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# WEAK NONMILD SOLUTIONS TO SOME SPDES

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ABSTRACT. We study the nonlinear stochastic heat equation driven by space–time white noise in the case that the initial datum $u_0$ is a (possibly signed) measure. In this case, one cannot obtain a mild random-field solution in the usual sense. We prove instead that it is possible to establish the existence and uniqueness of a weak solution with values in a suitable function space. Our approach is based on a construction of a generalized stochastic convolution via Young-type inequalities.

## 1. Introduction

Let us consider the nonlinear stochastic heat equation

$$\frac{\partial}{\partial t}u_t(x) = (\mathcal{L}u_t)(x) + \sigma\big(u_t(x)\big)\dot{W}(t,x) \quad (t \geq 0, x \in \mathbf{R}), \tag{1.1}$$

where: (i) $\mathcal{L}$ is the generator of a real-valued Lévy process $\{X_t\}_{t\geq 0}$ with Lévy exponent $\Psi$, normalized so that $\mathrm{E}e^{i\xi X_t} = e^{-t\Psi(\xi)}$ for every $\xi \in \mathbf{R}$ and $t \geq 0$; (ii) $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\sigma$; (iii) $\dot{W}$ is space–time white noise; and (iv) the initial datum $u_0$ is a signed Borel measure on $\mathbf{R}$.

Equation (1.1) arises in many different contexts; three notable examples are Bertini and Cancrini [1], Gyöngy and Nualart [16], and Carmona and Molchanov [7].

In the case that $u_0 : \mathbf{R} \to \mathbf{R}_+$ is a bounded measurable function, the theory of Dalang [11] shows that there exists a unique random-field mild solution

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$\{u_t(x)\}_{t\geq 0, x\in\mathbf{R}}$ provided that

$$\Upsilon(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\beta + 2\operatorname{Re}\Psi(\xi)} < \infty \quad \text{for some, hence all, } \beta > 0. \tag{1.2}$$

In general, Dalang's Condition (1.2) cannot be improved upon [11], [20].

Dalang's condition (1.2) implies also that the Lévy process $X$ has transition functions $p_t(x)$ [15, Lemma 8.1]; that is, for all measurable $f : \mathbf{R} \to \mathbf{R}_+$,

$$\mathrm{E} f(X_t) = \int_{-\infty}^{\infty} p_t(z) f(z)\,\mathrm{d}z \quad \text{for all } t > 0. \tag{1.3}$$

A *mild solution* in this setting is a random field $\{u_t(x)\}_{t\geq 0, x\in\mathbf{R}}$ that satisfies

$$u_t(x) = (P_t u_0)(x) + \int_{[0,t]\times\mathbf{R}} p_{t-s}(y-x)\sigma\big(u_s(y)\big) W(\mathrm{d}s\,\mathrm{d}y) \quad \text{a.s.} \tag{1.4}$$

for all $t \geq 0$ and $x \in \mathbf{R}$, where $\{P_t\}_{t\geq 0}$ denotes the semigroup associated to the process $X$, and the stochastic integral is understood as a Walsh martingale-measure stochastic integral [21]. Notice that (1.4) can be rewritten in the following form: For all $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$,

$$u_t(x) = (P_t u_0)(x) + \big(\tilde{p} * (\sigma \circ u)\dot{W}\big)_t(x) \quad \text{a.s.}, \tag{1.5}$$

where "$*$" denotes a space–time type "stochastic convolution" of $\tilde{p}$ with the martingale measure $(\sigma \circ u)\dot{W}$—see (2.1) below—and $\tilde{p}_t(x) := p_t(-x)$ for all $x \in \mathbf{R}$.

In the case that $u_0$ is not a bounded and measurable function, but instead a (possibly signed) Borel measure on $\mathbf{R}$, the solution $u$ cannot be defined as a random field, but has to be considered as a process with values in a certain space of generalized functions. It is also sometimes possible to consider random-field solutions with a measure-valued initial condition; see, for instance, Mueller [17] who considers the case that $\mathcal{L} \propto \Delta$, $\sigma(u) \propto u^\gamma$, and the equation is considered on a compact spatial interval up to a possible explosion time (when $\gamma > 1$).

When $u_0$ is a (possibly signed) Borel measure on $\mathbf{R}$, the stochastic convolution in (1.4) is not well defined in the sense of Walsh. Section 2 below is devoted to extending the definition of the stochastic convolution of a nonrandom process $\Gamma$ with respect to a martingale measure of the form $Z\dot{W}$ in the case that $Z$ takes values in a suitable Banach space $\mathbf{B}^k_{\beta,\eta}$ of random processes. The key step of this extension involves developing a kind of "stochastic Young inequality" (Proposition 2.2). Such an inequality appeared earlier in [10], in a different context, in order to obtain intermittency properties for equation (1.1) in the case that $u_0$ is a lower semicontinuous bounded function of compact support.

In Section 3, we establish the existence and uniqueness of a weak solution to (1.1). Namely, we prove that Dalang's condition (1.2) implies that if $u_0 = \mu$ is a (possibly signed) Borel measure on $\mathbf{R}$ that satisfies a suitable integrability condition (3.5), then there exists a unique $u \in \mathbf{B}^k_{\beta,\eta}$ such that $u$ almost surely satisfies (1.5) for almost every $t \geq 0$ and $x \in \mathbf{R}$. This solution is *not* a random field. Rather, it takes values in a certain space of generalized functions.

In Section 4, we prove that our condition for existence and uniqueness is sharp. And in Section 5, we mention briefly examples of initial data $u_0$ that lead to the existence and uniqueness of a weak solution to (1.1), together with further remarks that explain what happens if we study the 1-dimensional stochastic wave equation in place of the stochastic heat equation (1.1).

