\mathcal{W}_A$ in (3) equals $\Phi_*\mathcal{W} \upharpoonright F_A$. Conversely, if (3) holds, choose an orthonormal basis $\{h_k : k \geq 1\}$ in H so that, for each $k \geq 1$, either $Ah_k = 0$ or $h_k \perp \mathrm{Null}(A)$, and denote by $\mathcal{K}$ the set $\{k : h_k \perp \mathrm{Null}(A)\}$. Then $\{Ah_k : k \in \mathcal{K}\}$ is an orthonormal basis in H_A, and so $\sum_{k\in\mathcal{K}} \mathcal{I}_A(Ah_k)Ah_k$ is $\mathcal{W}_A$-almost surely convergent in F_A, where $\mathcal{I}_A$ is the Paley–Wiener map for $(H_A, F_A, \mathcal{W}_A)$. Since $Ah_k = 0$ for $k \notin \mathcal{K}$ and $\{\mathcal{I}_A(Ah_k) : k \in \mathcal{K}\}$ under $\mathcal{W}_A$ are mutually independent standard normal random variables, I have proved that (3) implies (2). □

Obviously, if $A : H \longrightarrow F$ is a linear map which is bounded with respect to $\|\cdot\|_E$ in the sense that $\|Ah\|_F \leq C\|h\|_E$ for some $C < \infty$, then the unique extension of A as a bounded linear map from E to F is a Wiener map determined by A. I will now discuss a more interesting source of A's which determine Wiener maps. Before stating the main result in this direction, I recall the following familiar fact (cf. [3] and [4]).

LEMMA 3.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, assume that the random variables $\{X_i : i \geq 1\} \subseteq L^2(\mathbb{P};\mathbb{R})$ span a Gaussian family, and set $C_{i,j} = \mathbb{E}^{\mathbb{P}}[X_i X_j]$. Then the joint distribution μ on $\mathbb{R}^{\mathbb{Z}^+}$ of $\{X_i : i \geq 1\}$ under $\mathbb{P}$ is absolutely continuous with respect to $\gamma_{0,1}^{\mathbb{Z}^+}$ if and only if the matrix $((C_{i,j}))_{i,j\in\mathbb{Z}^+}$ determines a bounded operator C on $\ell^2(\mathbb{Z}^+;\mathbb{R})$, $\mathrm{Null}(C) = 0$, and $\sum_{i,j=1}^{\infty}(\delta_{i,j} - C_{i,j})^2 < \infty$. Equivalently, $\mu \ll \gamma_{0,1}^{\mathbb{Z}^+}$ if and only if $((C_{i,j}))_{i,j\in\mathbb{Z}^+}$ determines a bounded, non-degenerate operator C on $\ell^2(\mathbb{Z}^+;\mathbb{R})$ such that $I - C$ is Hilbert–Schmidt.*

THEOREM 3.3. *Let $(E,H,\mathcal{W}_H)$ and $(F,G,\mathcal{W}_G)$ be a pair of infinite dimensional abstract Wiener spaces, let $A: H \longrightarrow G$ be a bounded linear map, and use $A^\dagger : G \longrightarrow H$ to denote its adjoint. If, for some $\lambda > 0$, $\lambda I - AA^\dagger$ is a Hilbert–Schmidt operator on G, then A determines a Wiener map from E to F.*

Proof. Clearly it suffices to handle the case when $\lambda = 1$ since the general case reduces to this one when A is replaced by $\lambda^{-\frac{1}{2}}A$. Thus, assume that $\lambda = 1$.

I will first show that there is an orthonormal basis for G such that $S = \lim_{m\to\infty} S_m$ exists in F $\mathcal{W}_H$-almost surely when $S_m = \sum_{i=1}^m \mathcal{I}(A^\dagger g_i)g_i$. To this end, note that, because $I - AA^\dagger$ is Hilbert–Schmidt, $\mathrm{Null}(A^\dagger) = \mathrm{Null}(AA^\dagger)$ is finite dimensional. Now choose an orthonormal basis for G $\{g_i : i \geq 1\}$ so that $g_i \in \mathrm{Null}(A^\dagger)$ if $1 \leq i \leq N = \dim(\mathrm{Null}(A^\dagger))$ and $g_i \perp \mathrm{Null}(A^\dagger)$ if $i > N$. Obviously, $S_m = 0$ if $1 \leq m \leq N$ and