## 2. Generalized stochastic convolutions

Let $\Gamma : (0,\infty) \times \mathbf{R} \to \mathbf{R}$ be measurable, and $Z := \{Z_t(x)\}_{t>0, x\in\mathbf{R}}$ be a predictable random field in the sense of Walsh [21, p. 292]. Let us define the *stochastic convolution* $\Gamma * Z\dot{W}$ of the process $\Gamma$ with the noise $Z\dot{W}$ as the predictable random field

$$(\Gamma * Z\dot{W})_t(x) := \int_{[0,t]\times\mathbf{R}} \Gamma_{t-s}(x-y) Z_s(y) W(\mathrm{d}s\,\mathrm{d}y). \tag{2.1}$$

The preceding is defined as a stochastic integral with respect to the martingale measure $Z\dot{W}$ in the sense of Walsh [21, Theorem 2.5], and is well defined in the sense of Walsh [21, Chapter 2] provided that the following condition holds for all $t > 0$ and $x \in \mathbf{R}$:

$$\left\|(\Gamma * Z\dot{W})_t(x)\right\|_2^2 = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \left[\Gamma_{t-s}(x-y)\right]^2 \left\|Z_s(y)\right\|_2^2 < \infty, \tag{2.2}$$

where, here and throughout the paper, $\|\cdot\|_k$ denotes the standard norm on $L^k(P)$. That is,

$$\|X\|_k := \left\{\mathrm{E}\left(|X|^k\right)\right\}^{1/k} \quad \text{for all } k \in [1,\infty) \text{ and } X \in L^k(\mathrm{P}). \tag{2.3}$$

Let $\mathbf{W}^2$ denote the collection of all predictable random fields $Z$ such that: (i) $Z_t(x) \in L^2(\mathrm{P})$ for all $t > 0$ and $x \in \mathbf{R}$; and (ii) For all $0 < t \leq \tau$ and $x \in \mathbf{R}$,

$$\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \left[\Gamma_{\tau-s}(x-y)\right]^2 \left\|Z_s(y)\right\|_2^2 < \infty. \tag{2.4}$$

We may think of the elements of $\mathbf{W}^2$ as *Walsh-integrable random fields.* And because Condition (2.4) implies (2.2), the preceding discussion tells us that the stochastic convolution $\Gamma * Z\dot{W}$ is a well-defined predictable random field for every $Z \in \mathbf{W}^2$.

Our present goal is to extend the definition of the stochastic convolution of $Z$ so that the extended stochastic convolution can be applied to more general random processes $Z$. Other extensions of this stochastic convolution have been developed for other purposes as well [9], [11], [12], [18].

Let us choose and fix a real number $k \in [2,\infty)$, and define $\mathbf{L}^k$ to be the collection of all predictable random fields $\{Z_t(x)\}_{t>0,x\in\mathbf{R}}$ such that $Z_t(x) \in$

$L^k(\mathrm{P})$ for all $t>0$ and $x\in\mathbf{R}$. Let $M(\mathbf{R})$ be the space of $\sigma$-finite Borel measures on $\mathbf{R}$. For every $\beta>0$, $\eta\in M(\mathbf{R})$, and $Z\in\mathbf{L}^k$, define

$$\mathcal{N}^k_{\beta,\eta}(Z):=\left(\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t\sup_{z\in\mathbf{R}}\int_{-\infty}^\infty \eta(\mathrm{d}x)\big\|Z_t(x-z)\big\|_k^2\right)^{1/2}. \tag{2.5}$$

Here and throughout, we use implicitly the following observation: If $Z$, $Z'\in\mathbf{L}^k$ satisfy $\mathcal{N}^k_{\beta,\eta}(Z-Z')=0$, then $Z$ and $Z'$ are modifications of one another. There is an obvious converse as well: If $Z$ and $Z'$ are modifications of one another, then $\mathcal{N}^k_{\beta,\eta}(Z-Z')=0$. We omit the elementary proof.

Our next proposition is a "stochastic Young's inequality," and plays a key role in our extension of Walsh-type stochastic convolutions. But first we introduce some notation and recall the following form of Burkholder's inequality that will be used here and throughout.

THEOREM 2.1 (The Burkholder–Davis–Gundy inequality [2], [3], [4]). *Let $\{M_t\}_{t\ge 0}$ be a continuous martingale. Then, for all $k\ge 1$ and for all $t>0$ there exists a constant $z_k$ such that*

$$\|M_t\|_k\le z_k\big\|\langle M\rangle_t\big\|_{k/2}^{1/2}, \tag{2.6}$$

*where $\langle M\rangle$ denotes the quadratic variation of $M$.*

Throughout this paper, we always choose the constant $z_k$ of Burkholder's inequality to denote the optimal constant in Burkholder's $L^k(\mathrm{P})$-inequality for continuous square-integrable martingales. The precise value of $z_k$ involves the zeros of Hermite polynomials; see Davis [14].

By the Itô isometry, $z_2=1$. Carlen and Kree [6, Appendix] have shown that $z_k\le 2\sqrt{k}$ for all $k\ge 2$, and moreover $z_k=(2+o(1))\sqrt{k}$ as $k\to\infty$.

We are ready to describe the main result of this section.

PROPOSITION 2.2 (A stochastic Young's inequality). *For every $k\in[2,\infty)$, $Z\in\mathbf{W}^2\cap\mathbf{L}^k$, $\eta\in M(\mathbf{R})$, and $\beta>0$,*

$$\mathcal{N}^k_{\beta,\eta}(\Gamma * Z\dot{W})\le z_k\left(\int_0^\infty \mathrm{e}^{-\beta t}\|\Gamma_t\|^2_{L^2(\mathbf{R})}\,\mathrm{d}t\right)^{1/2}\cdot\mathcal{N}^k_{\beta,\eta}(Z). \tag{2.7}$$

REMARK 2.3. We emphasize that $\mathbf{W}^2\cap\mathbf{L}^2=\mathbf{W}^2$.