$$S_m = \sum_{i=N+1}^{m} \mathcal{I}(A^\dagger g_i)g_i \quad \text{if } m > N.$$

Moreover, because $AA^\dagger$ is non-degenerate on $\mathrm{Null}(A^\dagger)^\perp$ and $I - AA^\dagger$ is Hilbert–Schmidt, Lemma 3.2 says that the joint distribution of $\{\mathcal{I}(A^\dagger g_i) : i > N\}$ under $\mathcal{W}_H$ is absolutely continuous with respect to the joint distribution of $\{\mathcal{J}(g_i) : i > N\}$ under $\mathcal{W}_G$, where $\mathcal{J}$ denotes the Paley–Wiener map for $(G,F,\mathcal{W}_G)$. Hence, because $\sum_{i=1}^\infty \mathcal{J}(g_i)g_i$ is $\mathcal{W}_G$-almost surely convergent in F, it follows that $\lim_{m\to\infty} S_m$ converges $\mathcal{W}_H$-almost surely in F.

To complete the proof, observe that each S_m is a centered, F-valued, Gaussian random variable under $\mathcal{W}_H$, and therefore that S is also. Thus, by Fernique's theorem, the fact that $S_m \longrightarrow S$ in F $\mathcal{W}_H$-almost surely implies that there exists an $\varepsilon > 0$ such that $\sup_{m\geq 1} E^{\mathcal{W}_H}[e^{\varepsilon \|S_m\|_F^2}] < \infty$, and therefore that $S_m \longrightarrow S$ in $L^1(\mathcal{W}_H;F)$. Now let $\{h_k : k \geq 1\}$ be an orthonormal basis for H, and let $\mathcal{F}_n$ be the σ-algebra generated by $\{\mathcal{I}(h_k) : 1 \leq k \leq n\}$. Then $\mathbb{E}^{\mathcal{W}_H}[S|\mathcal{F}_n] = \lim_{m\to\infty} \mathbb{E}^{\mathcal{W}_H}[S_m|\mathcal{F}_n]$ in $L^1(\mathcal{W}_H;F)$. At the same time,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{W}_H}[S_m|\mathcal{F}_n] \\
&\quad = \sum_{i=1}^m \mathbb{E}^{\mathcal{W}_H}\left[\mathcal{I}\big(A^\dagger g_i\big)|\mathcal{F}_n\right]g_i = \sum_{i=1}^m \left(\sum_{k=1}^n \big(A^\dagger g_i, h_k\big)_H \mathcal{I}(h_k)\right) \\
&\quad = \sum_{k=1}^n \mathcal{I}(h_k)\left(\sum_{i=1}^m (Ah_k, g_i)_G g_i\right) \longrightarrow \sum_{k=1}^n \mathcal{I}(h_k)Ah_k \quad \text{in } G \text{ as } m \to \infty.
\end{aligned}$$

Hence,

$$\mathbb{E}^{\mathcal{W}_H}[S|\mathcal{F}_n] = \sum_{k=1}^n \mathcal{I}(h_k)Ah_k \quad \mathcal{W}_H\text{-almost surely,}$$

and therefore $\sum_{k=1}^n \mathcal{I}(h_k)Ah_k$ converges in F to S $\mathcal{W}_H$-almost surely.

Now apply Theorem 3.1. □

As the next result makes explicit, in Theorem 3.3 the image $\Phi_*\mathcal{W}_H$ of $\mathcal{W}_H$ under Φ will not be absolutely continuous with respect to $\mathcal{W}_G$ unless $\lambda = 1$.

COROLLARY 3.4. *Let* $(H, E, \mathcal{W}_H)$ *and* $(G, F, \mathcal{W}_G)$ *be as in the proceding, and suppose that* $A : H \longrightarrow F$ *is a bounded, linear map. Then the following are equivalent.*

(1) *A determines a Wiener map $\Phi : E \longrightarrow F$ and $\Phi_*\mathcal{W}_H \ll \mathcal{W}_G$.*
(2) *A maps H boundedly into G,* $\mathrm{Null}(A^\dagger) = 0$, *and $I - AA^\dagger$ is Hilbert–Schmidt on G.*

Moreover, if (2), *and therefore* (1), *holds, then* $\mathcal{J}(g) \circ \Phi = \mathcal{I}(A^\dagger g)$ $\mathcal{W}_H$*-almost surely for each $g \in G$, where $\mathcal{I}$ and $\mathcal{J}$ denote the Paley–Wiener maps for* $(H, E, \mathcal{W}_H)$ *and* $(G, F, \mathcal{W}_G)$, *respectively.*