Before we prove Proposition 2.2, let us first describe how it can be used to extend stochastic convolutions. Proposition 2.2 will be proved after that extension is described.

Let $\mathbf{B}^k_{\beta,\eta}$ denote the completion of $\mathbf{W}^2\cap\mathbf{L}^k$ under the norm $\mathcal{N}^k_{\beta,\eta}$.[1] It follows then that $\mathbf{B}^k_{\beta,\eta}$ is a Banach space of predictable processes (identified up to evanescence) with norm $\mathcal{N}^k_{\beta,\eta}$.

[1] The latter is of course a norm on equivalence classes of modifications of random fields and not on random fields themselves. But we abuse notation as it is standard.

Proposition 2.2 immediately implies that if

$$\Upsilon(\beta) := \int_0^\infty \mathrm{e}^{-\beta t} \|\Gamma_t\|^2_{L^2(\mathbf{R})}\, \mathrm{d}t < \infty, \tag{2.8}$$

then $Z \mapsto \Gamma * Z\dot{W}$ has a unique extension to all $Z \in \mathbf{B}^k_{\beta,\eta}$, and the resulting extension—written still as $Z \mapsto \Gamma * Z\dot{W}$—defines a bounded linear operator from $\mathbf{B}^k_{\beta,\eta}$ into itself. And the operator norm is at most the square root of the *Dalang integral* $\Upsilon(\beta)$. [In the case that $\Gamma_t(x)$ denotes the transition density of a Lévy process with Lévy exponent $\Psi$, Plancherel's theorem implies that $\Upsilon(\beta)$ is the same Dalang integral as in (1.2); see (3.1) below as well.]

From now on, we deal solely with this extension of the stochastic convolution. However, we point out also that there is a great deal of variability in this extension, as the parameters $\beta > 0$, $k \in [2,\infty)$, and $\eta \in M(\mathbf{R})$ can take on many different values.

Let us conclude this section by establishing our stochastic Young's inequality.

The proof of Proposition 2.2 relies on an elementary estimate for Walsh-type stochastic integrals.

LEMMA 2.4. *For all real numbers $t > 0$, $x \in \mathbf{R}$, and $k \in [2,\infty)$, and for every $Z \in \mathbf{W}^2 \cap \mathbf{L}^k$,*

$$\big\|(\Gamma * Z\dot{W})_t(x)\big\|_k \le z_k \left(\int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \big[\Gamma_{t-s}(x-y)\big]^2 \big\|Z_s(y)\big\|_k^2\right)^{1/2}. \tag{2.9}$$

*Proof.* Condition (2.4) implies that if $0 < t \le \tau$, then

$$(\Gamma * Z\dot{W})_{t,\tau}(x) := \int_{[0,t]\times\mathbf{R}} \Gamma_{\tau-s}(x-y) Z_s(y) W(\mathrm{d}s\,\mathrm{d}y) \tag{2.10}$$

is well defined and in $L^2(\mathrm{P})$. Moreover,

$$\big\|(\Gamma * Z\dot{W})_{t,\tau}(x)\big\|_2 = \left(\int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \big[\Gamma_{\tau-s}(x-y)\big]^2 \big\|Z_s(y)\big\|_2^2\right)^{1/2}. \tag{2.11}$$

Walsh's theory of martingale measures [21, Theorem 2.5] tells us that the stochastic process $(0,\tau] \ni t \mapsto (\Gamma * Z\dot{W})_{t,\tau}(x)$ is a continuous $L^2(\mathrm{P})$-martingale. Therefore, Burkholder's inequality (Theorem 2.1) implies that

$$\big\|(\Gamma * Z\dot{W})_{t,\tau}(x)\big\|_k^k \le z_k^k \left\|\int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \big[\Gamma_{\tau-s}(x-y)\big]^2 \big[Z_s(y)\big]^2\right\|_{k/2}^{k/2}. \tag{2.12}$$

And it follows from Minkowski's inequality that $\|(\Gamma * Z\dot{W})_{t,\tau}(x)\|_k$ is bounded above by the right-hand side of the inequality (2.9). The lemma follows from this upon setting $\tau := t$. $\square$

*Proof of Proposition* 2.2. The original construction of Walsh implies that $\|(\Gamma * Z\dot{W})_t(x)\|_k$ defines a Borel-measurable function of $(t,x) \in (0,\infty) \times \mathbf{R}$. Indeed, it suffices to verify this measurability assertion in the case that $Z$ is a simple function in the sense of Walsh [21, p. 292], in which case the said measurability follows from a direct computation.

We may apply Lemma 2.4 with $x - z$ in place of the variable $x$, and then integrate $[\mathrm{d}\eta]$ to obtain

$$
\begin{aligned}
(2.13)\qquad & \int_{-\infty}^{\infty} \eta(\mathrm{d}x) \big\|(\Gamma * Z\dot{W})_t(x-z)\big\|_k^2 \\
&\le z_k^2 \int_{-\infty}^{\infty} \eta(\mathrm{d}x) \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \big[\Gamma_{t-s}(x-z-y)\big]^2 \big\|Z_s(y)\big\|_k^2 \\
&= z_k^2 \int_{-\infty}^{\infty} \eta(\mathrm{d}x) \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \big[\Gamma_{t-s}(y)\big]^2 \big\|Z_s(x-z-y)\big\|_k^2 \\
&\le z_k^2 \int_0^t \mathrm{d}s \|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2 \sup_{v \in \mathbf{R}} \int_{-\infty}^{\infty} \eta(\mathrm{d}x) \big\|Z_s(x-v)\big\|_k^2.
\end{aligned}
$$

Or equivalently,

$$
\begin{aligned}
(2.14)\qquad & \sup_{z \in \mathbf{R}} \int_{-\infty}^{\infty} \eta(\mathrm{d}x) \big\|(\Gamma * Z\dot{W})_t(x-z)\big\|_k^2 \\
&\le z_k^2 \int_0^t \mathrm{d}s \|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2 \sup_{z \in \mathbf{R}} \int_{-\infty}^{\infty} \eta(\mathrm{d}x) \big\|Z_s(x-z)\big\|_k^2.
\end{aligned}
$$

Multiply both sides by $\exp(-\beta t)$, integrate $[\mathrm{d}t]$ and use Laplace transforms properties for convolutions to obtain the result. □

PROPOSITION 2.5. *Suppose $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and $Z, Z^* \in \mathbf{B}_{\beta,\eta}^k$ for some $k \in [2,\infty)$, $\beta > 0$, and $\eta \in M(\mathbf{R})$. Then,*

$$
(2.15)\qquad \mathcal{N}_{\beta,\eta}^k\big(\sigma \circ Z - \sigma \circ Z^*\big) \le \mathrm{Lip}_\sigma \cdot \mathcal{N}_{\beta,\eta}^k\big(Z - Z^*\big).
$$

*Proof.* If $Z, Z^* \in \mathbf{W}^2 \cap \mathbf{L}^k$, then this is immediate. In the general case, we proceed by approximation: Let $Z^1, Z^2, \dots, Z^{1,*}, Z^{2,*}, \dots$ be in $\mathbf{W}^2 \cap \mathbf{L}^k$ such that $Z^n \to Z$ and $Z^{n,*} \to Z^*$ in $\mathbf{B}_{\beta,\eta}^k$, as $n \to \infty$. By going to a subsequence, if necessary, we can (and will!) also assume that

$$
(2.16)\qquad \mathcal{N}_{\beta,\eta}^k\big(Z^n - Z^{n+1}\big) + \mathcal{N}_{\beta,\eta}^k\big(Z^{n,*} - Z^{n+1,*}\big) \le 2^{-n} \quad \text{for all } n \ge 1.
$$

It follows also that for all $n \ge 1$,

$$
(2.17)\quad \mathcal{N}_{\beta,\eta}^k\big(\sigma \circ Z^n - \sigma \circ Z^{n+1}\big) + \mathcal{N}_{\beta,\eta}^k\big(\sigma \circ Z^{n,*} - \sigma \circ Z^{n+1,*}\big) \le \mathrm{Lip}_\sigma \cdot 2^{-n}.
$$

Of course, this implies immediately that $\sigma \circ Z^n$ and $\sigma \circ Z^{n,*}$ converge in $\mathbf{B}_{\beta,\eta}^k$. It suffices to prove that the mentioned limits are respectively $\sigma \circ Z$ and $\sigma \circ Z^*$.

But (2.17) implies that

$$\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \sum_{n=1}^\infty \sup_{z\in\mathbf{R}} \left( \int_{-\infty}^\infty \eta(\mathrm{d}x) \left\| \Delta_t^{n,i}(x-z) \right\|_k^2 \right) < \infty \tag{2.18}$$

for all $i = 1, 2, 3, 4$, where the $\Delta_t^{n,i}(x)$'s are defined as follows:

- $\Delta_t^{n,1}(x) := Z_t^n(x) - Z_t^{n+1}(x)$;
- $\Delta_t^{n,2}(x) := Z_t^{n,*}(x) - Z_t^{n+1,*}(x)$;
- $\Delta_t^{n,3}(x) := \sigma(Z_t^n(x)) - \sigma(Z_t^{n+1}(x))$; and
- $\Delta_t^{n,4}(x) := \sigma(Z_t^{n,*}(x)) - \sigma(Z_t^{n+1,*}(x))$.

Because $\sum_{n=1}^\infty \sup_{z\in\mathbf{R}}(\cdot) \geq \sup_{z\in\mathbf{R}} \sum_{n=1}^\infty(\cdot)$ in (2.18), it follows readily that for almost every pair $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$:

- $\lim_{n\to\infty} Z_t^n(x) = Z_t(x)$ almost surely;
- $\lim_{n\to\infty} Z_t^{n,*}(x) = Z_t^*(x)$ almost surely;
- $\lim_{n\to\infty} \sigma(Z_t^n(x)) = \sigma(Z_t(x))$ almost surely; and
- $\lim_{n\to\infty} \sigma(Z_t^{n,*}(x)) = \sigma(Z_t^*(x))$ almost surely.

(Note the order of the quantifiers!) We proved earlier that $\lim_{n\to\infty} \sigma\circ Z^n$ and $\lim_{n\to\infty} \sigma\circ Z^{n,*}$ exist in $\mathbf{B}_{\beta,\eta}^k$. The preceding shows that those limits are respectively $\sigma\circ Z$ and $\sigma\circ Z^*$. This completes the proof. $\square$

## 3. Existence and uniqueness

This section is devoted to the statement and proof of the existence and uniqueness of a weak solution to (1.1). We will make use of the generalized stochastic convolution developed in Section 2.

Before we proceed further, let us observe that from now on $\Gamma_t(x)$ of the previous section is chosen to be equal to the modified transition functions $\tilde{p}_t(x)$, in which case Dalang's integral can be computed from Plancherel's formula as follows:

$$\Upsilon(\beta) = \int_0^\infty \mathrm{e}^{-\beta t} \|p_t\|_{L^2(\mathbf{R})}^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathrm{d}\xi}{\beta + 2\operatorname{Re}\Psi(\xi)}. \tag{3.1}$$

In particular, the $\Upsilon$ of (2.8) and that of (1.2) are equal in the present setting.