Proof. First, assume that (1) holds. Choose $\{y_i^* : i \geq 1\} \subseteq F^*$ so that $\{g_i : i \geq 1\}$ is an orthonormal basis for G when $g_i = g_{y_i^*}$. Because $\Phi_*\mathcal{W}_H \ll \mathcal{W}_G$ and $\langle \Phi, y_i^* \rangle = \mathcal{J}(g_i) \circ \Phi$ $\mathcal{W}_H$-almost surely, the joint distribution of $\{\langle \Phi, y_i^* \rangle : i \geq 1\}$ under $\mathcal{W}_H$ is absolutely continuous with respect to the joint distribution of $\{\mathcal{J}(g_i) : i \geq 1\}$ under $\mathcal{W}_G$. Equivalently, the joint distribution of $\{\mathcal{I}(A^\top y_i^*) : i \geq 1\}$ under $\mathcal{W}_H$ is absolutely continuous with respect to $\gamma_{0,1}^{\mathbb{Z}^+}$. Hence, by Lemma 3.2, if $C_{i,j} = \langle A^\top y_i^*, A^\top y_j^* \rangle$, then the matrix $((C_{i,j}))_{i,j \in \mathbb{Z}^+}$ determines a bounded, non-degenerate operator C on $\ell^2(\mathbb{Z}^+; \mathbb{R})$ and $I - C$ is Hilbert–Schmidt. Starting from this and taking $M = 1 + \|I - C\|_{\mathrm{H.S.}}$, it is easy to check that $\|A^\top y^*\|_H \leq M \|g_{y^*}\|_G$ and therefore that there is a unique bounded, linear $A^\dagger : G \longrightarrow H$ such that $A^\dagger g_{y^*} = A^\top y^*$, and clearly $\|A^\dagger\|_{\mathrm{op}} \leq M$. Since this means that $|\langle Ah, y^* \rangle| \leq M \|h\|_H \|g_{y^*}\|_G$, it follows that A maps H boundedly into G and that $A^\dagger$ is the adjoint of A as a map from H to G. Furthermore, from the properties of C, it is clear that $AA^\dagger$ is non-degenerate and that $I - AA^\dagger$ is Hilbert–Schmidt on H.

Now assume that (2) holds. By Theorem 3.3, we know that A determines a Wiener map $\Phi : E \longrightarrow F$. Moreover, in the notation used above, the operator C is non-degenerate and $I - C$ is Hilbert–Schmidt. Hence, by Lemma 3.2, the joint distribution of $\{\mathcal{I}(A^\top y_i^*) : i \geq 1\}$ under $\mathcal{W}_H$ is absolutely continuous with respect to $\gamma_{0,1}^{\mathbb{Z}^+}$, and so the joint distribution of $\{\langle \Phi, y_i^* \rangle : i \geq 1\}$ under $\mathcal{W}_H$ is absolutely continuous with that of $\{\mathcal{J}(g_i) : i \geq 1\}$ under $\mathcal{W}_G$. Since this means that the distribution of Φ under $\mathcal{W}_H$ is absolutely continuous with respect to $\mathcal{W}_G$, $\Phi_*\mathcal{W}_H \ll \mathcal{W}_G$.

We now know that (1) $\Longleftrightarrow$ (2). Assume (2) and therefore (1) hold. Given $g \in G$, choose $\{y_n^* : n \geq 1\} \subseteq F^*$ so that $g_n \equiv g_{y_n^*} \longrightarrow g$ in G. Then, $\mathcal{J}(g_n) \longrightarrow \mathcal{J}(g)$ in $L^2(\mathcal{W}_G; \mathbb{R})$, and so I may and will assume that $\mathcal{J}(g_n) \longrightarrow \mathcal{J}(g)$ $\mathcal{W}_G$-almost surely, which, by absolute continuity, means that $\mathcal{J}(g_n) \circ \Phi \longrightarrow \mathcal{J}(g) \circ \Phi$ $\mathcal{W}_H$-almost surely. At the same time, $A^\dagger g_n \longrightarrow A^\dagger g$ in H, and so I may and will also assume that $\mathcal{I}(A^\dagger g_n) \longrightarrow \mathcal{I}(A^\dagger g)$ $\mathcal{W}_H$-almost surely. Hence, since $\mathcal{J}(g_n) \circ \Phi = \mathcal{I}(A^\dagger g_n)$ $\mathcal{W}_H$-almost surely, it follows that $\mathcal{J}(g) \circ \Phi = \mathcal{I}(A^\dagger g)$ $\mathcal{W}_H$-almost surely. $\square$

COROLLARY 3.5. *Again let $(H, E, \mathcal{W}_H)$ and $(G, F, \mathcal{W}_G)$ be as in Theorem 3.3, and assume that $\Phi : E \longrightarrow F$ is a Wiener map for which $\Phi_*\mathcal{W}_H \ll \mathcal{W}_G$. In addition, let Ψ be a Wiener map from F into a third real, separable Banach space K. Then $\Psi \circ \Phi$ is a Wiener map from E into K. Moreover, if Φ and Ψ are determined by A and B, respectively, then $\Psi \circ \Phi$ is determined by $B \circ A$.*

Proof. Because $\Phi_*\mathcal{W}_H \ll \mathcal{W}_G$, the set of $(x_1, x_2) \in E^2$ for which

$$\Psi\left(\frac{\Phi(x_1) + \Phi(x_2)}{\sqrt{2}}\right) \neq \frac{\Psi \circ \Phi(x_1) + \Psi \circ \Phi(x_2)}{\sqrt{2}}$$

has $\mathcal{W}_H^2$-measure 0. Hence, since

$$\Phi\left(\frac{x_1 + x_2}{\sqrt{2}}\right) = \frac{\Phi(x_1) + \Phi(x_2)}{\sqrt{2}}$$

for $\mathcal{W}_H^2$-almost every $(x_1, x_2) \in E^2$, it follows that $\Psi \circ \Phi$ is also a Wiener map.

Next, let $A : H \longrightarrow F$ be the bounded, linear map which determines Φ. By Corollary 3.4, A is a bounded map from H to G and $\mathcal{J}(g) \circ \Phi = \mathcal{I}(A^\dagger g)$ $\mathcal{W}_H$-almost surely for each $g \in G$. Hence, if $B : F \longrightarrow K$ determines Ψ, then, for any $z^* \in K^*$,

$$\langle \Psi \circ \Phi, z^* \rangle = \mathcal{J}\big(B^\top z^*\big) \circ \Phi = \mathcal{I}\big(A^\dagger B^\top z^*\big) \quad \mathcal{W}_H\text{-almost surely.}$$

But $A^\dagger B^\top = (B \circ A)^\top$, and so $B \circ A$ determines $\Psi \circ \Phi$. $\square$

COROLLARY 3.6. *Let $(H, E, \mathcal{W}_H)$ and $(G, F, \mathcal{W}_G)$ be as in Theorem 3.3, suppose that $\Phi : E \longrightarrow F$ is a Wiener map for which $\Phi_*\mathcal{W}_H \ll \mathcal{W}_G$, and let $A : H \longrightarrow G$ be the associated map described in* (2) *of Corollary 3.4. If A is non-degenerate and $I - A^\dagger A$ is Hilbert–Schmidt on H, then there exists a Wiener map $\Psi : F \longrightarrow E$ such that $\Psi_*\mathcal{W}_G \ll \mathcal{W}_H$ and Ψ is the inverse of Φ in the sense that $\Psi \circ \Phi(x) = x$ for $\mathcal{W}_H$-almost every $x \in E$ and $\Phi \circ \Psi(y) = y$ for $\mathcal{W}_G$-almost every $y \in F$.*

Proof. I begin by showing that A has a bounded inverse $A^{-1} : G \longrightarrow H$ and that $I - A^{-1}(A^{-1})^\dagger$ is Hilbert–Schmidt on H. Indeed, because A is non-degenerate and $I - A^\dagger A$ is Hilbert–Schmidt, $AA^\dagger$ has a pure point spectrum $\{\lambda_i : i \geq 1\} \subseteq (0, \infty)$ for which 1 is the only possible accumulation point. Hence, there exists an $\varepsilon > 0$ such that $\lambda_i \geq \varepsilon$ for all $i \geq 1$, and therefore A has a bounded inverse $A^{-1} : G \longrightarrow H$. Furthermore, the Hilbert–Schmidt norm of $I - A^{-1}(A^{-1})^\dagger$ is

$$\sum_{i=1}^{\infty} \left(1 - \frac{1}{\lambda_i}\right)^2 \leq \varepsilon^{-2} \sum_{i=1}^{\infty} (1 - \lambda_i)^2 < \infty,$$

and so $I - A^{-1}(A^{-1})^{\dagger}$ is Hilbert–Schmidt on H.

Given the preceding, Corollary 3.4 says that A^{-1} determines a Wiener map $\Psi : F \longrightarrow E$ such that $\Psi_* \mathcal{W}_G \ll \mathcal{W}_H$, and Corollary 3.5 says that $\Psi \circ \Phi$ is a Wiener map which is determined by $A^{-1} \circ A = I$. Hence, by uniqueness, $\Psi \circ \Phi$ is $\mathcal{W}_H$-almost surely equal to the identity map on E. Similarly, $\Phi \circ \Psi$ must be $\mathcal{W}_G$-almost surely equal to the identity map on F. □

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