Next, we identify our notion of "solution" to (1.1) in the case that $u_0 = \mu$ is a measure.

Suppose first that $u_0$ is a nice *function* and (1.1) has a mild solution $u$ with initial datum $u_0$. Then for all $t > 0$ and $x \in \mathbf{R}$,

$$\mathrm{P}\left\{ u_t(x) = (P_t u_0)(x) + \left(\tilde{p} * (\sigma\circ u)\dot{W}\right)_t(x) \right\} = 1. \tag{3.2}$$

Consequently, Fubini's theorem tells us that every mild solution $u$ to (1.1) with initial function $\mu := u_0$ is a *weak solution* in the sense that the following holds with probability one (note the order of the quantifiers!):

$$u_t(x) = (P_t\mu)(x) + \left(\tilde{p} * (\sigma\circ u)\dot{W}\right)_t(x) \quad \text{for a.e. } (t, x) \in \mathbf{R}_+ \times \mathbf{R}. \tag{3.3}$$

It is easy to see that the preceding agrees with Walsh's definition of a weak solution [21, p. 309].

Now we consider (1.1) in the case that $u_0 = \mu$ is a possibly-signed Borel measure on $\mathbf{R}$.

Let us suppose that Dalang's condition (2.8) holds, and consider an arbitrary $u \in \mathbf{B}^k_{\beta,\eta}$. Since $\sigma$ is Lipschitz continuous, it follows that $\sigma \circ u \in \mathbf{B}^k_{\beta,\eta}$. Therefore, we can conclude that the stochastic convolution $\tilde{p} * (\sigma \circ u)\dot{W}$ is a well-defined mathematical object, as was shown in the previous section. Consequently, we can try to find a solution $u$ to (1.1) with $u_0 = \mu$ by seeking to find $u \in \mathbf{B}^k_{\beta,\eta}$ such that

$$u = P_\bullet \mu + \tilde{p} * (\sigma \circ u)\dot{W}, \tag{3.4}$$

where the equality is understood as equality of elements of $\mathbf{B}^k_{\beta,\eta}$. Of course, we implicitly are assuming that $P_\bullet \mu \in \mathbf{B}^k_{\beta,\eta}$ as well. That condition is clearly satisfied if

$$\int_0^\infty \mathrm{e}^{-\beta s}\,\mathrm{d}s \sup_{z \in \mathbf{R}} \left( \int_{-\infty}^\infty \eta(\mathrm{d}x) \left| (P_s\mu)(x - z) \right|^2 \right) < \infty. \tag{3.5}$$

Then, $u$ is a solution of function-space type to (1.1) with $u_0 = \mu$. But it has more structure than that. Indeed, suppose that: (i) (3.5) holds; and (ii) There exists $u \in \mathbf{B}^k_{\beta,\eta}$ that satisfies (3.4). Then the preceding discussion shows also that $u$ is a weak solution to (1.1) in the sense of Walsh [21, p. 309]. And it would be hopeless to try to prove that such a $u$ is a mild solution, as there is no natural way to define $u_t(x)$ for all $t > 0$ and $x \in \mathbf{R}$.

Throughout the remainder of this section, we choose $\eta \in M(\mathbf{R})$. In the case that $\sigma(0) \neq 0$, then we assume additionally that $\eta$ is a finite measure.

THEOREM 3.1. *Consider* (1.1) *subject to* $u_0 = \mu$, *where* $\mu$ *is a signed measure that satisfies* (3.5). *If* (1.2) *holds and*

$$\Upsilon(\beta) < \frac{1}{(z_k \mathrm{Lip}_\sigma)^2}, \tag{3.6}$$

*then there exists a solution* $u \in \mathbf{B}^k_{\beta,\eta}$ *that satisfies* (3.4). *Moreover*, $u$ *is unique in* $\mathbf{B}^k_{\beta,\eta}$; *that is, if there exists another weak solution* $v$ *that is in* $\mathbf{B}^k_{\beta,\eta}$ *for some* $k \geq 2$, *then* $v$ *is a modification of* $u$.

*Proof.* First, we argue that we can always choose $\beta$ such that (3.6) holds.

Indeed, Condition (3.5) implies that $\mathcal{N}^k_{\beta,\eta}(P_\bullet u_0) < \infty$ for all $\beta > 0$ and $k \in [2, \infty)$. Also, because of Dalang's condition (1.2), and by the monotone convergence theorem, $\lim_{\alpha \to \infty} \Upsilon(\alpha) = 0$. Therefore, we can combine these two observations to deduce that (3.6) holds for all $\beta$ large, where $1/0 := \infty$. Throughout the remainder of the proof, we hold fixed a $\beta$ that satisfies (3.6).

Set $u_t^{(0)} := 0$, and iteratively define

$$u^{(n+1)} := P_\bullet \mu + \tilde{p} * \left( \left[ \sigma \circ u^{(n)} \right] \dot{W} \right). \tag{3.7}$$

These $u^{(n+1)}$'s are all well defined elements of $\mathbf{B}_{\beta,\eta}^k$. In fact, it follows from Proposition 2.2 that for all $n \geq 0$,

$$\mathcal{N}_{\beta,\eta}^k\big(u^{(n+1)}\big) \leq \mathcal{N}_{\beta,\eta}^k(P_\bullet \mu) + z_k\sqrt{\Upsilon(\beta)}\mathcal{N}_{\beta,\eta}^k\big(\sigma \circ u^{(n)}\big). \tag{3.8}$$

And because $|\sigma(z)| \leq |\sigma(0)| + \mathrm{Lip}_\sigma|z|$ for all $z \in \mathbf{R}$,

$$\begin{aligned}\mathcal{N}_{\beta,\eta}^k&\big(u^{(n+1)}\big) \\ &\leq \mathcal{N}_{\beta,\eta}^k(P_\bullet \mu) + z_k\sqrt{\Upsilon(\beta)}\big[\big|\sigma(0)\big| \cdot \mathcal{N}_{\beta,\eta}^k(\mathbf{1}) + \mathrm{Lip}_\sigma \cdot \mathcal{N}_{\beta,\eta}^k\big(u^{(n)}\big)\big],\end{aligned} \tag{3.9}$$

where $\mathbf{1}_t(x) := 1$ for all $t > 0$ and $x \in \mathbf{R}$. In particular, $u^{(l)} \in \mathbf{B}_{\beta,\eta}^k$ for all $l \geq 0$, by induction. This is clear if $\sigma(0) = 0$; and if $\sigma(0) \neq 0$, then it is also true because $\mathcal{N}_{\beta,\eta}^k(\mathbf{1}) = \sqrt{\eta(\mathbf{R})/\beta} < \infty$, thanks to the finiteness assumption on $\eta$ [for the case $\sigma(0) \neq 0$]. Moreover, (3.6) and induction together show more; namely, that $\sup_{n\geq 0}\mathcal{N}_{\beta,\eta}^k(u^{(n)}) < \infty$.

A similar computation, this time using also Proposition 2.5, shows that for all $n \geq 1$,

$$\mathcal{N}_{\beta,\eta}^k\big(u^{(n+1)} - u^{(n)}\big) \leq z_k \mathrm{Lip}_\sigma\sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k\big(u^{(n)} - u^{(n-1)}\big). \tag{3.10}$$

And (3.6) implies that $\sum_{n=0}^\infty \mathcal{N}_{\beta,\eta}^k(u^{(n+1)} - u^{(n)}) < \infty$, therefore, $\{u^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence in $\mathbf{B}_{\beta,\eta}^k$. Let $u := \lim_{n\to\infty} u^{(n)}$, where the limit takes place in $\mathbf{B}_{\beta,\eta}^k$. According to Proposition 2.2,

$$\begin{aligned}\mathcal{N}_{\beta,\eta}^k\big(\tilde{p} * u^{(n)}\dot{W} - \tilde{p} * u\dot{W}\big) &\leq z_k\sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k\big(u^{(n)} - u\big) \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.11}$$

It follows readily from these remarks that $\mathcal{N}_{\beta,\eta}^k(u - P_\bullet\mu + \tilde{p} * (\sigma \circ u)\dot{W}) = 0$. That is, $u$ satisfies (3.3) for almost all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$; see also (3.4). This proves the first part of the theorem.

In order to prove the second part, let us suppose that there exists another "weak solution" $v \in \mathbf{B}_{\beta,\eta}^k$. Then, $\delta := u - v \in \mathbf{B}_{\beta,\eta}^k$ and

$$\delta = \tilde{p} * \big([\sigma \circ u]\dot{W}\big) - \tilde{p} * \big([\sigma \circ v]\dot{W}\big) = \tilde{p} * \big([\sigma \circ u - \sigma \circ v]\dot{W}\big). \tag{3.12}$$

(The second identity follows from the very construction of our stochastic convolution, using the fact that $Z \mapsto \tilde{p} * Z\dot{W}$ is a bounded *linear* map from $\mathbf{B}_{\beta,\eta}^k$ to itself.) Propositions 2.2 and 2.5 together imply the following:

$$\begin{aligned}\mathcal{N}_{\beta,\eta}^k(\delta) &\leq z_k\sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k(\sigma \circ u - \sigma \circ v) \\ &\leq z_k \mathrm{Lip}_\sigma\sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k(\delta).\end{aligned} \tag{3.13}$$

Thanks to (3.6), $\mathcal{N}_{\beta,\eta}^k(u - v) = \mathcal{N}_{\beta,\eta}^k(\delta) = 0$. This readily implies that $u$ and $v$ are modifications of one another, as well. $\square$

## 4. On Condition (3.6)

Let us consider the measure $\eta_m \in M(\mathbf{R})$ defined by

$$\eta_m(\mathrm{d}x) = \mathrm{e}^{-|x|/m}\,\mathrm{d}x, \tag{4.1}$$

where $m > 0$ is fixed. If $\sigma(0) = 0$, then we may take $m := \infty$, whence $\eta(\mathrm{d}x) = \mathrm{d}x$.

THEOREM 4.1. *Suppose* (1.1) *has a solution* $u \in \bigcap_{m>0} \mathbf{B}^2_{\beta,\eta_m}$ *with* $u_0 = \mu$ *for a nonvoid signed Borel measure* $\mu$ *on* $\mathbf{R}$ *with* $|\mu|(\mathbf{R}) < \infty$. *Suppose also that* $\mathrm{L}_\sigma := \inf_{z\in\mathbf{R}} |\sigma(z)/z| > 0$. *Then,* $\beta$ *satisfies* $\Upsilon(\beta) < \mathrm{L}_\sigma^{-2}$.

*Proof.* Let $\mathcal{M}_\beta$ be the norm defined by

$$\mathcal{M}_\beta(Z) := \left(\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \int_{-\infty}^\infty \mathrm{e}^{-|x|/m}\,\mathrm{d}x \big\|Z_t(x)\big\|_2^2\right)^{1/2}. \tag{4.2}$$

Notice that $M_\beta$ is similar to $\mathcal{N}_{\beta,\eta_m}$, but is missing a supremum on $Z$ in the space variable; cf. (2.5). Moreover, $\mathcal{M}_\beta(u) \le \mathcal{N}^2_{\beta,\eta_m}(u) < \infty$. Note that if $H, Z \in \mathbf{B}^2_{\beta,\eta_m}$ with one of them—say $H$—random and the other one deterministic, then we have $[\mathcal{M}_\beta(H+G)]^2 = [\mathcal{M}_\beta(H)]^2 + [\mathcal{M}_\beta(G)]^2$. This is a direct computation if $H, G \in \mathbf{W}^2$; the general case follows from approximation (we omit the details because the method appears already during the course of the proof of Proposition 2.5). It follows that

$$\big[\mathcal{M}_\beta(u)\big]^2 = \big[\mathcal{M}_\beta(P_\bullet\mu)\big]^2 + \big[\mathcal{M}_\beta\big(\tilde{p} * \big([\sigma\circ u]\dot{W}\big)\big)\big]^2. \tag{4.3}$$

The method of proof of Proposition 2.5, together with the simple bound, $\mathrm{e}^{-|x|/m} \ge \mathrm{e}^{-|x-y|/m}\cdot \mathrm{e}^{-|y|/m}$, shows also that

$$\mathcal{M}_\beta\big(\tilde{p} * \big([\sigma\circ u]\dot{W}\big)\big) \ge \mathrm{L}_\sigma \mathcal{M}_\beta(\tilde{p} * u\dot{W}). \tag{4.4}$$

But

$$\mathcal{M}_\beta(\tilde{p} * Z\dot{W}) = \left(\int_0^\infty \mathrm{e}^{-\beta t}\|p_t\|^2_{L^2(\mathbf{R})}\,\mathrm{d}t\right)^{1/2} \cdot \mathcal{M}_\beta(Z). \tag{4.5}$$

(Again one proves this first for nice $Z$'s and then take limits.) Therefore,

$$\mathcal{M}_\beta\big(\tilde{p} * \big([\sigma\circ u]\dot{W}\big)\big) \ge \mathrm{L}_\sigma\sqrt{\Upsilon(\beta)}\cdot \mathcal{M}_\beta(u). \tag{4.6}$$

Combine this with (4.3) to find that

$$\big[\mathcal{M}_\beta(u)\big]^2 \ge \big[\mathcal{M}_\beta(P_\bullet\mu)\big]^2 + \mathrm{L}_\sigma^2\Upsilon(\beta)\big[\mathcal{M}_\beta(u)\big]^2. \tag{4.7}$$

Now suppose, to the contrary, that $\Upsilon(\beta) \ge \mathrm{L}_\sigma^{-2}$. Then, it follows that $\mathcal{M}_\beta(P_\bullet\mu) = 0$ regardless of the value of $m$; that is, for all $m > 0$,

$$\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \int_{-\infty}^\infty \mathrm{e}^{-|x|/m}\,\mathrm{d}x \big|(P_t\mu)(x)\big|^2 = 0. \tag{4.8}$$

Let $m \uparrow \infty$ and apply the monotone convergence theorem, and then the Plancherel theorem, in order to deduce that

$$0 = \int_0^\infty e^{-\beta t} \|P_t\mu\|^2_{L^2(\mathbf{R})} \, dt = \frac{1}{2\pi} \int_0^\infty e^{-\beta t} \, dt \int_{-\infty}^\infty d\xi e^{-2t \operatorname{Re}\Psi(\xi)} \left|\hat{\mu}(\xi)\right|^2 = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{|\hat{\mu}(\xi)|^2}{\beta + 2\operatorname{Re}\Psi(\xi)} \, d\xi. \tag{4.9}$$

Since $\Psi$ is never infinite, the preceding implies that $\mu \equiv 0$, which is a contradiction. It follows that $\Upsilon(\beta) < \mathrm{L}_\sigma^{-2}$. □

Theorem 4.1 implies also that Condition (3.6) is sharp: Consider the case that $\mathrm{Lip}_\sigma = \mathrm{L}_\sigma$. [This is the case, for instance, for the parabolic Anderson problem where $\sigma(x) \propto x$, or when $\sigma$ has sharp linear growth.] Then in this case Theorem 3.1 and Theorem 4.1 together imply that (3.6) is a necessary and sufficient condition for the existence of a weak solution to (1.1) that has values in $\bigcap_{m\geq 1} \mathbf{B}^k_{\beta,\eta_m}$.

## 5. Examples and remarks

EXAMPLE 5.1 (A parabolic Anderson model). Let $\sigma(x) = \lambda x$. In that case, the solution $u$ corresponds to the conditional expected density at time $t \geq 0$ of a branching Lévy process starting with distribution $u_0$, given white-noise random branching. The case that $\sigma(0) = 0$ and $u_0$ is a function with compact support is studied in [10], in which intermittency properties are derived. Here, $u_0$ can be a compactly supported measure (not necessarily a function). If we let $\eta$ denote the one-dimensional Lebesgue measure, then (3.5) becomes

$$\int_{-\infty}^\infty \frac{|\hat{\mu}(\xi)|^2}{\beta + 2\operatorname{Re}\Psi(\xi)} \, d\xi < \infty. \tag{5.1}$$

For instance, if $\mu = \delta_0$, then condition (5.1) is precisely Dalang's condition (1.2), and (1.1) admits a weak solution. In this way, we can now define in a more standard manner the solution of the parabolic Anderson model with $u_0 = \delta_0$, which was studied in Bertini and Cancrini [1].

REMARK 5.2 (A nonlinear stochastic wave equation). It is possible to apply similar techniques to the study of the following nonlinear stochastic wave equation driven by the Laplacian:

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 \frac{\partial^2}{\partial x^2} u_t(x) + \sigma\big(u_t(x)\big) \dot{W}(t,x) \quad (t \geq 0, x \in \mathbf{R}). \tag{5.2}$$

If the initial conditions $u_0$ and $v_0$ are nice functions, then the solution to (5.2) can be written as

$$u_t(x) = \big(\Gamma'_t * u_0\big)(x) + (\Gamma_t * v_0)(x) + \int_{[0,t]\times\mathbf{R}} \Gamma_{t-s}(y-x)\sigma\big(u_s(y)\big)W(\mathrm{d}s\,\mathrm{d}y), \tag{5.3}$$

where $\Gamma$ is the fundamental solution of the 1-dimensional wave equation, namely

$$\Gamma_t(x) := \frac{1}{2}\mathbf{1}_{[-\kappa t,\kappa t]}(x) \quad \text{for } t > 0 \text{ and } x \in \mathbf{R}, \tag{5.4}$$

and $\Gamma'_t$ denotes the weak spatial derivative of $\Gamma_t$. Then, the existence and uniqueness of a weak solution to (5.2) in the case that $u_0$ and $v_0$ are (possibly signed) Borel measures on $\mathbf{R}$ can be established using the techniques of this paper, since the definition of the generalized stochastic convolution applies as well with the 1-D wave propagator $\Gamma$ above. The conditions on the initial conditions have to be adapted to insure that $\Gamma'_t * u_0$ and $\Gamma_t * v_0$ both are in $\mathbf{B}^k_{\beta,\eta}$, but are similar to (3.5). The details on this are left to the reader, as the stochastic wave equation in dimension one has been widely studied already [5], [8], [13], [19], [20], [21], [22].

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## References


[1] L. Bertini and N. Cancrini, *The stochastic heat equation: Feynman–Kac formula and intermittence*, J. Statist. Physics **78** (1994), 1377–1402. MR 1316109

[2] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504. MR 0208647

[3] D. L. Burkholder, B. J. Davis and R. F. Gundy, *Integral inequalities for convex functions of operators on martingales*, Proceedings of the sixth Berkeley symposium on mathematical statistics and probability, vol. II, University of California Press, Berkeley, CA, 1972, pp. 223–240. MR 0400380

[4] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. **124** (1970), 249–304. MR 0440695

[5] E. M. Cabaña, *The vibrating string forced by white noise*, Z. Wahrsch. Verw. Gebiete **15** (1970), 111–130. MR 0279909

[6] E. Carlen and P. Kree, *$L^p$ estimates for multiple stochastic integrals*, Ann. Probab. **19** (1991), 354–368. MR 1085341

[7] R. A. Carmona and S. A. Molchanov, *Parabolic Anderson problem and intermittency*, Mem. Amer. Math. Soc., vol. 108, Amer. Math. Soc., Providence, RI, 1994. MR 1185878

[8] R. A. Carmona and D. Nualart, *Random nonlinear wave equations: Propagation of singularities*, Ann. Probab. **16** (1988), 730–751. MR 0929075

[9] D. Conus and R. C. Dalang, *The nonlinear stochastic wave equation in high dimensions*, Electron. J. Probab. **13** (2008), 629–670. MR 2399293

[10] D. Conus and D. Khoshnevisan, *On the existence and position of the farthest peaks of a family of parabolic and hyperbolic SPDE's*, to appear in Probab. Theory Related Fields (2012).

[11] R. C. Dalang, *Extending martingale measure stochastic integral with applications to spatially homogeneous SPDE's*, Electron. J. Probab. **4** (1999), 1–29. MR 1684157

[12] R. C. Dalang and C. Mueller, *Some nonlinear S.P.D.E.'s that are second order in time*, Electron. J. Probab. **8** (2003), 1–21 (electronic). MR 1961163

[13] R. C. Dalang, *The stochastic wave equation*, A minicourse on stochastic partial differential equations, Lecture Notes in Mathematics, vol. 1962, Springer, Berlin, 2009, pp. 39–71. MR 2508773

[14] B. Davis, *On the $L^p$ norms of stochastic integrals and other martingales*, Duke Math. J. **43** (1976), 697–704. MR 0418219

[15] M. Foondun, D. Khoshnevisan and E. Nualart, *A local-time correspondence for stochastic partial differential equations*, Trans. Amer. Math. Soc. **363** (2011), 2481–2515. MR 2763724

[16] I. Gyöngy and D. Nualart, *On the stochastic Burgers' equation in the real line*, Ann. Probab. **27** (1999), 782–802. MR 1698967

[17] C. Mueller, *Singular initial conditions for the heat equation with noise*, Ann. Probab. **24** (1996), 377–398. MR 1387640

[18] D. Nualart and L. Quer-Sardanyons, *Existence and smoothness of the density for spatially homogeneous SPDEs*, Potential Anal. **27** (2007), 281–299. MR 2336301

[19] S. Peszat, *The Cauchy problem for a nonlinear stochastic wave equation in any dimension*, J. Evol. Equ. **2** (2002), 383–394. MR 1930613

[20] S. Peszat and J. Zabczyk, *Nonlinear stochastic wave and heat equations*, Probab. Theory Related Fields **116** (2000), 421–443. MR 1749283

[21] J. B. Walsh, *An Introduction to Stochastic Partial Differential Equations*, Ecole d'Etè de Probabilités de St-Flour, XIV, 1984, Lecture Notes in Mathematics, vol. 1180, Springer-Verlag, Berlin, 1986, pp. 265–439. MR 0876085

[22] J. B. Walsh, *On numerical solutions of the stochastic wave equation*, Illinois J. Math. **50** (2006), 991–1018. MR 2247851

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